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Asymptotics for Strongly Nonlinear Dynamical Systems

Strongly nonlinear dynamical systems are analysed using a new asymptotic approach. Periodic oscillations of the systems with a polynomial nonlinearity are considered and a period of strong nonlinear oscillations of the mathematical pendulum is estimated analytically.

1. Introduction

The first analytical investigations of nonlinear dynamical systems were rather limited to the systems with small nonlinearities or to those with small dynamical processes [1,2]. However, recent results show that these limitations can be omitted. A formal perturbation parameter can be introduced as well as the recently developed, so called small δ method can be used [3]. In this paper we present how the small δ method can be used to detect the periodic oscillations of discrete nonlinear dynamical systems governed by ordinary differential equations.
We consider the homogeneous nonlinear differential equation with the attached initial conditions

$$\ddot{x} + x^n = 0, \quad x(0) = 1, \quad \dot{x}(0) = 0. \tag{1}$$

Using the small δ method, equation (1) is transformed into the following one

$$\ddot{x} + x^{1+2\delta} = 0. \tag{2}$$

Taking into account the following series

$$(x^2)^\delta = 1 + \delta\ln(x^2) + \frac{\delta^2}{2}[\ln(x^2)]^2 + \dots \tag{3}$$

the solution to equation (2) is sought in the form $x = \sum_{n=0}^{\infty} \delta^n x_n$.

Besides, we introduce a standard change of time following the formula and the introduced parameter ω is defined by the series

$$t = \tau/\omega, \qquad \omega^2 = 1 + \alpha_1\delta + \alpha_2\delta^2 + \dots \tag{4}$$

Substituting the introduced quantities into equation (2) we get (after splitting with respect to δ) the following recurrent system of equations

$$\ddot{x}_0 + x_0 = 0, \quad x_0(0) = 1, \quad \dot{x}_0(0) = 0; \tag{5}$$

$$\ddot{x}_1 + x_1 = -x_0\ln(x_0)^2 - \alpha_1\ddot{x}_0, \quad x_1(0) = \dot{x}_1(0) = 0; \tag{6}$$

$$\ddot{x}_2 + x_2 = -[x_1\ln(x_0^2) + 2x_1] - x_0[\ln(x_0^2)]^2 - \alpha_2\ddot{x}_0 - \alpha_1\ddot{x}_1, \quad x_2(0) = \dot{x}_2(0) = 0; \tag{7}$$

....

From (5) one obtains $x_0 = \cos t$, and from (6) we get

$$\ddot{x}_1 + x_1 = -\cos t\ln(\cos^2 t) + \alpha_1\cos t \equiv L_0. \tag{8}$$

The most natural method of constructing the solution to equation (8) is related to the Fourier series of the first term of the right-hand side equation in order to cancel the resonance behavior by a proper choice of α_1. A condition of secular terms absence in the final solution may be written as follows:

$$\int_0^{\pi/2} L_0\cos t\,dt = 0, \text{ and one obtains } \alpha_1 = 1 - 2\ln 2.$$

Therefore, a period of the solution can be defined as follows: $T \approx 2\pi[1 + \delta(\ln 2 - 0.5)]$.
For $n = 3$ ($\delta = 1$) the above formula yields $T = 6.8070$, where the exact period value $T = 7.4164$ (the error exceeds 8%). The next approximation gives practically the exact value ($T = 7.4111$).

2. Mathematical pendulum

We consider the mathematical pendulum governed by the equation with the corresponding boundary conditions

$$\ddot{x} + \sin x = 0, \; x(0) = 1, \quad \dot{x}(0) = 0. \tag{9}$$

The quasi-linear approach does not guarantee the required accuracy of the solution to the above boundary value problem. Taking into account the following approximations

$$\sin x = x - \frac{1}{3!}x^{1+2\delta} + \frac{1}{5}x^{1+4\delta} - \frac{1}{7!}x^{1+6\delta} + \dots, \tag{10}$$

$$x^{2\delta} = 1 + \delta\ln(x^2) + \frac{1}{2}\delta^2\ln^2 x^2 + \frac{1}{6}\delta^3\ln^3 x^2 + \dots,$$

the equation (9) is transformed into the form

$$\ddot{x} + x = \frac{1}{3!}xx^{2\delta} - \frac{1}{5!}xx^{4\delta} + \frac{1}{7!}xx^{6\delta} + \dots \tag{11}$$

