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STATIONARY AND NONSTATIONARY ONE-FREQUENCY PERIODIC OSCILLATIONS IN NONLINEAR AUTONOMOUS DISCRETE-CONTINUOUS SYSTEMS WITH TIME DELAY

J. AWREJCEWICZ (ŁÓDŹ)(*)

The paper presents an analytical method of determinating one-frequency periodic oscillations in nonlinear autonomous discrete-continuous mechanical systems with time delay, on the basis of the asymptotic approach. The periodic solutions are sought in the form of some particular asymptotic series with respect to two independent bifurcation parameters – one is related to nonlinearity and the other to delay. Some technical problems, which can only be solved using this approach, are demonstrated. The method is illustrated in a mechanical example which includes a self-excited oscillations of a beam connected with a discrete one degree-of-freedom system.

1. Introduction

One of the important problems of mechanical and automatic control engineering is active control of the oscillations of the mechanical objects by means of control units, which can frequently be treated as inertial systems with concentrated parameters and time delay [1]. The subject to control can be nonlinear mechanical systems with concentrated (further referred to as discrete mechanical systems) or distributed parameters. The latter, referred to as continuous systems, are dealt within this paper.

In real control systems of this type, the control unit influences the object subject to control and the state of the controlled object is monitored only in a certain isolated points. It is usually possible to find controlled objects, which are governed by partial differential nonlinear equations as well as control units, which can be modelled by ordinary nonlinear differential equations.

As has been mentioned above, the systems governed by nonlinear partial and ordinary equations have many technical applications and they are considered in this work. It is a continuation of earlier work, where the two-variable asymptotic expansions technique has been used to analyze periodic oscillations in nonlinear parametrically excited mechanical systems [2–4], bifurcated oscillations [5, 6] as well as oscillations in discrete-continuous systems. The presented research develops the approach from [7], where similar systems were sought in the form of power series of two independent perturbation parameters. The recurrent set of linear differential equations obtained by means of comparing the expressions found at the same powers of two perturbation parameters were then solved using the harmonic balance method. The approach, however, enables one to analyze only the steady states of the considered mechanical systems. The technique developed and illustrated here is more universal. By the use of such a method the steady and

(*)Currently with the Institute of Technical Mechanics, University of Braunschweig, Germany.

unsteady (transient) oscillation can be analyzed and, as it will be shown in a future paper, static-type catastrophes during oscillations can be detected.

The presented technique is a generalization of classical asymptotic methods, which are widely treated in the literature [8–17], to the analysis of discrete-continuous mechanical systems governed by partial and ordinary nonlinear equations with two independent parameters.

2. Method

Let us consider a discrete-continuous system governed by the following equations:

$$(2.1)\quad \begin{aligned} \frac{\partial^2 u(t,x)}{\partial t^2} &= L_x^{(2m)}\{u(t,x)\} + \varepsilon f_1\{x, u(t,x), y(t-\mu)\}, \\ \frac{dy(t)}{dt} &= \sum_{p=0}^{P} A_p y(t-\tau_p) + \varepsilon F_1\{y(t-\mu), u(t-\mu, t-\xi)\} \end{aligned}$$

subject to the following non-homogeneous boundary conditions

$$(2.2)\quad L_x^{(h,j)}\{u(t,x)\}|_{x\in S} = \varepsilon g_{hj}\{y(t-\mu)\}, \qquad h = 1,\ldots,m.$$

The coordinate t denotes time and $t \in R$; x is the vector of the coordinates and $x \in (G \cup S)$, while S is the limiting set of G; $u(t,x)$ is a certain scalar function determined in the set $R \times G$ and $L_x^{(h,j)}$ is a linear operator of order $2m$ on x; $L_x^{(2m)}$ is the linear differential operator of $j \leq 2m-1$; y and F_1 are vectors of an m-dimensional space; A_p are constant matrices of $(m \times m)$ order; F_1, f_1 and g_{hj} are functions of $y(t-\mu)$, $u(t-\mu,\xi)$, $\xi \in (G \cup S)$, while τ_p and μ are time delays. Finally, we assume that ε and μ are small positive parameters.

Thanks to this mathematical formulation of the problem, the presented analytical approach can be further used for many different discrete-continuous mechanical systems governed by Eqs. (2.1). Thus we will continue our consideration first in general form, and then, in order to demonstrate the physical insight of the problem, we will illustrate the method with an example from the area of mechanics.

The problem including non-homogeneous boundary conditions (2.2) can be reduced [1, 7] to one of homogeneous boundary conditions. Thus we analyze the following system:

$$(2.3)\quad \begin{aligned} \frac{\partial^2 \nu(t,x)}{\partial t^2} &= L_x^{(2m)}\{\nu(t,x)\} + \varepsilon f_1\{x, \nu(t,x), y(t-\mu)\}, \\ \frac{dy(t)}{dt} &= \sum_{p=0}^{P} A_p y(t-\tau_p) + \varepsilon F_1\{y(t-\mu), \nu(t-\mu), \xi\}, \end{aligned}$$

where $\nu(t,x)$ fulfils the homogeneous boundary conditions

$$(2.4)\quad L_x^{(h,j)}\{\nu(t,x)\}|_{x\in S} = 0, \qquad h = 1,\ldots,m.$$

From the first equation system (2.3), and for $\varepsilon = 0$, we obtain

$$L_x^{(2m)}\{X(x)\} + \sigma X(x) = 0, \qquad (2.5)$$
$$L_x^{(h,j)}|_{x\in S} = 0, \qquad h = 1,\dots,m,$$

while from the other we obtain the following characteristic equation:

$$D(\rho) = \det\left\{\sum_{p=0}^{P} A_p e^{-\tau_p \rho} - E\rho\right\}. \qquad (2.6)$$

In the considered dynamical system, oscillations will appear if $\sigma_s = \omega_{\nu s}^2$ and/or if the characteristic equation (2.6) has imaginary eigenvalues $\rho_k = \pm i\omega_{yk}$.

In this paper we shall consider the case where $\sigma_1 = \omega_{\nu_1}^2 = \omega_1^2$ and the other eigenvalues of the first equation of the system (2.5) amount to $\sigma_s \neq \{(p/q)\omega_1\}^2$, where p and q are integers. Moreover, it is assumed that the characteristic equation (2.6) does not possess imaginary eigenvalues. We seek a one-frequency solution of the dynamic system (1) with the frequency approaching ω_1 for $\varepsilon \to 0$ and $\mu \to 0$. To this aim the approach suggested by Krylov–Bogolubov–Mitropolski will be used. We look for a solution in the form

$$\nu(t,x) = a(t)X_1(x)\cos\psi t + \sum_{k=1}^{K}\sum_{l=0}^{L}\varepsilon^k\mu^l V_{kl}\{x, a(t), \psi(t)\}, \qquad (2.7)$$
$$y(t) = \sum_{k=1}^{K}\sum_{l=0}^{L} y_{kl}\{a(t), \psi(t)\},$$

where

$$\frac{da}{dt} = \sum_{k=1}^{K}\sum_{l=0}^{L}\varepsilon^k\mu^l A_{kl}\{a(t)\}, \qquad (2.8)$$
$$\frac{d\psi}{dt} = \omega_1 + \sum_{k=1}^{K}\sum_{l=0}^{L}\varepsilon^k\mu^l B_{kl}\{a(t)\},$$

and $X_1(x)$ is the solution of the boundary problem (2.5). From the first equation of (2.7) we obtain

$$\frac{\partial\nu}{\partial t} = \left\{\frac{da}{dt}\cos\psi - a\frac{d\psi}{dt}\sin\psi\right\}X_1(x) + \sum_{k=1}^{K}\sum_{l=0}^{L}\varepsilon^k\mu^l\left\{\frac{\partial V_{kl}}{\partial a}\frac{da}{dt} + \frac{\partial V_{kl}}{\partial\psi}\frac{d\psi}{dt}\right\},$$

$$\frac{\partial^2\nu}{\partial t^2} = \left\{\frac{d^2a}{dt^2}\cos\psi - 2\frac{da}{dt}\frac{d\psi}{dt}\sin\psi - a\frac{d^2\psi}{dt^2}\sin\psi - \right. \qquad (2.9)$$
$$\left. -a\left(\frac{d\psi}{dt}\right)^2\cos\psi\right\}X_1(x) + \sum_{k=1}^{K}\sum_{l=0}^{L}\varepsilon^k\mu^l\left\{\frac{\partial^2 V_{kl}}{\partial a^2}\left(\frac{da}{dt}\right)^2 + \right.$$
$$\left. +2\frac{\partial^2 V_{kl}}{\partial a\partial\psi}\frac{d\psi}{dt}\frac{da}{dt} + \frac{\partial V_{kl}}{\partial a}\frac{d^2a}{dt^2} + \frac{\partial^2 V_{kl}}{\partial\psi^2}\left(\frac{d\psi}{dt}\right)^2 + \frac{\partial V_{kl}}{\partial\psi}\frac{d^2\psi}{dt^2}\right\}.$$

From the first equation of (2.3) we calculate

$$\frac{\partial^2 \nu}{\partial t^2} - L_x^{(2m)}\{\nu(t,x)\} = \left\{ \frac{d^2 a}{dt^2} X_1(x) - \left\{ \left(\frac{d\psi}{dt} \right)^2 X_1(x) + \right.\right. \tag{2.10}$$

$$\left.\left. + L_x^{(2m)}\{X_1(x)\} \right\} a \right\} \cos\psi - \left\{ 2\frac{da}{dt}\frac{d\psi}{dt} + a\frac{d^2\psi}{dt^2} \right\} X_1(x)\sin\psi +$$

$$+ \sum_{k=1}^{K} \sum_{l=0}^{L} \varepsilon^k \mu^l \left\{ \frac{\partial^2 V_{kl}}{\partial a^2} \left(\frac{da}{dt} \right)^2 + 2\frac{\partial^2 V_{kl}}{\partial a \partial \psi} \frac{d\psi}{dt}\frac{da}{dt} + \frac{\partial V_{kl}}{\partial a}\frac{d^2 a}{dt^2} + \right.$$

$$\left. + \frac{\partial^2 V_{kl}}{\partial \psi^2} \left(\frac{d\psi}{dt} \right)^2 + \frac{\partial V_{kl}}{\partial \psi}\frac{d^2\psi}{dt^2} - L_x^{(2m)}\{V_{kl}\} \right\}.$$

From the second equation of (2.7) we obtain

$$\frac{dy}{dt} = \sum_{k=1}^{K} \sum_{l=0}^{L} \varepsilon^k \mu^l \left\{ \frac{\partial y_{kl}}{\partial a}\frac{da}{dt} - \frac{\partial y_{kl}}{\partial \psi}\frac{d\psi}{dt} \right\}. \tag{2.11}$$

