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ANALYTICAL CONDITION FOR THE EXISTENCE OF AN IMPLICIT TWO-PARAMETER FAMILY OF PERIODIC ORBITS IN THE RESONANCE CASE

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1. INTRODUCTION

Recently, a growing tendency has appeared in non-linear dynamics to analyze two-parameter families of the ordinary differential equations. Many interesting phenomena, both from the theoretical and the applied point of view, are expected in the two-parameter families which cannot be found in one parameter families [1, 2]. Among others, these include hysteresis points, isola centers or multiple bifurcation points. The analysis of two-parameter families gives the possibility of tracing explicitly how coalescing points such as isola centers, hysteresis centers, bifurcations or Hopf centers occur naturally as organizing centers [2].

Thus, the problem of how to find a two-parameter family of solutions is of fundamental importance. The analytical two-parameter solution enables one to solve many different questions (for example, one can find the critical boundary of the family considered or one can calculate a branching set of parameters).

In this letter a local analytical method is developed to obtain a two-parameter family of solutions which lies near the known periodic orbit. Considerations are limited to the case in which the dynamical system is excited periodically, and excitations and non-linearities are expressed by two independent small parameters. The most important case is also analyzed; i.e., when both excitations are in resonance with the linear part of the system. Of course, the presented method can be extended to the k-control space ($k > 2$).

This approach develops the author's earlier works [3–6], in which two-parameter solutions for coupled oscillators with parametric excitations and discrete–continuous systems with time delays were analyzed. The present method is related to the asymptotic methods widely described by, among others, Malkin [7], Bogoliubov and Mitropolskii [8], Hale [9], Nayfeh and Mook [10], Nayfeh [11], Zubov [12] and Bajaj and Johnson [13].

2. METHOD AND RESULTS

Consider the dynamical system

$$\dot{X} = F(t, X, \varepsilon, \mu), \tag{1}$$

where (a) the vector function F is given for $t \in (-\infty, +\infty)$, $X \in R^n$ and is continuous in its independent variables; (b) F fulfils the Lipschitz condition $\|F(t, X) - F(t, Y)\| \leqslant C\|X - Y\|$, where C is a certain constant; (c) $F(t + 2\pi, X) = F(t, X)$ for $\varepsilon = \mu$; (d) the solution of $X = X(t, X^0, 0)$ is determined by $\forall t \geqslant 0$, for $X = X^0$ if $t = t_0$. We shall seek

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periodic solutions $Y(t, \varepsilon, \mu)$ dependent on ε and μ such that $Y \to X^0$ if $\varepsilon \to 0+0$ and $\mu \to +0$. The problem can be reduced to the analysis of the system

$$\dot{Y} = R(t)Y + \Phi(t) + \varepsilon F^{(1)}(t, Y, \varepsilon, \mu) + \mu F^{(2)}(t, Y, \varepsilon, \mu),$$
$$F^{(i)}(t) = F^{(i)}(t + 2\pi), \quad \Phi(t) = \Phi(t + 2\pi), \qquad i = 1, 2. \tag{2}$$

We shall consider the resonance case assuming that simple elementary divisors correspond to the multiplied resonance eigenvalues. Let us assume that from among the $\lambda_1, \ldots, \lambda_n$ eigenvalues there are k zero ones and 21 values with the form $\pm iMN^{-1}$ (M and N being natural numbers) which will be denoted by v_l. There is always [14] such a linear transformation of vector Y with real and constant coefficients, that the system (2) can be reduced to the form

