Journal of Technical Physics, J. Tech. Phys., 37, 3-4, 277–280, 1996.
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CHAOTIC DYNAMICS EXHIBITED BY TWO-DIMENSIONAL MAPS

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The Chossat–Golubitsky (C-G) and the Tinkerbell (T) maps serve as examples to exhibit some aspects of chaotic dynamics and to give an explanation based mainly on numerical techniques applied. The C-G map exhibits periodic, chaotic and hyperchaotic dynamics and many different bifurcations, which are discussed and illustrated. On the basis of the consideration of the T map, a peculiar organization of unstable orbits, a role of their unstable manifolds and their relation to a fractal structure is discussed.

1. Introduction

There are many examples of physical systems governed by invertible or even non-invertible two-dimensional maps, including those of mechanics and electronics. The two-dimensional maps allow one to understand and explain many key features of non-linear dynamics, including prediction and control of periodic, quasi-periodic and chaotic orbits, location of many different types of bifurcation, prediction of chaos occurrence and its death, explanations of boundary and interior crises or sudden changes of the chaotic attractor size, basins of the attractor and basin boundaries, metamorphoses, as well as a role of saddle straddle trajectories.

The numerical methods used in our investigations are presented in the monographs [1–4] and are not repeated here.

2. A Peculiar Bifurcation Scenario of the C-G Map

We consider first the Chossat-Golubtsky map of the form [5]:

$$F_P: \; Z \to (A|Z^2| + B\,\mathrm{Re}\,(Z^I) + C)Z + D(Z^*)^{I-1}, \tag{1}$$

where $Z = X + iY$ ($i^2 = -1$), $Z^* = X - iY$, $P = (A, B, C, D, I)$, $A, B, C, D \in \mathbb{R}$ and $I \in \mathbb{Z}$ (integer). $B \in (-0.6, 0.4)$ serves us as a control parameter, whereas the other parameters are fixed: $A = 1.0$, $C = -1.775$, $D = 0.5$, $I = 5$. This map exhibits the so-called symmetry-increasing bifurcation.

Figure 1 presents a bifurcation diagram corresponding to the already mentioned changes of the control parameter. The period-2 stable orbit bifurcates via a Hopf bifurcation into a quasiperiodic attractor which is presented for $B = -0.4$. It loses its stability with an increase of B and a strange chaotic attractor is born (presented for $B = -0.32$), which for $B = -0.3$ has a leaf shape. A stability of the chaotic attractor has been changed and a period-4 motion occurs (see for example $B = -0.2$). The periodic orbits lose their stability and four pieces of chaotic attractor are born (see $B = -0.18$).