Moreover, taking Eqs.(2.8) into account, we calculate

$$\frac{d^2 a}{dt^2} = \left\{ \sum_{k=1}^{K} \sum_{l=0}^{L} \varepsilon^k \mu^l \frac{dA_{kl}}{da} \right\} \left\{ \sum_{k=1}^{K} \sum_{l=0}^{L} \varepsilon^k \mu^l A_{kl} \right\} = \tag{2.12}$$

$$= \varepsilon^2 A_{10}\frac{dA_{10}}{da} + \varepsilon^2\mu \left\{ \frac{dA_{10}}{da} A_{11} + \frac{dA_{11}}{da} A_{10} \right\} +$$

$$+\varepsilon^3 \left\{ \frac{dA_{20}}{da} A_{10} + \frac{dA_{10}}{da} A_{20} \right\} + O(\varepsilon^k \mu^l;\; k+l=4),$$

$$\left(\frac{d\psi}{dt} \right)^2 X_1(x) + L_x^{(2m)}\{X_1(x)\} = \left\{ \sum_{k=1}^{K} \sum_{l=0}^{L} \varepsilon^k \mu^l B_{kl}(a) \right\}^2 X_1 + \tag{2.13}$$

$$+2\omega_1 \sum_{k=1}^{K} \sum_{l=0}^{L} \varepsilon^k \mu^l B_{kl}(a) X_1 = 2\varepsilon\omega_1 B_{10} X_1 +$$

$$+\varepsilon^2\{2\omega_1 B_{20} + B_{10}^2\} X_1 + 2\varepsilon\mu\omega_1 B_{11} X_1 + 2\varepsilon^2\mu\{B_{10}B_{11} + \omega_1 B_{21}\} X_1 +$$

$$+ 2\varepsilon\mu^2\omega_1 B_{12} X_1 + 2\varepsilon^3\{\omega_1 B_{30} + B_{20}B_{10}\} + O(\varepsilon^k \mu^l;\; k+l=4),$$

because in accordance with the first equation of Eqs. (2.5) we have $X_1(x)\omega_1^2 + L_x^{(2m)}\{X(x)\} = 0$ and

$$\frac{da}{dt}\frac{d\psi}{dt} = \sum_{k=1}^{K} \sum_{l=0}^{L} \varepsilon^k \mu^l A_{kl} \left\{ \omega_1 + \sum_{k=1}^{K} \sum_{l=0}^{L} \varepsilon^k \mu^l B_{kl} \right\} = \varepsilon\omega_1 A_{10} + \tag{2.14}$$

$$+\varepsilon^2\{\omega_1 A_{20} + A_{10}B_{10}\} + \varepsilon\mu\omega_1 A_{11} + \varepsilon^2\mu\{\omega_1 A_{21} + A_{11}B_{10} + A_{10}B_{11}\} +$$

$$+\varepsilon\mu^2 A_{12}\omega_1 + \varepsilon^3\{A_{30}\omega_1 + A_{20}B_{10} + A_{10}B_{20}\} + O(\varepsilon^k \mu^l;\; k+l=4),$$

(2.15)
$$\frac{d^2\psi}{dt^2} = \left\{\sum_{k=1}^{K}\sum_{l=0}^{L}\varepsilon^k\mu^l\frac{dB_{kl}}{da}\right\}\left\{\sum_{k=1}^{K}\sum_{l=0}^{L}\varepsilon^k\mu^l A_{kl}\right\} =$$
$$= \varepsilon^2\frac{dB_{10}}{da}A_{10} + \varepsilon^2\mu\left\{\frac{dB_{10}}{da}A_{11} + \frac{dB_{11}}{da}A_{10}\right\} +$$
$$+\varepsilon^3\left\{\frac{dB_{10}}{da}A_{20} + \frac{dB_{20}}{da}A_{10}\right\} + O(\varepsilon^k\mu^l;\ k+l=4),$$

(2.16)
$$\left(\frac{da}{dt}\right)^2 = \left\{\sum_{k=1}^{K}\sum_{l=0}^{L}\varepsilon^k\mu^l A_{kl}\right\}^2 = \varepsilon^2 A_{10}^2 + 2\varepsilon^2\mu A_{10}A_{11} +$$
$$+2\varepsilon^3 A_{20}A_{10} + O(\varepsilon^k\mu^l;\ k+l=4),$$

(2.17)
$$\left(\frac{d\psi}{dt}\right)^2 = \left\{\omega_1 + \sum_{k=1}^{K}\sum_{l=0}^{L}\varepsilon^k\mu^l B_{kl}\right\}^2 = \omega_1^2 + 2\varepsilon\omega_1 B_{10} + 2\varepsilon\mu B_{11}\omega_1 +$$
$$+\varepsilon^2(2\omega_1 B_{20} + B_{10}^2) + 2\varepsilon\mu^2 B_{12}\omega_1 + 2\varepsilon^2\mu(\omega_1 B_{21} + B_{11}B_{10}) +$$
$$+2\varepsilon^3(\omega_1 B_{30} + B_{20}B_{10}) + O(\varepsilon^k\mu^l;\ k+l=4).$$

Since y and ν can be expressed as power series,

(2.18)
$$y(t-\mu) = \sum_{n=0}^{N}\frac{1}{n!}\frac{d^n y(t)}{dt^n}(-\mu)^n,$$
$$\nu(t-\mu,\xi) = \sum_{n=0}^{N}\frac{1}{n!}\frac{d^n\nu(t,\xi)}{dt^n}(-\mu)^n,$$

then the functions εf and εF after expansion in a power series of small parameters μ and ε and reduction to $n=1$ will assume the form

(2.19)
$$\varepsilon f\{x,\nu(t,x),y(t-\mu)\} = \varepsilon\{f\{x,\nu,y,\dot{y}_1\} + \varepsilon\left\{\frac{\partial f}{\partial\nu}\frac{d\nu}{d\varepsilon} +\right.$$
$$\left.+\sum_{l=1}^{m}\frac{\partial f}{\partial y_l}\frac{dy_l}{d\varepsilon} + \sum_{k=1}^{m}\frac{\partial f}{\partial\dot{y}_{1k}}\frac{d\dot{y}_{1k}}{d\varepsilon}\right\} + \mu\left\{\frac{\partial f}{\partial\nu}\frac{d\nu}{d\mu} + \sum_{l=1}^{m}\frac{\partial f}{\partial y_l}\frac{dy_l}{d\mu} +\right.$$
$$\left.+\sum_{k=1}^{m}\frac{\partial f}{\partial\dot{y}_{1k}}\frac{d\dot{y}_{1k}}{d\mu}\right\} + \varepsilon^2\left\{\frac{\partial^2 f}{\partial\nu^2}\left(\frac{d\nu}{d\varepsilon}\right)^2 + \frac{\partial f}{\partial\nu}\frac{d^2\nu}{d\varepsilon^2} + \sum_{l=1}^{m}\frac{\partial^2 f}{\partial y_1^2}\left(\frac{dy_l}{d\varepsilon}\right)^2 +\right.$$
$$+\sum_{k=1}^{m}\frac{\partial^2 f}{\partial\dot{y}_{1k}^2}\left(\frac{d\dot{y}_{1k}}{d\varepsilon}\right)^2 + \sum_{l=1}^{m}\frac{\partial f}{\partial y_l}\frac{d^2 y_l}{d\varepsilon^2} + \sum_{k=1}^{m}\frac{\partial f}{\partial\dot{y}_{1k}}\frac{d^2\dot{y}_{1k}}{d\varepsilon^2} +$$
$$\left.+2\sum_{l=1}^{m}\frac{\partial^2 f}{\partial\nu\partial y_l}\frac{d\nu}{d\varepsilon}\frac{dy_l}{d\varepsilon} + 2\sum_{l=1}^{m}\frac{\partial^2 f}{\partial\nu\partial\dot{y}_l}\frac{d\nu}{d\varepsilon}\frac{d\dot{y}_{lk}}{d\varepsilon} + \sum_{k=1}^{m}\sum_{l=1}^{m}\frac{\partial^2 f}{\partial y_l\partial\dot{y}_{1k}}\frac{dy_l}{d\varepsilon}\frac{d\dot{y}_{1k}}{d\varepsilon}\right\} +$$
$$+\mu^2\left\{\frac{\partial^2 f}{\partial\nu^2}\left(\frac{d\nu}{d\mu}\right)^2 + \frac{\partial f}{\partial\nu}\frac{d^2\nu}{d\mu^2} + \sum_{l=1}^{m}\frac{\partial^2 f}{\partial y_l^2}\left(\frac{dy_l}{d\mu}\right)^2 + \sum_{l=1}^{m}\frac{\partial f}{\partial y}\frac{d^2 y_l}{d\mu^2} +\right.$$

(2.19) [cont.]

$$+\sum_{k=1}^{m}\frac{\partial^2 f}{\partial \dot{y}_{lk}^2}\left(\frac{d\dot{y}_{1k}}{d\mu}\right)^2+\sum_{k=1}^{m}\frac{\partial f}{\partial \dot{y}_{1k}}\frac{d^2\dot{y}_{1k}}{d\mu^2}+2\sum_{l=1}^{m}\frac{\partial^2 f}{\partial\nu\partial y_l}\frac{d\nu}{d\mu}\frac{dy_l}{d\mu}+$$

$$+2\sum_{k=1}^{m}\frac{\partial^2 f}{\partial\nu\partial y_{1k}}\frac{d\nu}{d\mu}\frac{d\dot{y}_{1k}}{d\mu}+2\sum_{k=1}^{m}\sum_{l=1}^{m}\frac{\partial^2 f}{\partial y_l\partial \dot{y}_{1k}}\frac{dy_l}{d\mu}\frac{d\dot{y}_{1k}}{d\mu}\Bigg\}+$$

$$+\varepsilon\mu\Bigg\{2\frac{\partial^2 f}{\partial\nu^2}\frac{d\nu}{d\mu}\frac{d\nu}{d\varepsilon}+2\frac{\partial f}{\partial\nu}\frac{d^2\nu}{d\mu d\varepsilon}+2\sum_{l=1}^{m}\frac{\partial f}{\partial y_l}\frac{d^2 y_l}{d\mu d\varepsilon}+2\sum_{k=1}^{m}\frac{\partial f}{\partial \dot{y}_{1k}}\frac{d^2\dot{y}_{1k}}{d\mu d\varepsilon}+$$

$$+2\sum_{l=1}^{m}\frac{\partial^2 f}{\partial y_l^2}\frac{dy_l}{d\mu}\frac{dy_l}{d\varepsilon}+2\sum_{k=1}^{m}\frac{\partial^2 f}{\partial \dot{y}_{1k}^2}\frac{d\dot{y}_{1k}}{d\mu}\frac{d\dot{y}_{1k}}{d\varepsilon}+\sum_{l=1}^{m}\frac{\partial^2 f}{\partial\nu\partial y_l}\left(\frac{d\nu}{d\varepsilon}\frac{dy_l}{d\mu}+\frac{d\nu}{d\mu}\frac{dy_l}{d\varepsilon}\right)+$$

$$+\sum_{k=1}^{m}\frac{\partial^2 f}{\partial\nu\partial \dot{y}_{1k}}\left(\frac{d\nu}{d\varepsilon}\frac{d\dot{y}_{1k}}{d\mu}+\frac{d\nu}{d\mu}\frac{d\dot{y}_{1k}}{d\varepsilon}\right)+\sum_{k=1}^{m}\sum_{l=1}^{m}\frac{\partial^2 f}{\partial y_l\partial \dot{y}_{1k}}\Bigg(\frac{dy_l}{d\varepsilon}\frac{d\dot{y}_{1k}}{d\mu}+$$

$$+\frac{dy_l}{d\mu}\frac{d\dot{y}_{1k}}{d\varepsilon}\Bigg)\Bigg\}\Bigg\}+O(\varepsilon^k\mu^l;\ k+l=4),$$