Therefore, the problem is reduced to the analysis of the equation

$$\ddot{x} + \omega^2 x = \delta Q(x), \tag{12}$$

where:

$$\omega^2 = \sin 1,$$

$$Q(x) = x\left\{\frac{1}{3!}\left(\ln x^2 + \frac{1}{2}\delta[\ln x^2]^2 + \frac{1}{6}\delta^2[\ln x^2]^3 + \dots\right) - \frac{1}{5!}\left(\ln x^4 + \frac{1}{2}[\ln x^4]^2 + \frac{1}{6}\delta^2[\ln x^4]^3 + \dots\right) + \right. \tag{13}$$

$$\left. + \frac{1}{7!}\left(\ln x^6 + \frac{1}{2}\delta[\ln x^6]^2 + \frac{1}{6}\delta^2[\ln x^6]^3 + \dots\right)\right\}.$$

Taking into account the following classical relations

$$\tau = \alpha t, \quad \alpha^2 = 1 + \delta\alpha_1 + \delta^2\alpha_2 + \dots, \quad x = x_0 + \delta x_1 + \delta^2 x_2 + \dots, \tag{14}$$

the right-hand side of equation (12) can be presented in the form

$$Q(x) = x_0\omega_1^2\ln x_0^2 + \delta[x_0\omega_2^2(\ln x_0^2)^2 + x_1\omega_1^2\ln x_0^2] + \delta^2[x_0\omega_3^2(\ln x_0^2)^3 + x_1\omega_2^2(\ln x_0^2)^2 + x_2\omega_1^2\ln x_0^2] + \dots \tag{15}$$

where:

$$\omega_1^2 = \frac{1}{3!} - \frac{2}{5!} + \frac{3}{7!} - \dots, \qquad \omega_2^2 = \frac{1}{2!}\left(\frac{1}{3!} - \frac{2^2}{5!} + \frac{3^2}{7!} - \dots\right), \qquad \omega_3^2 = \frac{1}{3!}\left(\frac{1}{3!} - \frac{2^3}{5!} + \frac{3^3}{7!} - \dots\right). \tag{16}$$

The left hand-side of equation (12) reads:

$$L = (1 + \alpha_1\delta + \alpha_2\delta^2 + \alpha_3\delta^3)(\ddot{x}_0 + \delta\ddot{x}_1 + \delta^2\ddot{x}_2 + \delta^3\ddot{x}_3) + \omega^2(x_0 + \delta x_1 + \delta^2 x_2 + \delta^3 x_3). \tag{17}$$

After splitting with respect to δ, the following recurrent set of equations is obtained:

$$\delta^1: \ \ddot{x}_1 + \omega^2 x_1 = x_0\omega_1^2\ln x_0^2 - \alpha_1\ddot{x}_0, \quad \delta^2: \ \ddot{x}_2 + \omega^2 x_2 = x_0\omega_2^2(\ln x_0^2)^2 - \ddot{x}_1\alpha_1 - \alpha_2\ddot{x}_0, \tag{18}$$

....

Above $\omega_1^2 = 0.5(\sin 1 - \cos 1) = 0.150584$.

The zero order solution has the form $x_0 = \cos\omega t$. Using the method of avoiding secular terms, one gets $\alpha_1 = \frac{\omega_1^2}{\omega^2}(2\ln 2 - 1)$. For $\delta = 1$ we get the formula for the period of oscillations $T = \frac{2\pi}{\omega}[1 - \frac{\omega_1^2}{\omega^2}(\ln 2 - 0.5)]$.

The next successive approximations can be obtained in a similar way. The zero order approximation gives $T = 6.8$ and this approximation is better than the usually used value of $T = 6.28$. In the first order approximation one obtains practically the exact value of $T = 6.57$ (numerically computed "exact" $T = 6.6$ and the error is 0.5%).

Therefore, the method of small δ can be treated as the adequate one for the approximate integration of equation (9) which does not include explicitly small parameters.

3. References

1 A. H. NAYFEH, Perturbation methods. Wiley Interscience, New York, 1973.

2 J. AWREJCEWICZ, I. V. ANDRIANOV AND L. I. MANEVITCH, Asymptotic approaches in nonlinear dynamics: new trends and applications. Springer Verlag, Heidelberg, 1998.

3 C. M. BENDER, K. M. MILTON, S. S. PINSKY AND L. M. SIMMONDS, J. Math. Phys., 30(7), 1445 (1989).

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