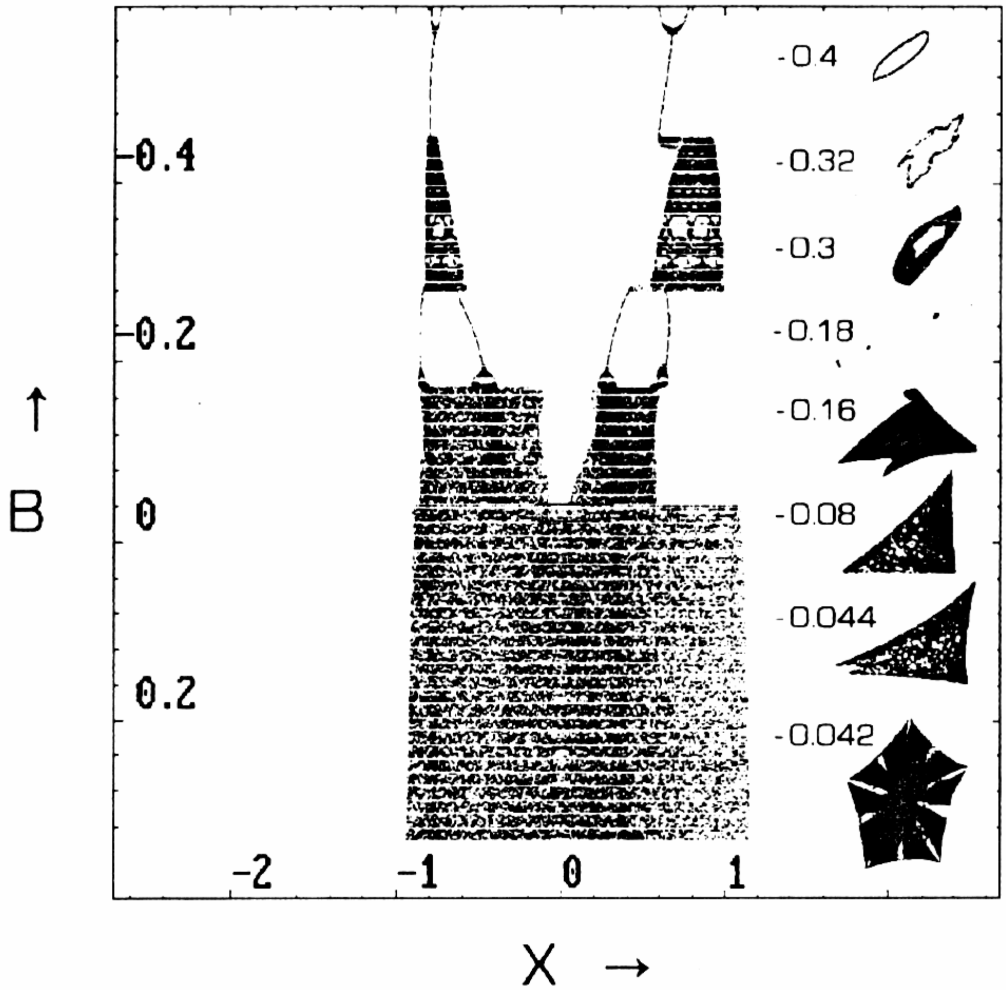

FIG. 1. The bifurcation structure of the Chossat–Golubitsky map with some attractors corresponding to the fixed values of B parameter.

The sudden change of the bifurcation diagram close to $B = -0.17$ can be explained here in a following manner. Namely, an occurrence of ten-piece dolphin-like symmetric attractor (one piece is presented for $B = -0.16$) is implied by the stability gained by the above mentioned invariant non-attracting set. This is caused by as a sudden increase in a number of unstable periodic orbits added to the four-piece-like chaotic attractor. With a further increase of the control parameter, the chaotic attractor changes its structure to the triangle-like ten-piece symmetric one (again one piece is presented for $B = -0.08$). We have investigated some of the periodic orbits embedded in this chaotic attractor and we have found remarkable order in their organization. The chaotic attractor bifurcates into the Besicovitch and Kakeya-like set of points [6], presented for $B = -0.044$ in Fig. 1. For $B > -0.044$, the chaotic attractor undergoes a symmetric collision and instead of ten different pieces of the attractor, a new one-piece symmetric umbrella-like chaotic attractor is born, which is presented in Fig. 1 for $B = -0.042$.

To explain a sudden change of the attractor size, we have calculated a set of the saddle period-8 orbits. With a high probability, a simultaneous saddle-node bifurcation takes place between this orbit and a saddle orbit lying on the boundary of the umbrella-shape points. This collision results in the occurrence of a large umbrella-like attractor, and a merging of the previously separated parts of the attractor occurs.

3. Unstable Orbits and Manifolds Exhibited by the T Map

We have considered the Tinkerbell map defined by

$$X \to X^2 - Y^2 + AX + BY,$$
$$Y \to 2XY + CX + DY, \quad (2)$$

where A, B, C, D are the control parameters, here taken as $A = 0.9$, $B = -0.6$, $C = 2.0$, $D = 0.5$, and the exemplary numerical results are presented in Fig. 2.