(2.20)

$$\varepsilon F\{x,\nu(t-\mu,\xi),y(t-\mu)\}=\varepsilon\Bigg\{F\{x,\nu,\dot{\nu}_1,y,\dot{y}_1\}+$$

$$+\varepsilon\left\{\frac{\partial F}{\partial\nu}\frac{d\nu}{d\varepsilon}+\sum_{l=1}^{m}\frac{\partial F}{\partial y_l}\frac{dy_l}{d\varepsilon}+\sum_{k=1}^{m}\frac{\partial F}{\partial \dot{y}_{1k}}\frac{\dot{d}\,y_{1k}}{d\varepsilon}\right\}+\frac{\partial F}{\partial\dot{\nu}_1}\frac{\dot{d}\,\nu_1}{d\varepsilon}+$$

$$+\mu\left\{\frac{\partial F}{\partial\nu}\frac{\partial\nu}{\partial\mu}+\sum_{l=1}^{m}\frac{\partial F}{\partial y_l}\frac{dy_l}{d\mu}+\sum_{k=1}^{m}\frac{\partial F}{\partial \dot{y}_{1k}}\frac{\dot{d}\,y_{1k}}{d\mu}+\frac{\partial F}{\partial\dot{\nu}_1}\frac{\dot{d}\,\nu_1}{d\mu}\right\}+$$

$$+\varepsilon^2\Bigg\{\frac{\partial^2 F}{\partial\nu^2}\left(\frac{d\nu}{d\varepsilon}\right)^2+\frac{\partial F}{\partial\nu}\frac{d^2\nu}{d\varepsilon^2}+\sum_{l=1}^{m}\frac{\partial^2 F}{\partial y_l^2}\left(\frac{dy_l}{d\varepsilon}\right)^2+$$

$$+\sum_{l=1}^{m}\frac{\partial F}{\partial y_l}\frac{d^2 y_l}{d\varepsilon^2}+\sum_{k=1}^{m}\frac{\partial^2 F}{\partial \dot{y}_{1k}^2}\left(\frac{d\dot{y}_{1k}}{d\varepsilon}\right)^2+\sum_{k=1}^{m}\frac{\partial F}{\partial \dot{y}_{1k}}\frac{d^2\dot{y}_{1k}}{d\varepsilon^2}+\frac{\partial^2 F}{\partial\dot{\nu}_1^2}\left(\frac{\dot{d}\,\nu_1}{d\varepsilon}\right)^2+$$

$$+\frac{\partial F}{\partial\dot{\nu}_1}\frac{d^2\dot{\nu}_1}{d\varepsilon^2}+2\sum_{l=1}^{m}\frac{\partial^2 F}{\partial\nu\partial y_l}\frac{d\nu}{d\varepsilon}\frac{dy_l}{d\varepsilon}+2\sum_{k=1}^{m}\frac{\partial^2 F}{\partial\nu\partial \dot{y}_{1k}}\frac{d\nu}{d\varepsilon}\frac{d\dot{y}_{1k}}{d\varepsilon}+$$

$$+2\sum_{k=1}^{m}\sum_{l=1}^{m}\frac{\partial^2 F}{\partial y_l\partial \dot{y}_{1k}}\frac{dy_l}{d\varepsilon}\frac{d\dot{y}_{1k}}{d\varepsilon}+2\frac{\partial^2 F}{\partial\nu\partial\dot{\nu}_1}\frac{d\nu}{d\varepsilon}\frac{d\dot{\nu}_1}{d\varepsilon}+$$

$$+2\sum_{l=1}^{m}\frac{\partial^2 F}{\partial y_l\partial\dot{\nu}_1}\frac{\partial\dot{\nu}_1}{\partial\varepsilon}\frac{dy_l}{d\varepsilon}+2\sum_{k=1}^{m}\frac{\partial^2 F}{\partial \dot{y}_{1k}\,\partial\dot{\nu}_1}\frac{d\dot{y}_{1k}}{d\varepsilon}\frac{d\dot{\nu}_1}{d\varepsilon}\Bigg\}+$$

$$+\mu^2\Bigg\{\frac{\partial^2 F}{\partial\nu^2}\left(\frac{d\nu}{d\mu}\right)^2+\frac{\partial F}{\partial\nu}\frac{d^2\nu}{d\mu^2}+\sum_{l=1}^{m}\frac{\partial^2 F}{\partial y_l^2}\left(\frac{dy_l}{d\mu}\right)^2+\sum_{l=1}^{m}\frac{\partial F}{\partial y_l}\frac{d^2 y_l}{d\mu^2}+$$

(2.20) [cont.]

$$+\sum_{k=1}^{m}\frac{\partial^2 F}{\partial \dot{y}_{1k}^2}\left(\frac{d\dot{y}_{1k}}{d\mu}\right)^2+\frac{\partial^2 F}{\partial \dot{\nu}_1^2}\left(\frac{d\dot{\nu}_1}{d\mu}\right)^2+\frac{\partial F}{\partial \dot{\nu}_1}\frac{d^2\dot{\nu}_1}{d\mu^2}+$$

$$+2\sum_{l=1}^{m}\frac{\partial^2 F}{\partial \nu \partial y_l}\frac{d\nu}{d\mu}\frac{dy_l}{d\mu}+2\sum_{k=1}^{m}\frac{\partial^2 F}{\partial \nu \partial \dot{y}_{1k}}\frac{d\nu}{d\mu}\frac{d\dot{y}_{1k}}{d\mu}+$$

$$+2\sum_{k=1}^{m}\sum_{l=1}^{m}\frac{\partial^2 F}{\partial y_l \partial \dot{y}_{1k}}\frac{dy_l}{d\mu}\frac{d\dot{y}_{1k}}{d\mu}+2\frac{\partial^2 F}{\partial \nu \partial \dot{\nu}_1}\frac{d\nu}{d\mu}\frac{d\dot{\nu}_1}{d\mu}+2\sum_{l=1}^{m}\frac{\partial^2 F}{\partial y_l \partial \dot{\nu}_1}\frac{dy_l}{d\mu}\frac{d\dot{\nu}_1}{d\mu}+$$

$$+2\sum_{k=1}^{m}\frac{\partial^2 F}{\partial \dot{y}_{1k}\partial \dot{\nu}_1}\frac{d\dot{y}_{1k}}{d\mu}\frac{d\dot{\nu}_1}{d\mu}\Bigg\}+$$

$$+\varepsilon\mu\Bigg\{2\frac{\partial^2 F}{\partial \nu^2}\frac{d\nu}{d\mu}\frac{d\nu}{d\varepsilon}+2\sum_{l=1}^{m}\frac{\partial^2 F}{\partial y^2}\frac{dy_l}{d\mu}\frac{dy_l}{d\varepsilon}+2\sum_{k=1}^{m}\frac{\partial^2 F}{\partial \dot{y}_{1k}^2}\frac{d\dot{y}_{1k}}{d\mu}\frac{d\dot{y}_{1k}}{d\varepsilon}+$$

$$+2\frac{\partial^2 F}{\partial \dot{\nu}_1^2}\frac{d\dot{\nu}_1}{d\mu}\frac{d\dot{\nu}_1}{d\varepsilon}+\sum_{l=1}^{m}\frac{\partial^2 F}{\partial \nu \partial y_l}\left(\frac{d\nu}{d\varepsilon}\frac{dy_l}{d\mu}+\frac{d\nu}{d\mu}\frac{dy_l}{d\varepsilon}\right)+$$

$$+\sum_{k=1}^{m}\frac{\partial^2 F}{\partial \nu \partial \dot{y}_{1k}}\left(\frac{d\nu}{d\varepsilon}\frac{d\dot{y}_{1k}}{d\mu}+\frac{d\nu}{d\mu}\frac{d\dot{y}_{1k}}{d\varepsilon}\right)+\sum_{k=1}^{m}\sum_{l=1}^{m}\frac{\partial^2 F}{\partial y_l \partial \dot{y}_{1k}}\left(\frac{dy_l}{d\varepsilon}\frac{d\dot{y}_{1k}}{d\mu}+\frac{dy_l}{d\mu}\frac{d\dot{y}_{1k}}{d\varepsilon}\right)+$$

$$+\sum_{l=1}^{m}\frac{\partial^2 F}{\partial y_l \partial \dot{\nu}_1}\left(\frac{dy_l}{d\varepsilon}\frac{d\dot{\nu}_1}{d\mu}+\frac{dy_l}{d\mu}\frac{d\dot{\nu}_1}{d\varepsilon}\right)+\sum_{k=1}^{m}\frac{\partial^2 F}{\partial \dot{y}_{1k}\partial \dot{\nu}_1}\left(\frac{d\dot{y}_{1k}}{d\varepsilon}\frac{d\dot{\nu}_1}{d\mu}+\frac{d\dot{\nu}_1}{d\varepsilon}\frac{d\dot{y}_{1k}}{d\mu}\right)+$$

$$+\frac{\partial^2 F}{\partial \nu \partial \dot{\nu}_1}\left(\frac{d\nu}{d\varepsilon}\frac{d\dot{\nu}_1}{d\mu}+\frac{d\dot{\nu}_1}{d\varepsilon}\frac{d\dot{\nu}}{d\mu}\right)+2\sum_{l=1}^{m}\frac{\partial F}{\partial y_l}\frac{\partial^2 y_l}{\partial \mu \partial \varepsilon}+2\frac{\partial F}{\partial \nu}\frac{\partial^2 \nu}{\partial \mu \partial \varepsilon}+$$

$$+2\frac{\partial F}{\partial \dot{\nu}_1}\frac{\partial^2 \dot{\nu}_1}{\partial \mu \partial \varepsilon}+2\sum_{k=1}^{m}\frac{\partial F}{\partial \dot{y}_{1k}}\frac{\partial^2 \dot{y}_{1k}}{\partial \mu \partial \varepsilon}\Bigg\}+O(\varepsilon^k\mu^l;\ k+l=4),$$

where $\dot{y}_1 = -\mu(dy/dt)$ and $\dot{\nu}_1 = -\mu(d\nu/dt)$. Then, in accordance with Eqs. (2.9) and (2.11) we obtain

$$\dot{y}_1 = -\sum_{k=1}^{K}\sum_{l=1}^{L}\varepsilon^k\mu^l\left\{\frac{\partial y_{kl}}{\partial a}\frac{da}{dt}+\frac{\partial y_{kl}}{\partial \psi}\frac{d\psi}{dt}\right\},$$