$$\mathrm{d}u_s/\mathrm{d}t = \phi_s(t) + \varepsilon \bar{F}_s^{(1)}(t, U, X, Y, Z, \varepsilon, \mu) + \mu \bar{F}_s^{(2)}(t, U, X, Y, Z, \varepsilon, \mu),$$
$$\mathrm{d}x_p/\mathrm{d}t = -v_p x_p + \alpha_p(t) + \varepsilon \bar{A}_p^{(1)}(t, U, X, Y, Z, \varepsilon, \mu) + \varepsilon \bar{A}_p^{(2)}(t, U, X, Y, Z, \varepsilon, \mu),$$
$$\mathrm{d}y_p/\mathrm{d}t = v_p y_p + \eta_p(t) + \varepsilon \bar{E}_p^{(1)}(t, U, X, Y, Z, \varepsilon, \mu) + \mu \bar{E}_p^{(2)}(t, U, X, Y, Z, \varepsilon, \mu),$$
$$\mathrm{d}z_r/\mathrm{d}t = \sum_{i=1}^{m} p_{ri} + \theta_r(t) + \varepsilon \bar{\Gamma}_r^{(1)}(t, U, X, Y, Z, \varepsilon, \mu) + \mu \bar{\Gamma}_r^{(2)}(t, U, X, Y, Z, \varepsilon, \mu),$$
$$s = 1, \ldots, k, \qquad p = 1, \ldots, l, \qquad r = 1, \ldots, m, \qquad k + 2l + m = n. \tag{3}$$

The matrix $[P_{ri}]$ has no resonance eigenvalues, and the following have been assumed in equations (3):

$$U = (u_1, \ldots, u_k), \qquad X = (x_1, \ldots, x_l), \qquad Y = (y_1, \ldots, y_l), \qquad Z = (z_1, \ldots, z_m).$$

The following exchange of variables is performed:

$$x_p = \bar{x}_p \cos v_p t + \bar{y}_p \sin v_p t, \quad y_p = \bar{x}_p \sin v_p t + \bar{y}_p \cos v_p t, \qquad p = 1, \ldots, l. \tag{4}$$

From equation (3) one obtains

$$\mathrm{d}v_s/\mathrm{d}t = W_s(t) + \varepsilon F_s^{(1)}(t, V, Z, \varepsilon, \mu) + \mu F_s^{(2)}(t, V, Z, \varepsilon, \mu),$$
$$\mathrm{d}z_r/\mathrm{d}t = \sum_{i=1}^{m} p_{ri} z_i + \theta_r(t) + \varepsilon \Gamma_r^{(1)}(t, V, Z, \varepsilon, \mu) + \mu \Gamma_r^{(2)}(t, V, Z, \varepsilon, \mu), \tag{5}$$
$$s = 1, \ldots, k + 2l, \qquad r = 1, \ldots, m, \qquad v_s = u_s, \qquad v_{k+p} = \bar{x}_p, \qquad v_{k+1+p} = \bar{y}_p,$$

and $s = 1, \ldots, k$, $p = 1, \ldots, l$. For $\varepsilon = \mu = 0$ the system (5) has a family of periodic solutions, with a period $2\pi N$, dependent on $2(k + 2l)$ constants if the following conditions are satisfied:

$$\int_0^{2\pi N} W_s(\tau)\, \mathrm{d}\tau = 0, \qquad s = 1, \ldots, k + 2l. \tag{6}$$

If conditions (6) are satisfied, then the family of periodic solutions being sought is

$$v_s = v_s^{(1)} + v_s^{(2)} = C_s^{(1)} + C_s^{(2)} + 2\int_0^t W_s(\tau)\, \mathrm{d}\tau,$$
$$Z = Z^{(1)} + Z^{(2)} = 2\mathrm{e}^{Pt}C + 2\int_0^t \mathrm{e}^{P(t-\tau)}\Theta(\tau)\, \mathrm{d}\tau, \tag{7}$$

where

$$\Theta(t) = (\theta_1(t), \ldots, \theta_m(t)), \qquad C = (I - \mathrm{e}^{2\pi NP})^{-1}\int_0^{2\pi N} \mathrm{e}^{P(2\pi N - \tau)}\Theta(\tau)\, \mathrm{d}\tau, \tag{8}$$

and $C_s^{(1)}$ is connected with the parameter ε, and $C_s^{(2)}$ with the parameter μ.