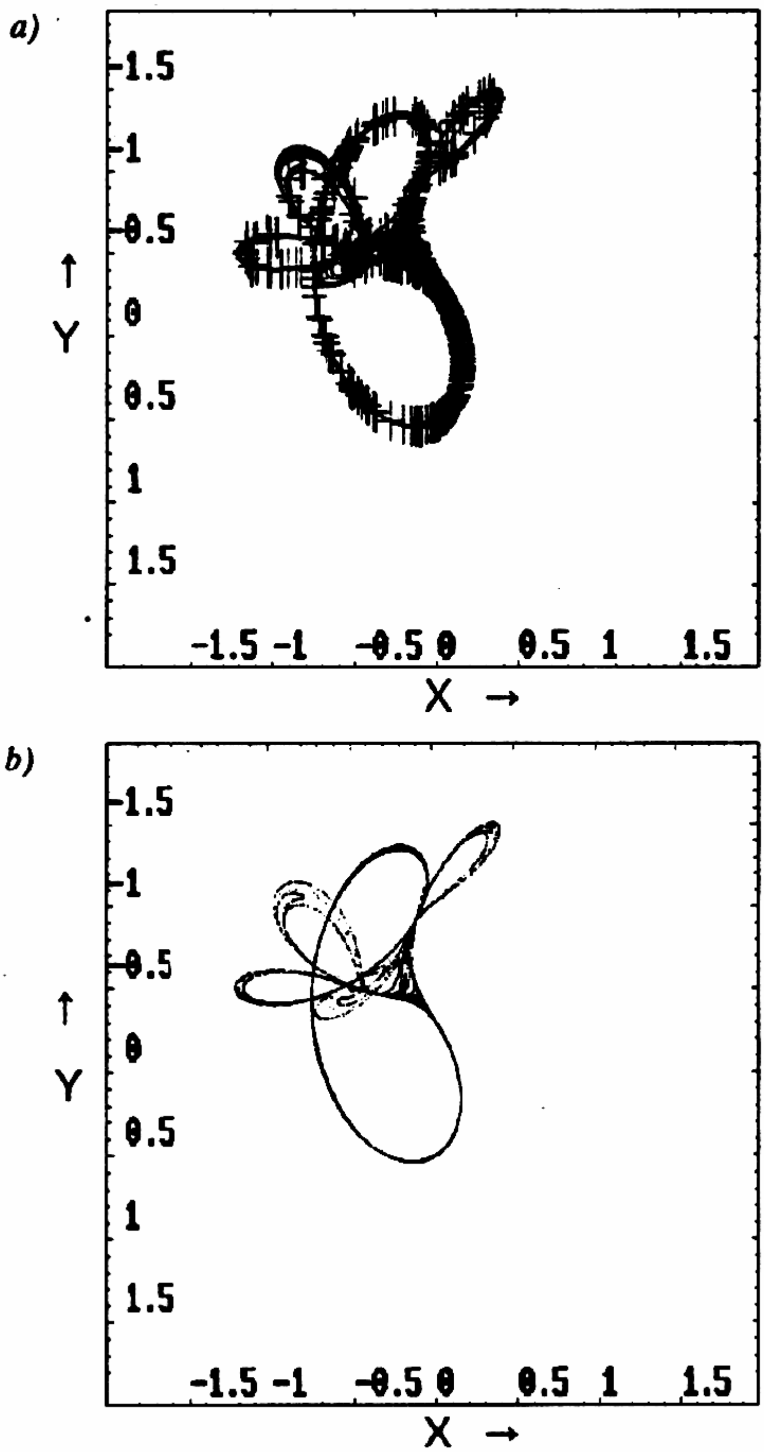

FIG. 2. Period-6 saddles (a) and the Tinkerbell chaotic attractor (b).

The calculations have been organized in such a manner that we have first randomly looked for the periodic orbits with the assummed period, and then we have calculated the unstable manifolds (at least some of them). We have found two period-1 orbits, which are (repellers) $(0,0)$ and $(1.028, -1.32)$ with their eigenvalues $\sigma_{1,2} = 0.7 \pm 1.077i$ and $\sigma_{1,2} = 2.755 \pm 1.1i$, respectevely. All of the period-2 orbits found are repellers and their unstable manifolds undergo big changes (loops) after their final approaching the set similar to the Tinkerbell attractor. Note also that the fixed points are lying rather far from the chaotic attractor. The period-4 orbits found are lying closer to the chaotic attractor. The period-3 orbits are lying on a special part of the set of points of the period-4.

On the basis of the simulation result we have discovered a certain organization of the fixed points with different periods. The period 2, 3 and 4 points of unstable manifolds create a barrier-like structure which causes the boundaries of the real trajectory (note the existence of repellers). Higher periodic orbits are saddles and they are lying on (or they are very close to) the set of points of the Tinkerbell attractor. Our numerical observation has led us to the conclusion that a set of points of the chaotic attractor is a *common* set of all unstable manifolds of all orbits. Therefore, it means that a set of the Tinkerbell attractor points belongs simultaneously to all infinitely many unstable manifolds of different periodic orbits.

Acknowledgment

This work was carried out during J. A. stay at the ENTPE in Lyon, France.

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Received April 4, 1996.