(2.21)

$$\dot{\nu}_1 = \mu\left\{a\frac{d\psi}{dt}\sin\psi-\frac{da}{dt}\cos\psi\right\}X_1(x)-\sum_{k=1}^{K}\sum_{l=1}^{L}\varepsilon^k\mu^l\left\{\frac{\partial V_{kl}}{\partial a}\frac{da}{dt}+\frac{\partial V_{kl}}{\partial \psi}\frac{d\psi}{dt}\right\}.$$

The derivatives of the functions in Eq. (2.19) are calculated at the point $\mu = \varepsilon = 0$, $\nu_0(t,x) = a(t)X_1(x)\cos\psi$, $y_0 = 0$ and $y_0 = 0$, while the derivatives in Eq. (2.20) are calculated at the point $\mu = \varepsilon = 0$, $\nu_0(t,\xi) = a(t)X_1(\xi)\cos\psi$ and $y_0 = 0$.

Having accounted for Eqs. (2.7)–(2.11) in Eqs. (2.19) and (2.20), we obtain

$$\text{(2.22)}\quad \varepsilon f\{x,\nu(t,x),\,y(t-\mu)\} = \varepsilon\left\{f(x,\nu,y)+\varepsilon\left\{\frac{\partial f}{\partial\nu}V_{10}+\sum_{l=1}^{m}\frac{\partial f}{\partial y_l}y_{(10)l}\right\}+\right.$$
$$+\varepsilon^2\left\{\frac{\partial^2 f}{\partial\nu^2}V_{10}^2+\frac{\partial f}{\partial\nu}V_{20}+\sum_{l=1}^{m}\frac{\partial^2 f}{\partial y_l^2}y_{(10)l}^2+\sum_{l=1}^{m}\frac{\partial f}{\partial y_l}y_{(20)l}+2\sum_{l=1}^{m}\frac{\partial^2 f}{\partial\nu\partial y_l}V_{10}y_{(10)l}\right\}+$$
$$\left.+2\varepsilon\mu\left\{\frac{\partial f}{\partial\nu}V_{11}+\sum_{l=1}^{m}\frac{\partial f}{\partial y_l}y_{(11)l}-\sum_{k=1}^{m}\frac{\partial f}{\partial\dot{y}_{1k}}\frac{\partial\dot{y}_{(11)k}}{\partial\psi}\omega_1\right\}\right\}+O(\varepsilon^k\mu^l;\ k+l=4);$$

$$\text{(2.23)}\quad \varepsilon F\{x,\nu(t-\mu,\xi),\,y(t-\mu)\} = \varepsilon\left\{F(x,\nu,y)+\varepsilon\left\{\frac{\partial F}{\partial\nu}V_{10}+\sum_{l=1}^{m}\frac{\partial F}{\partial y_l}y_{(10)l}\right\}+\right.$$
$$+\mu\frac{\partial F}{\partial\dot{\nu}_1}aX_1\omega_1\sin\psi+\varepsilon^2\left\{\frac{\partial^2 F}{\partial\nu^2}V_{10}^2+\frac{\partial F}{\partial\nu}V_{20}+\sum_{l=1}^{m}\frac{\partial^2 F}{\partial y_l^2}y_{(10)l}^2+\right.$$
$$\left.+\sum_{l=1}^{m}\frac{\partial F}{\partial y_l}y_{(20)l}+2\sum_{l=1}^{m}\frac{\partial^2 F}{\partial\nu\partial y_l}V_{10}y_{(10)l}\right\}+\mu^2\frac{\partial^2 F}{\partial\dot{\nu}_1^2}aX_1\omega_1\sin\psi+$$
$$+\varepsilon\mu\left\{\sum_{l=1}^{m}\frac{\partial^2 F}{\partial y\partial\dot{\nu}_1}a\omega_1X_1(\xi)y_{(10)l}\sin\psi+\frac{\partial^2 F}{\partial\nu\partial\dot{\nu}_1}V_{10}X_1(\xi)a\omega_1\sin\psi+\right.$$
$$+2\sum_{l=1}^{m}\frac{\partial F}{\partial y_l}y_{(11)l}+2\frac{\partial F}{\partial\nu}V_{11}+2\frac{\partial F}{\partial\dot{\nu}_1}\left\{B_{10}X_1(\xi)a\sin\psi-\right.$$
$$\left.\left.\left.-A_{10}X_1(\xi)\cos\psi-\frac{\partial V_{11}}{\partial\psi}\omega_1\right\}-\sum_{k=1}^{m}\frac{\partial F}{\partial\dot{y}_{1k}}\frac{\partial y_{(11)k}}{\partial\psi}\omega_1\right\}\right\}+O(\varepsilon^k\mu^l;\ k+l=4).$$

From Eq. (2.10), after accounting for Eqs. (2.14)–(2.17), we obtain

$$\text{(2.24)}\quad \frac{\partial^2\nu}{\partial t^2}-L_x^{(2m)}\{\nu(t,x)\}=\varepsilon\left\{\left\{\omega_1^2\frac{\partial^2 V_{10}}{\partial\psi^2}-L_x^{(2m)}\{V_{10}\}\right\}-\right.$$
$$\left.-2\omega_1B_{10}X_1a\cos\psi-2\omega_1A_{10}X_1\sin\psi\right\}+\varepsilon^2\left\{\left\{\omega_1^2\frac{\partial^2 V_{20}}{\partial\psi^2}-L_x^{(2m)}\{V_{20}\}\right\}+\right.$$
$$+2\omega_1B_{10}\frac{\partial^2 V_{10}}{\partial\psi^2}+2\omega_1A_{10}\frac{\partial^2 V_{10}}{\partial a\partial\psi}+\left\{A_{10}\frac{dA_{10}}{da}-\{2\omega_1B_{20}+B_{10}^2\}a\right\}X_1\cos\psi-$$
$$\left.-\left\{2\omega_1A_{20}+2A_{10}B_{10}+\frac{dB_{10}}{da}A_{10}\right\}X_1\sin\psi\right\}+$$
$$+\varepsilon\mu\left\{\left\{\omega_1^2\frac{\partial^2 V_{11}}{\partial\psi^2}-L_x^{(2m)}\{V_{11}\}\right\}-2\omega_1B_{11}X_1a\cos\psi-2\omega_1A_{11}X_1\sin\psi\right\}+$$
$$+\varepsilon^2\mu\left\{\left\{\omega_1^2\frac{\partial^2 V_{21}}{\partial\psi^2}-L_x^{(2m)}\{V_{21}\}\right\}+2\omega_1B_{10}\frac{\partial^2 V_{11}}{\partial\psi^2}+2\omega_1B_{11}\frac{\partial^2 V_{10}}{\partial\psi^2}+\right.$$
$$+2\omega_1A_{10}\frac{\partial^2 V_{11}}{\partial a\partial\psi}+2\omega_1A_{11}\frac{\partial^2 V_{10}}{\partial a\partial\psi}+\left\{\frac{dA_{10}}{da}A_{11}+\frac{dA_{11}}{da}A_{10}-\right.$$

(2.24) [cont.]

$$-2(B_{10}B_{11}+\omega_1 B_{21})a\Big\}X_1\cos\psi-\Big\{2(\omega_1 A_{21}+A_{11}B_{10}+A_{10}B_{11})+$$

$$+a\left(\frac{dB_{10}}{da}A_{11}+\frac{dB_{11}}{da}A_{10}\right)\Big\}X_1\sin\psi\Big\}+\varepsilon\mu^2\left\{\left\{\omega_1^2\frac{\partial^2 V_{12}}{\partial\psi^2}+L_x^{(2m)}\{V_{12}\}\right\}-\right.$$

$$-2\omega_1 B_{12}X_1 a\cos\psi-2A_{12}X_1\omega_1\sin\psi\Big\}+\varepsilon^3\left\{\left\{\omega_1^2\frac{\partial^2 V_{30}}{\partial\psi^2}-L_x^{(2m)}\{V_{30}\}\right\}+\right.$$

$$+2\omega_1 B_{10}\frac{\partial^2 V_{20}}{\partial\psi^2}+(2\omega_1 B_{20}+B_{10}^2)\frac{\partial^2 V_{10}}{\partial\psi^2}+$$

$$+A_{10}\frac{dB_{10}}{da}\frac{\partial V_{10}}{\partial\psi}+A_{10}\frac{dA_{10}}{da}\frac{\partial V_{10}}{\partial a}+2\omega_1 A_{10}\frac{\partial^2 V_{20}}{\partial a\partial\psi}+$$

$$+2(\omega_1 A_{20}+A_{10}B_{10})\frac{\partial^2 V_{10}}{\partial a\partial\psi}+A_{10}^2\frac{\partial^2 V_{10}}{\partial a^2}+$$

$$+\left\{\left(\frac{dA_{20}}{da}A_{10}+\frac{dA_{10}}{da}A_{20}\right)+2(\omega_1 B_{30}+B_{20}B_{10})a\right\}X_1\cos\psi-$$

$$-\left\{2(A_{30}\omega_1+A_{20}B_{10}+A_{10}B_{20})+a\left(\frac{dB_{10}}{da}A_{20}+\frac{dB_{20}}{da}A_{10}\right)\right\}X_1\sin\psi\Big\}+$$

$$+O(\varepsilon^k\mu^l;\ k+l=4).$$

From the second equation of system (2.3), after taking Eqs. (2.8) and (2.11) into account, we get

$$\frac{dy}{dt}-\sum_{p=0}^{P}A_p y(t-\tau_p)=\varepsilon\left\{\frac{\partial y_{10}(a,\psi)}{\partial\psi}\omega_1-\sum_{p=0}^{P}A_p y_{10}(a,\psi-\tau_p\omega_1)\right\}+ \tag{2.25}$$

$$+\varepsilon^2\left\{\frac{\partial y_{10}(a,\psi)}{\partial a}A_{10}+\frac{\partial y_{20}(a,\psi)}{\partial\psi}\omega_1+\frac{\partial y_{10}(a,\psi)}{\partial\psi}B_{10}-\right.$$

$$\left.-\sum_{p=0}^{P}A_p y_{20}(a,\psi-\tau_p\omega_1)\right\}+\varepsilon\mu\left\{\frac{\partial y_{11}}{\partial\psi}\omega_1-\sum_{p=0}^{P}y_{11}(a,\psi-\tau_p\omega_1)\right\}+$$

$$+\varepsilon^2\mu\left\{\frac{\partial y_{11}}{\partial a}A_{10}+\frac{\partial y_{11}}{\partial\psi}B_{10}+\frac{\partial y_{21}}{\partial\psi}\omega_1-\sum_{p=0}^{P}A_p y_{21}(a,\psi-\tau_p\omega_1)\right\}+$$

$$+\varepsilon\mu^2\left\{\frac{\partial y_{12}}{\partial\psi}\omega_1-\sum_{p=0}^{P}A_p y_{12}(a,\psi-\tau_p\omega_p)\right\}+\varepsilon^3\left\{\frac{\partial y_{10}}{\partial a}A_{20}+\right.$$

$$+\frac{\partial y_{20}}{\partial a}A_{10}+\frac{\partial y_{30}}{\partial\psi}\omega_1+\frac{\partial y_{20}}{\partial\psi}B_{10}+\frac{\partial y_{10}}{\partial\psi}B_{20}-$$

$$\left.-\sum_{p=0}^{P}A_p y_{30}(a,\psi-\tau_p\omega_1)\right\}+O(\varepsilon^k\mu^l;\ k+l=4).$$