We look for a general solution of equations (5) with the following initial conditions:

$$v_s = C_s^{(1)} + C_s^{(2)} + d_s^{(1)} + d_s^{(2)}, \qquad Z = 2C + G^{(1)} + G^{(2)}, \tag{9}$$

for $t = 0$, $s = 1, \ldots, k + 2l$. According to equations (5), we find that

$$v_s(t) = C_s^{(1)} + C_s^{(2)} + d_s^{(1)} + d_s^{(2)} + 2\int_0^t W_s(\tau)\,d\tau + \varepsilon\int_0^t F_s^{(1)}(\tau, V, Z)\,d\tau + \mu\int_0^t F_s^{(2)}(\tau, V, Z)\,d\tau,$$

$$Z(t) = e^{Pt}(2C + G^{(1)} + G^{(2)}) + 2\int_0^t e^{P(t-\tau)}\Theta(\tau)\,d\tau + \varepsilon\int_0^t e^{P(t-\tau)}\Gamma^{(1)}(\tau, V, Z)\,d\tau + \mu\int_0^t e^{P(t-\tau)}\Gamma^{(2)}(\tau, V, Z)\,d\tau. \tag{10}$$

After accounting for periodicity in equations (10) we obtain

$$\varepsilon S_s^{(1)} + \mu S_s^{(2)} = \varepsilon\int_0^{2\pi N} F_s^{(1)}(\tau, V, Z, \varepsilon, \mu)\,d\tau + \mu\int_0^{2\pi N} F_s^{(2)}(\tau, V, Z, \varepsilon, \mu)\,d\tau,$$

$$s = 1, \ldots, k + 2l, \tag{11}$$

$$(e^{2\pi NP} - I)(G^{(1)} + G^{(2)}) + \varepsilon\int_0^{2\pi NP} e^{(2\pi N - \tau)}\Gamma^{(1)}(\tau, V, Z, \varepsilon, \mu)\,d\tau + \mu\int_0^{2\pi NP} e^{2\pi N - \tau}\Gamma^{(2)}(\tau, V, Z, \varepsilon, \mu)\,d\tau = 0. \tag{12}$$

In order to have $D^{(1)} \to 0$, $D^{(2)} \to 0$ and $G^{(1)} \to 0$, $G^{(2)} \to 0$, at $\varepsilon \to +0$, $\mu \to +0$ (where $D^{(1)} = (d_1^{(1)}, \ldots, d_{k-2l}^{(1)})$, $D^{(2)} = (d_1^{(2)}, \ldots, d_{k+2l}^{(2)})$), it is necessary to satisfy the conditions

$$S_s^{(1)}(d_1^{(2)}, \ldots, d_{k-2l}^{(2)}) = \int_0^{2\pi N} F_s^{(1)}(\tau, V, Z, \varepsilon, \mu)\,d\tau = 0,$$

$$S_s^{(2)}(d_1^{(2)}, \ldots, d_{k+2l}^{(2)}) = \int_0^{2\pi N} F_s^{(2)}(\tau, V, Z, \varepsilon, \mu)\,d\tau = 0, \qquad s = 1, \ldots, k + 2l. \tag{13}$$

Equation (12) expresses $G^{(1)}$ and $G^{(2)}$ as implicit functions of the quantities ε, μ, $d_1^{(1)}, \ldots, d_{k+2l}^{(1)}$ and $d_1^{(2)}, \ldots, d_{k+2l}^{(2)}$. The integrals in equations (13) are calculated on the basis of the periodic solution (7). Not every periodic solution of equations (7) can be the limit solution for the dynamical system considered as $\varepsilon \to +0$ and $\mu \to +0$. Only if the solutions of the forms (9) satisfy conditions (13) and the inequalities

$$D(S_1^{(1)}, \ldots, S_{k+2l}^{(1)})/D(d_1^{(1)}, \ldots, d_{k+2l}^{(1)}) \neq 0,$$

$$D(S_2^{(2)}, \ldots, S_{k+2l}^{(2)})/D(d_1^{(2)}, \ldots, d_{k+2l}^{(2)}) \neq 0 \tag{14}$$

are satisfied, does the initial system of equations have a $2\pi N$ periodic solution for all sufficiently small $\varepsilon > 0$ and $\mu > 0$, which approaches without restriction (for $\varepsilon \to +0$ and $\mu \to +0$) the $2\pi N$ periodic solution of the non-linear system (5).

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