When comparing the terms found at the same powers of $\varepsilon^k\mu^l$ we determine a sequence of recurrent linear differential equations, which are given in Appendix A.

After expanding the function $f_{(\cdot)}$ into a Fourier series one obtains

$$f_{(\cdot)} = \sum_{n=1}^{\infty} \{ b_{(\cdot)n}(a) \cos n\psi + c_{(\cdot)n}(a) \sin n\psi \} , \tag{2.26}$$

where

$$\begin{aligned} b_{(\cdot)n}(a) &= \frac{1}{2\pi l} \int_0^l dx \int_0^{2\pi} f_{(\cdot)}(x, a, \psi) X_1(x) \cos n\psi \, d\psi, \\ c_{(\cdot)n}(a) &= \frac{1}{2\pi l} \int_0^l dx \int_0^{2\pi} f_{(\cdot)}(x, a, \psi) X_1(x) \sin n\psi \, d\psi. \end{aligned} \tag{2.27}$$

If we equate the coefficients of $X_1(x) \sin \psi$ and $X_1(x) \cos \psi$ to zero, we obtain A_{kl} and B_{kl}, which are given in Appendix B. According to Eq. (2.8) we get

$$\begin{aligned} \Phi(a) = \frac{da}{dt} &= \varepsilon A_{10} + \varepsilon^2 A_{20} + \varepsilon^3 A_{30} + \varepsilon\mu A_{11} + \\ &\quad + \varepsilon^2 \mu A_{21} + \varepsilon\mu^2 A_{12} + O(\varepsilon^k \mu^l;\ k + l = 4), \\ \omega(a) = \frac{d\psi}{dt} &= \omega_1 + \varepsilon B_{10} + \varepsilon^2 B_{20} + \varepsilon^3 B_{30} + \varepsilon\mu B_{11} + \\ &\quad + \varepsilon^2 \mu B_{21} + \varepsilon\mu^2 B_{12} + O(\varepsilon^k \mu^l;\ k + l = 4), \end{aligned} \tag{2.28}$$

at the initial conditions $a(t_0) = a_0$, $\psi(t_0) = \psi_0$.

From the first equation of (2.28) we obtain the dependence $a(t)$, which upon introduction into the later equation of (2.28) enables us to determine the dependence $\psi\{a(t)\}$. Thanks to this it is possible to analyze the slow transient processes leading to steady state. The latter are analyzed by assuming that $da/dt = 0$, which leads to the algebraic equation

$$G(a, \varepsilon, \mu) = A_{10} + \varepsilon A_{20} + \varepsilon^2 A_{30} + \mu A_{11} + \varepsilon\mu A_{21} + \mu^2 A_{12} = 0. \tag{2.29}$$

If the calculations are limited up to order ε, we get from Eq. (2.29)

$$A_{10} = 0, \tag{2.30}$$

which enables us to find: (a) one isolated solution; (b) few isolated solutions; (c) no solutions. However, sometimes the phase flow of the considered starting equations can be very sensitive to changes in the amplitude a and/or the parameters ε and μ. For these reasons the full equation (2.29) should be taken into consideration. The solution of Eq. (2.30) can serve as a first approximation for the numerical solution of the full equation (2.29).

Now we briefly indicate the variety of problems which can be solved using this approach, and that can not be solved by the use of a single perturbation method.

A. Suppose that the parameter ε undergoes slight changes, which are impossible to avoid. We want to control such changes by treating μ as a control parameter. Inserting $a = a^0 = \text{const}$ into Eq. (2.29) we can find $G(\varepsilon, \mu, a^0) = G(\varepsilon, \mu) = 0$. Thus, in accordance with the changes of ε we can find the values of μ in order to maintain a constant amplitude. The possible solutions of the problem are discussed in detail in Appendix C, which is based on reference [18].

B. Suppose that we would like to have $a = a(\varepsilon)$ and because the shape of $a(\varepsilon)$ should be fixed a priori. The problem is then again reduced to the implicit algebraic functions of second order; further discussion is presented in Appendix C.

C. Different branching phenomena can be expected. We can find the hysteresis variety points defined by the following equations:

$$\begin{aligned} G(a,\varepsilon,\mu) &= 0, \\ G_a(a,\varepsilon,\mu) &= 0, \\ G_{aa}(a,\varepsilon,\mu) &= 0. \end{aligned} \tag{2.31}$$

If it is possible to eliminate the amplitude a from one of Eqs. (2.31), then the other two enable us to find the hysteresis points. The bifurcation and isolated variety points are defined by the following three equations:

$$\begin{aligned} G(a,\varepsilon,\mu) &= 0, \\ G_a(a,\varepsilon,\mu) &= 0, \\ G_\varepsilon(a,\varepsilon,\mu) &= 0. \end{aligned} \tag{2.32}$$

As mentioned above, Eqs. (2.32) can possess several different solutions for a. Thus M-multiply limit variety can be defined by the following equations:

$$\begin{aligned} G(a_1,\varepsilon,\mu) &= 0, \\ &\vdots \\ G(a_m,\varepsilon,\mu) &= 0, \\ &\vdots \\ G_a(a_1,\varepsilon,\mu) &= 0, \\ &\vdots \\ G_a(a_m,\varepsilon,\mu) &= 0. \end{aligned} \tag{2.33}$$

Using μ as a parameter, we can control the branching phenomena mentioned above.

D. We can find the (ε, μ) set of parameters for which no real solutions of Eq. (2.29) exist. Thus, a domain of the assumed solution (2.7) can be defined in the two-parameter space.

E. Suppose that we want to change the amplitude of oscillations, but the frequency of oscillations should not undergo any changes(or it should be controlled only by the linear part of the equations). In order to fulfil such requirements we have:

$$\begin{aligned} G(a,\varepsilon,\mu) &= A_{10} + \varepsilon A_{20} + \varepsilon^2 A_{30} + \mu A_{11} + \varepsilon\mu A_{21} + \mu^2 A_{12} = 0, \\ H(a,\varepsilon,\mu) &= B_{10} + \varepsilon B_{20} + \varepsilon^2 A_{30} + \mu B_{11} + \varepsilon\mu B_{21} + \mu^2 B_{12} = 0. \end{aligned} \tag{2.34}$$

After eliminating a from one of equations (2.34) there remains one equation, which defines the implicit algebraic function of second order in ε and μ. One can freely choose one parameter and then calculate the value of the second one. Thus, by such an appropriate choice of the parameters ε and μ the amplitude of the one-frequency oscillations will change, however, the frequency ω_1 will always remain constant.

Finally, we have to emphasize that our calculations have been limited to the order of $O(\mu^3)$, because higher powers of μ do not cause a good approximation for the series (2.18).

3. Example

We consider the following example from the field of solids and structures. An elastic beam of constant cross-section is connected by a spring k_2 with a discrete one degree-of-freedom system (see Fig. 1). We assume that the linear coupling stiffness in-

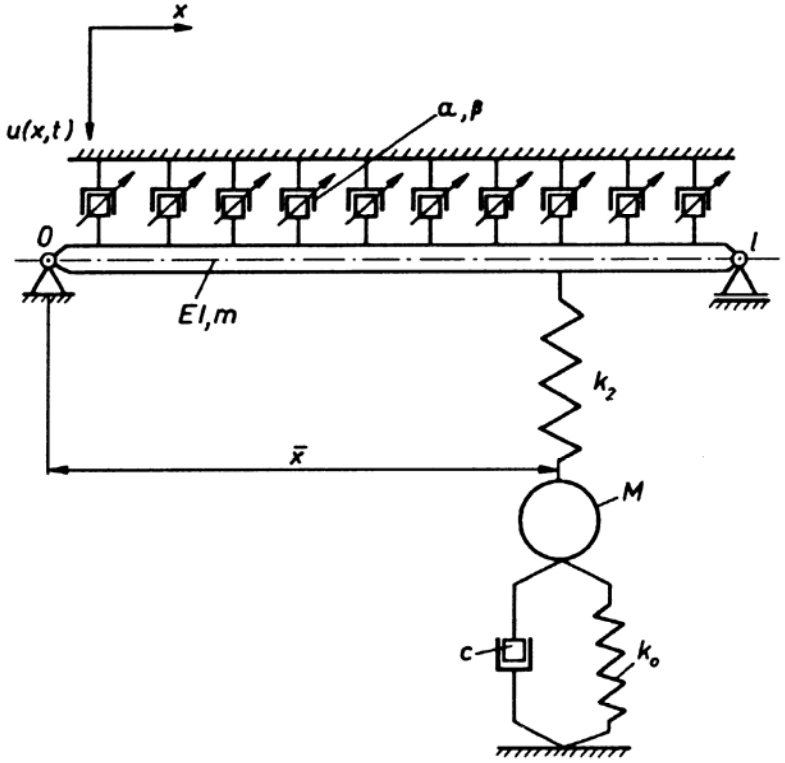

FIG. 1. Calculation example: Self-excited vibrations of a beam connected with a one degree-of-freedom system.

volves a time delay and the nonlinearities, the time delay, and the amplitude of oscillations are small. Our system is an autonomous one, and the Van der Pol damping acting on the beam is responsible for oscillations. Within the framework of the usual assumptions of the elementary theory of bending we obtain the following set of governing equations

$$
\begin{aligned}
lEI\frac{\partial^4 u}{\partial x^4} + ml\frac{\partial^2 u}{\partial t^2} &= l(\alpha - \beta u^2)\frac{\partial u}{\partial t} - k_2\{y(t,\overline{x}) - \delta(x-\overline{x})y(t-\hat{\mu})\}, \\
M\ddot{y} &= -c\dot{y} - (k_0 + k_2) + k_2 u(t-\hat{\mu},\overline{x}),
\end{aligned}
\tag{3.1}
$$

where the damping coefficient α and β and the mass m are taken per unit length, and $\hat{\mu}$ is a time delay. The other standard parameters are given in Fig. 1. We have the following boundary conditions:

$$u(x,t)|_{x=0} = u(x,t)_{x=l} = 0, \tag{3.2}$$
$$\left.\frac{\partial^2 u(x,t)}{\partial x^2}\right|_{x=0} = \left.\frac{\partial^2 u(x,t)}{\partial x^2}\right|_{x=l} = 0.$$

In addition to nonlinear mechanical systems with time delay (see, for example [19,20], the governing delay nonlinear differential equations can be found in problems related to biology, blood circulation, and control systems [21–23]. Therefore we transform the dimensional equations (3.1) to a nondimensional form. Thanks to this we reduce the number of valid parameters, and our further calculations are valid not only for the mechanical system shown in Fig. 1, but for others as well. The new nondimensional governing equations are

$$\frac{\partial^2 w(\tau,\xi)}{\partial \tau^2} + \rho^4 \frac{\partial^4 w(\tau,\xi)}{\partial \xi^4} = \varepsilon(1 - w^2(\tau,\xi))\frac{\partial w(\tau,\xi)}{\partial \tau} - \varepsilon A w(\tau,\bar{\xi}) + \varepsilon B \delta(\xi - \bar{\xi})\eta(\tau - \mu), \tag{3.3}$$
$$\frac{d^2\eta(\tau)}{d\tau^2} = -\varepsilon C \frac{d\eta}{d\tau} + \varepsilon F w(\tau - \mu, \bar{\xi}),$$

and the new boundary conditions are

$$w(\xi,\tau)|_{\xi=0} = w(\xi,\tau)|_{\xi=l} = 0, \tag{3.4}$$
$$\left.\frac{\partial^2 w(\xi,\tau,)}{\partial \tau^2}\right|_{\xi=0} = \left.\frac{\partial^2 w(\xi,\tau)}{\partial \tau^2}\right|_{\xi=1} = 0.$$

The nondimensional parameters are defined as follows:

$$\tau = \Omega t, \quad \mu = \Omega \hat{\mu}, \quad \xi = x l^{-1}, \quad w = (\beta\alpha^{-1})^{\frac{1}{2}} u, \tag{3.5}$$
$$\varepsilon = \alpha(m\Omega)^{-1}, \quad \rho^4 = EIm^{-1}\Omega^{-2}l^{-4}, \quad A = k_2\alpha^{-1}l^{-1}\Omega^{-1},$$
$$B = k_2\alpha^{-\frac{1}{2}}\beta^{\frac{1}{2}}\Omega^{-1}, \quad C = cm\alpha^{-1}M^{-1}, \quad D = (k_0 + k_2)M^{-1}\Omega^{-2},$$
$$F = k_2 m\alpha^{-\frac{1}{2}}\beta^{-\frac{1}{2}}\Omega^{-1}l^{-1}M^{-1}.$$

In order to avoid tedious calculations we assume that

$$\eta(\tau - \mu) = \eta(\tau) - \mu\frac{d\eta}{d\tau}, \tag{3.6}$$
$$w(\tau - \mu, \bar{\xi}) = w(\tau) - \mu\frac{\partial w(\tau,\bar{\xi})}{\partial \tau}.$$

Taking Eqs. (3.6) into consideration, we obtain from (3.2) the following set of equations:

$$\frac{\partial^2 w(\tau,\xi)}{\partial \tau^2} + \rho^4 \frac{\partial^4 w(\tau,\xi)}{\partial \xi^4} = \varepsilon(1 - w^2(\tau,\xi))\frac{\partial w(\tau,\xi)}{\partial \tau} - \varepsilon A w(\tau,\bar{\xi}) + \varepsilon B \delta(\xi - \bar{\xi})\eta(\tau) - \varepsilon\mu B\delta(\xi - \bar{\xi})\frac{d\eta}{d\tau}, \tag{3.7}$$
$$\frac{d^2\eta(\tau)}{d\tau^2} + D\eta = -\varepsilon C\frac{d\eta}{d\tau} + \varepsilon F w(\tau,\bar{\xi}) - \varepsilon\mu F\frac{\partial w(\tau,\bar{\xi})}{\partial \tau}.$$

From the first equation of (3.7), and for $\varepsilon = 0$, we determine the frequency $\nu = (n\pi\rho)^2$, $n \in N$. Limiting our calculations to $n = 1$, and with regard to the earlier section, the solutions of (3.7) are sought in the form

$$\begin{aligned} w(\xi,\tau) &= a(t)\sin\pi\xi\cos\psi(\tau) + \varepsilon W_{10}(\xi,a,\psi) + \varepsilon^2 W_{20}(\xi,a,\psi) + \\ &\quad + \varepsilon^3 W_{30}(\xi,a,\psi) + \varepsilon\mu W_{11}(\xi,a,\psi) + \varepsilon^2\mu W_{21}(\xi,a,\psi), \\ \eta(\tau) &= \varepsilon\eta_{10}(a,\psi) + \varepsilon^2\eta_{20}(a,\psi) + \varepsilon^3\eta_{30}(a,\psi) + +\varepsilon\mu\eta_{11}(a,\psi + \varepsilon^2\mu\eta_{21}(a,\psi), \end{aligned} \tag{3.8}$$

where $W_{kl}(\xi,a,\psi)$ and $\eta_{kl}(a,\psi)$ are the limited and periodic (with regard to ψ) functions to be obtained. The unknown amplitude $a(\tau)$ and phase $\psi(\tau)$ are calculated from

$$\begin{aligned} \frac{da}{dt} &= \varepsilon A_{10}(a) + \varepsilon^2 A_{20}(a) + \varepsilon^3 A_{30}(a) + \varepsilon\mu A_{11}(a) + \varepsilon^3\mu A_{21}(a), \\ \frac{d\psi}{dt} &= \nu + \varepsilon B_{10}(a) + \varepsilon^2 B_{20}(a) + \varepsilon_{30} B_{30}(a) + \varepsilon\mu B_{11} + \varepsilon^2\mu B_{21}(a). \end{aligned} \tag{3.9}$$

Proceeding in an analogous way to that shown in Sec. 2, we find the sequence of recurrence linear equations given in Appendix D. The solution of this set of equations gives us

$$\begin{aligned} W_{10} &= -\frac{1}{1280\nu}a^3\sin 3\pi\xi\sin\psi - \frac{3}{128\nu}a^3\sin\pi x\sin 3\psi - \frac{1}{1152\nu}a^3\sin 3\pi\xi\sin 3\psi, \\ \eta_{10} &= \frac{Fa}{(D-\nu^2)}\sin\pi\bar{\xi}\cos\psi, \qquad \bar{\xi}\in(0,1), \\ W_{20} &= A_1\sin 3\pi\xi\sin\psi + A_2\sin\pi\xi\sin 3\psi + A_3\sin 3\pi\xi\sin 3\psi + \\ &\quad + A_4\sin 3\pi\xi\cos\psi + A_5\sin\pi\xi\cos 3\psi + A_6\sin 3\pi\xi\cos 3\psi, \end{aligned}$$

$$\begin{aligned} A_1 &= \frac{1}{80\nu^2}\left\{-\frac{a^3}{1280\nu}A\sin\pi\bar{\xi} - \frac{a^3}{32\nu}A\sin\pi\bar{\xi} + \frac{a^3}{1280\nu}A\right\}, \\ A_2 &= \frac{1}{8\nu^2}\left\{\frac{27a^3}{128\nu}A\sin\pi\bar{\xi} - \frac{3a^3}{32\nu}A\sin\pi\bar{\xi} - \frac{3a^3}{128\nu}A\right\}, \\ A_3 &= \frac{1}{72\nu^2}\left\{-\frac{9a^3}{1152\nu}A\sin\pi\bar{\xi} - \frac{a^3}{32\nu}A\sin\pi\bar{\xi} + \frac{a^3}{1152\nu}A\right\}, \\ A_4 &= \frac{1}{80\nu^2}\left\{\frac{a^3}{640}\right\}, \\ A_5 &= \frac{1}{8\nu^2}\left\{\frac{9}{64}a^3\right\}, \\ A_6 &= \frac{1}{72\nu^2}\left\{\frac{3}{1152}a^3\right\}, \\ \eta_{20} &= \left\{2\nu A_{10}\frac{F}{(D-\nu^2)^2} + a\nu\frac{FC}{(D-\nu^2)^2}\right\}\sin\pi\bar{\xi}\sin\psi + \\ &\quad + 2\nu a B_{10}\frac{F}{(D-\nu^2)^2}\sin\pi\bar{\xi}\cos\psi - \frac{a^3F}{1280\nu(D-\nu^2)}\sin 3\pi\bar{\xi}\sin\psi - \\ &\quad - \frac{3a^3F}{128\nu(D-9\nu^2)}\sin\pi\bar{\xi}\sin 3\psi - \frac{a^3F}{1152\nu(D-9\nu^2)}\sin 3\pi\bar{\xi}\sin 3\psi, \end{aligned} \tag{3.10}$$

(3.10) [cont.]

$$W_{11} = 0,$$

$$\eta_{11} = \frac{Fa\nu}{(D-\nu^2)} \sin \pi\bar{\xi} \sin \psi,$$

$$A_{10} = \frac{1}{2}a\left(1 - \frac{3}{16}a^2\right),$$

$$B_{10} = \frac{1}{2\nu} A \sin \pi\bar{\xi},$$

$$A_{20} = 0,$$

$$B_{20} = -\frac{1}{\nu}\left\{\frac{1}{8} + \frac{3}{64}a^2 + \frac{1}{8\nu^2}A^2 \sin^2 \pi\bar{\xi} + \frac{BF}{2(D-\nu^2)} \sin \pi\bar{\xi}\right\},$$

$$A_{30} = -\frac{3a^3}{256\nu^4}A^2 \sin^2 \pi\bar{\xi} - \frac{3BFa^3}{64\nu^2(D-\nu^2)} \sin \pi\bar{\xi} + \frac{3a^3}{256\nu^2} + \frac{BFa}{2\nu^2(D-\nu^2)} \sin \pi\bar{\xi} - \frac{BFa\left(1 - \frac{3}{16}a^2\right)}{2(D-\nu^2)^2} \sin \pi\bar{\xi} - \frac{CBFa}{2(D-\nu^2)^2} \sin \pi\bar{\xi},$$

$$B_{30} = \frac{a}{2\nu^3}\left\{\frac{1}{8} + \frac{3}{64}a^2 + \frac{1}{8\nu^2}A^2 \sin^2 \pi\bar{\xi} - \frac{BF}{2(D-\nu^2)} \sin \pi\bar{\xi}\right\} - \frac{BF}{4\nu^2(D-\nu^2)^2} A^2 \sin^3 \pi\bar{\xi},$$

$$A_{11} = 0,$$

$$B_{11} = 0,$$

$$A_{21} = -\frac{BFa}{2\nu(D-\nu^2)}(\nu + 1) \sin \pi\bar{\xi},$$

$$B_{21} = 0.$$

In the calculations we have not taken into account harmonics of order greater than three, and we have omitted the power of the amplitudes which were greater than three. Let us consider the stationary state which leads to the following algebraic equation:

(3.11) $$A_{10} + \varepsilon^2 A_{30} + \varepsilon\mu A_{21} = 0.$$

Because we have limited the calculations to the first power of μ in our example, we can use the general discussion given in Appendix D by substituting μ for a and considering further the implicit function (3.11) with regard to a and ε.

From Eq. (3.11) one obtains

(3.12) $$\varepsilon^2 \frac{BF}{\nu^2(D-\nu^2)}\{D - \nu^2(C+2)\} \sin \pi\bar{\xi} - \varepsilon \frac{BF(\nu+1)\mu}{\nu(D-\nu^2)} \sin \pi\bar{\xi} - \frac{3}{16}a^2 + 1 = 0.$$

We have also determined

(3.13)
$$W = \frac{3BF}{16\nu^2(D-\nu^2)^2} \sin \pi\bar{\xi}\left\{(\nu^2(C+2) - D) + \mu^2 \frac{BF(\nu+1)^2}{4} \sin \pi\bar{\xi}\right\},$$

$$V = \frac{3BF}{16\nu^2(D-\nu^2)^2}\{\nu^2(C+2) - D\} \sin \pi\bar{\xi}.$$

These results allow us to come to some conclusions important from the point of view of possible applications.

If $\nu^2(C+2) > D$, then in the considered system, a one-frequency periodic solution does not exist, because W is always greater than 0.

If $\nu^2(C+2) = D$, then $W > 0$, and the curve $\varepsilon(a)$ is a parabola.

If $\nu^2(C+2) < D$ and $W \neq 0$, then the curve $\varepsilon(a)$ is an equilateral hyperbola.

Finally if $\nu^2(C+2) < D$ and $W = 0$, we have two intersecting lines.

4. Concluding Remarks

This paper has presented a local analytical method for determining the periodic one-frequency oscillations in dynamical nonlinear discrete-continuous systems with delay. This method employed the classical KBM technique (Krylov–Bogolubov–Mitropolski) and, in a new approach, the solution is sought in the form of certain power series in terms of two independent perturbation parameters ε and μ. The former is connected with nonlinearity and the latter with time delay. It is assumed that both parameters are small, and the amplitude of oscillations is small.

Thanks to this method the problem of analyzing the transient nonstationary states leading to the steady state has been reduced to the analysis of two first order differential equations. The first is an equation with separable variables, and its solution after its introduction into the second enables us to determine how the frequency of the sought solution changes in time, and the influence of the parameters μ and ε, which appear explicitly in the solution. A general discussion of the benefits of using the two-perturbation technique is provided. Such problems, important from the point of view of applications, are demonstrated. These problems can not be solved by the use of a classical single-perturbation technique.

In order to demonstrate physical insight, an example from the field of mechanics is considered. Based on this example, we demonstrated that the presented analytical approach leads to important results. Most importantly, we can find a set of parameters for which the one-frequency periodic solution does or does not exist. Additionally, in the example system we calculated fixed sets of parameters for which the amplitude of oscillations and the control parameter ε form a parabola, an equilateral hyperbola, or two intersecting lines.

Appendix A

The sequence of recurrent linear differential equations is of the following form:

$$\varepsilon: \qquad \omega_1^2 \frac{\partial^2 V_{10}(x,a,\psi)}{\partial \psi^2} = L_x^{(2m)}\{V_{10}\} + 2\omega_1 B_{10} X_1 a \cos\psi + \\ + 2\omega_1 A_{10} X_1 \sin\psi + f_\varepsilon(x,a,\psi),$$

$$\omega_1 \frac{\partial y_{10}(a,\psi)}{\partial \psi} = \sum_{p=0}^{P} A_p y_{10}(a, \psi - \tau_p \omega_1) + F_\varepsilon(a,\psi);$$

$\varepsilon^2:$
$$\omega_1^2 \frac{\partial^2 V_{20}(x,a,\psi)}{\partial \psi^2} = L_x^{(2m)}\{V_{20}\} + 2\omega_1 B_{20} X_1 a \cos\psi + \\ +2\omega_1 A_{20} X_1 \sin\psi + f_{\varepsilon^2}(x,a,\psi),$$
$$\omega_1 \frac{\partial y_{20}(a,\psi)}{\partial \psi} = \sum_{p=0}^{P} A_p y_{20}(a, \psi - \tau_p \omega_1) + F_{\varepsilon^2}(a,\psi);$$

$\varepsilon\mu:$
$$\omega_1^2 \frac{\partial^2 V_{11}(x,a,\psi)}{\partial \psi^2} = L_x^{(2m)}\{V_{11}\} + 2\omega_1 B_{11} X_1 a \cos\psi + \\ +2\omega_1 A_{11} X_1 \sin\psi + f_{\varepsilon\mu}(x,a,\psi),$$
$$\omega_1 \frac{\partial y_{11}(a,\psi)}{\partial \psi} = \sum_{p=0}^{P} A_p y_{11}(a, \psi - \tau_p \omega_1) + F_{\varepsilon\mu}(a,\psi);$$

$\varepsilon^3:$
$$\omega_1^2 \frac{\partial^2 V_{30}(x,a,\psi)}{\partial \psi^2} = L_x^{(2m)}\{V_{30}\} + 2\omega_1 B_{30} X_1 a \cos\psi + \\ +2\omega_1 A_{30} X_1 \sin\psi + f_{\varepsilon^3}(x,a,\psi),$$
$$\omega_1 \frac{\partial y_{30}(a,\psi)}{\partial \psi} = \sum_{p=0}^{P} A_p y_{20}(a, \psi - \tau_p \omega_1) + F_{\varepsilon^3}(a,\psi);$$

$\varepsilon\mu^2:$
$$\omega_1^2 \frac{\partial^2 V_{12}(x,a,\psi)}{\partial \psi^2} = L_x^{(2m)}\{V_{12}\} + 2\omega_1 B_{12} X_1 a \cos\psi + \\ +2\omega_1 A_{12} X_1 \sin\psi + f_{\varepsilon\mu^2}(x,a,\psi),$$
$$\omega_1 \frac{\partial y_{12}(a,\psi)}{\partial \psi} = \sum_{p=0}^{P} A_p y_{12}(a, \psi - \tau_p \omega_1) + F_{\varepsilon\mu^2}(a,\psi);$$

$\varepsilon^2\mu:$
$$\omega_1^2 \frac{\partial^2 V_{21}(x,a,\psi)}{\partial \psi^2} = L_x^{(2m)}\{V_{21}\} + 2\omega_1 B_{21} X_1 a \cos\psi + \\ +2\omega_1 A_{21} X_1 \sin\psi + f_{\varepsilon^2\mu}(x,a,\psi),$$
$$\omega_1 \frac{\partial y_{21}(a,\psi)}{\partial \psi} = \sum_{p=0}^{P} A_p y_{21}(a, \psi - \tau_p \omega_1) + F_{\varepsilon^2\mu}(a,\psi);$$

where

$$f_\varepsilon = f(x, \nu_0);$$
$$F_\varepsilon = F(x, \nu_0);$$
$$f_{\varepsilon^2} = \frac{\partial f}{\partial \nu} V_{10} + \sum_{l=1}^{m} \frac{\partial f}{\partial y_l} y_{(10)l} - 2\omega_1 B_{10} \frac{\partial^2 V_{10}}{\partial \psi^2} - 2\omega_1 A_{10} \frac{\partial^2 V_{10}}{\partial a \partial \psi} - \\ - \left(A_{10}\frac{dA_{10}}{da} + B_{10}^2 a\right) X_1 \cos\psi - \left(2A_{10}B_{10} + a\frac{dB_{10}}{da} A_{10}\right) X_1 \sin\psi;$$
$$F_{\varepsilon^2} = \frac{\partial F}{\partial \nu} V_{10} + \sum_{l=1}^{m} \frac{\partial F}{\partial y_l} y_{(10)l} - \sum_{l=1}^{m} \frac{\partial y_{(10)l}}{\partial a} A_{10};$$

$$
\begin{aligned}
f_{\varepsilon\mu} &= 0;\\
F_{\varepsilon\mu} &= \frac{\partial F}{\partial \dot{\nu}_1} a\omega_1 X_1 \sin\psi;\\
f_{\varepsilon^3} &= \frac{\partial^2 f}{\partial \nu^2} V_{10}^2 + \frac{\partial f}{\partial \nu} V_{20} + \sum_{l=1}^{m} \frac{\partial^2 f}{\partial y_l^2} y_{(10)l}^2 + \sum_{l=1}^{m} \frac{\partial f}{\partial y_l} y_{(20)l} -\\
&\quad -2\omega_1 B_{10} \frac{\partial^2 V_{20}}{\partial \psi^2} - (2\omega_1 B_{20} + B_{10}^2) \frac{\partial^2 V_{10}}{\partial \psi^2} - \frac{\partial B_{10}}{da} \frac{\partial V_{10}}{\partial \psi} A_{10} -\\
&\quad -A_{10} \frac{dA_{10}}{da} \frac{\partial V_{10}}{\partial a} - 2\omega_1 A_{10} \frac{\partial^2 V_{20}}{\partial a \partial \psi} - 2(\omega_1 A_{20} + A_{10} B_{10}) \frac{\partial^2 V_{10}}{\partial a \partial \psi} -\\
&\quad -A_{10}^2 \frac{\partial^2 V_{10}}{\partial a^2} - \left\{ \frac{dA_{20}}{da} A_{10} + \frac{dA_{10}}{da} A_{20} + 2B_{20} B_{10} a \right\} X_1 \cos\psi +\\
&\quad + \left\{ 2(A_{20} B_{10} + A_{10} B_{20}) + a \left(\frac{dB_{10}}{da} A_{10} + \frac{dB_{20}}{da} A_{10} \right) \right\} X_1 \sin\psi;\\
F_{\varepsilon^3} &= \frac{\partial^2 F}{\partial \nu^2} V_{10}^2 + \frac{\partial F}{\partial \nu} V_{20} + \sum_{l=1}^{m} \frac{\partial^2 F}{\partial y_l^2} y_{(10)l}^2 + \sum_{l=1}^{m} \frac{\partial F}{\partial y_l} y_{(20)l} +\\
&\quad +2 \sum_{l=1}^{m} \frac{\partial^2 F}{\partial \nu \partial y_l} V_{10} y_{(10)l} - \frac{\partial y_{10}}{\partial a} A_{20} - \frac{\partial y_{20}}{\partial a} A_{10} - \frac{\partial y_{20}}{\partial \psi} B_{10} - \sum_{l=1}^{m} \frac{\partial y_{(10)l}}{\partial \psi} B_{20};\\
f_{\varepsilon\mu^2} &= 0;\\
F_{\varepsilon\mu^2} &= \frac{\partial^2 F}{\partial \dot{\nu}^2} a\omega_1 X_1 \sin\psi;\\
f_{\varepsilon^2\mu} &= 2 \left\{ \frac{\partial f}{\partial \nu} V_{11} + \sum_{l=1}^{m} \frac{\partial f}{\partial y_l} y_{(11)l} - \sum_{k=1}^{m} \frac{\partial f}{\partial \dot{y}_{1k}} \frac{\partial y_{(11)k}}{\partial \psi} \omega_1 \right\} -\\
&\quad -2\omega_1 B_{10} \frac{\partial^2 V_{11}}{\partial \psi^2} - 2\omega_{11} B_{11} \frac{\partial^2 V_{10}}{\partial \psi^2} - 2\omega_1 A_{10} \frac{\partial^2 V_{11}}{\partial a \partial \psi} -\\
&\quad -2\omega_1 A_{11} \frac{\partial^2 V_{10}}{\partial a \partial \psi} - \left(\frac{dA_{10}}{da} A_{11} + \frac{dA_{11}}{da} A_{10} - 2B_{10} B_{11} a \right) X_1 \cos\psi +\\
&\quad + \left\{ 2(A_{11} B_{10} + A_{10} B_{11}) + a \left(\frac{dB_{10}}{da} A_{11} + \frac{dB_{11}}{da} A_{10} \right) \right\} X_1 \sin\psi;\\
F_{\varepsilon^2\mu} &= \sum_{l=1}^{m} \frac{\partial^2 F}{\partial y_l \partial \dot{\nu}_1} y_{(10)l} a\omega_1 X_1(\xi) \sin\psi + \frac{\partial^2 F}{\partial \nu \partial \dot{\nu}_1} V_{10} a\omega_1 X_1(\xi) \sin\psi + 2 \sum_{l=1}^{m} \frac{\partial F}{\partial y_l} y_{(11)l} +\\
&\quad +2 \frac{\partial F}{\partial \nu} V_{11} + 2 \frac{\partial F}{\partial \dot{\nu}_1} \left(aB_{10} X_1(\xi) \sin\psi - A_{10} X_1(\xi) \cos\psi - \frac{\partial V_{11}}{\partial \psi} \omega_1 \right) -\\
&\quad -2 \sum_{l=1}^{m} \frac{\partial F}{\partial \dot{y}_{1l}} \frac{\partial y_{(11)l}}{\partial \psi} \omega_1 - \sum_{l=1}^{m} \frac{\partial y_{(11)l}}{\partial a} A_{10} - \sum_{l=1}^{m} \frac{\partial y_{(11)l}}{\partial \psi} B_{10}.
\end{aligned}
$$

Appendix B

$$A_{10}(a) = -\frac{c_{(\varepsilon)1}(a)}{2\omega_1}, \qquad B_{10}(a) = -\frac{b_{(\varepsilon)1}(a)}{2\omega_1 a},$$

$$A_{20}(a) = -\frac{c_{(\varepsilon^2)1}(a)}{2\omega_1}, \qquad B_{20}(a) = -\frac{b_{(\varepsilon^2)1}(a)}{2\omega_1 a},$$

$$A_{30}(a) = -\frac{c_{(\varepsilon^3)1}(a)}{2\omega_1}, \qquad B_{30}(a) = -\frac{b_{(\varepsilon^3)1}(a)}{2\omega_1 a},$$

$$A_{11}(a) = -\frac{c_{(\varepsilon\mu)1}(a)}{2\omega_1}, \qquad B_{11}(a) = -\frac{b_{(\varepsilon\mu)1}(a)}{2\omega_1 a},$$

$$A_{12}(a) = -\frac{c_{(\varepsilon\mu^2)1}(a)}{2\omega_1}, \qquad B_{12}(a) = -\frac{b_{(\varepsilon\mu^2)1}(a)}{2\omega_1 a},$$

$$A_{21}(a) = -\frac{c_{(\varepsilon^2\mu)1}(a)}{2\omega_1}, \qquad B_{21}(a) = -\frac{b_{(\varepsilon^2\mu)1}(a)}{2\omega_1 a}.$$

Appendix C

Equation (2.29) is transformed into the form

(C.1) $$A_{30}\varepsilon^2 + 2A'_{21}\varepsilon\mu + A_{12}\mu^2 + 2A'_{20}\varepsilon + 2A'_{11}\mu + A_{10} = 0,$$

where

(C.2) $$A'_{21} = \frac{1}{2}A_{21}, \qquad A'_{20} = \frac{1}{2}A_{20}, \qquad A'_{11} = \frac{1}{2}A_{11}.$$

Equation (C.1) presents implicit second-order algebraic functions if A_{30}, A'_{21} and A_{12} are not equal to zero at the same time. The form of the function is determined by the following expressions:

(C.3) $$W = \det\begin{pmatrix} A_{30} & A'_{21} & A'_{20} \\ A'_{21} & A_{12} & A'_{11} \\ A'_{20} & A'_{11} & A_{10} \end{pmatrix}, \qquad V = \det\begin{pmatrix} A_{30} & A'_{21} \\ A'_{21} & A_{12} \end{pmatrix}, \qquad S = A_{30} + A_{12},$$
$$W_{22} = A_{30}A_{10} - (A'_{20})^2, \qquad W_{11} = A_{12}A_{10} - (A'_{11})^2.$$

By means of shifting the origin of the coordinate system and turning the axis, it is possible to obtain the following functional forms (expressions W, V, S are the invariants of such shifts and turns):

1. $V > 0$, $AW < 0$. Curve (C.1) is the ellipse $\varepsilon^2/A^2 + \mu^2/B^2 = 1$.

2. $V > 0$, $W = 0$. Equation (C.1) can be transformed to $\varepsilon^2/A^2 + \mu^2/B^2 = 0$ and the solution is point (0,0);

3. $V > 0$, $AW > 0$. Curve (C.1) is an imaginary ellipse (no real curve exists).

4. $V < 0$, $W \neq 0$. Equation (C.1) is the equilateral hyperbola $\varepsilon^2/A^2 - \mu^2/B^2 = 1$.

5. $V < 0$, $W = 0$. The solution of Eq. (C.1) is a pair intersecting lines $\varepsilon^2/A^2 - \mu^2/B^2 = 0$.

6. $V = 0$, $W \neq 0$. The curve governed by Eq. (C.1) is a parabola $\mu^2 = 2p\varepsilon$.

7. $V = 0$, $W = 0$, $W_{11} < 0$ or $W_{22} > 0$. Equation (C.1) presents a pair of parallel lines $\mu^2 - A^2 = 0$.

8. $V = 0$, $W = 0$, $W_{11} > 0$ or $W_{22} > 0$. The solution of Eq. (C.1) are imaginary parallel lines $\mu^2 + A^2 = 0$ (no real curve exists).

9. $V = 0$, $W = 0$, $W_{11} = 0$ or $W_{22} = 0$. The solution of Eq. (C.1) is a double line $\mu^2 = 0$.

The coefficients of Eq. (C.1) are functions of the amplitude a and their values are determined by the functions $f_{(\cdot)}$.

Appendix D

(D.1) ε:

$$\nu^2 \frac{\partial^2}{\partial \psi^2} + \rho^4 \frac{\partial^4 W_{10}}{\partial \xi^4} - 2\nu B_{10} a \cos\psi \sin\pi\xi - 2\nu A_{10} \sin\psi \sin\pi\xi =$$
$$= -a\nu \sin\pi\xi \sin\psi + \nu a^3 \sin^3\pi\xi \sin\psi \cos^2\psi - aA \sin\pi\bar{\xi} \sin\psi \cos\psi,$$

$$\nu^2 \frac{\partial^2 \eta_{10}}{\partial \psi^2} + D\eta_{10} = aF \sin\pi\bar{\xi} \cos\psi;$$

(D.2) ε^2:

$$\nu^2 \frac{\partial^2 W_{20}}{\partial \psi^2} + \rho^4 \frac{\partial^4 W_{20}}{\partial \xi^4} - 2\nu B_{20} a \cos\psi \sin\pi\xi - 2\nu A_{20} \sin\psi \sin\pi\xi =$$
$$= -2\nu B_{10} \frac{\partial^2 W_{10}}{\partial \psi^2} - 2\nu A_{10} \frac{\partial^2 W_{10}}{\partial a \partial \psi} - \left(A_{10} \frac{dA_{10}}{da} - B_{10}^2 a\right) \cos\psi \sin\pi\xi +$$
$$+ \left(2A_{10}B_{10} + a \frac{dB_{10}}{da} A_{10}\right) \sin\psi \sin\pi\xi + A_{10} \cos\psi \sin\pi\xi -$$
$$-aB_{10} \sin\psi \sin\pi\xi + \nu \frac{\partial W_{10}}{\partial \psi} - A_{10} a^2 \cos^2\psi \sin^2\pi\xi +$$
$$+2a^2 W_{10} \nu \sin\psi \cos\psi \sin^2\pi\xi + a^3 B_{10} \sin\psi \cos^2\psi \sin^3\pi\xi -$$
$$-\frac{\partial W_{10}}{\partial \psi} \nu a^2 \cos^2\psi \sin^2\pi\xi - AW_{10} + B\delta(\xi - \bar{\xi})\eta_{10},$$

$$\nu^2 \frac{\partial^2 \eta_{20}}{\partial \psi^2} + D\eta_{20} = -2\nu A_{10} \frac{\partial^2 \eta_{10}}{\partial a \partial \psi} - 2\nu B_{10} \frac{\partial^2 \eta_{10}}{\partial \psi^2} - C \frac{\partial \eta_{10}}{\partial \psi} + FW_{10};$$

(D.3) ε^3:

$$\nu^2 \frac{\partial^2 W_{30}}{\partial \psi^2} + \rho^4 \frac{\partial^4 W_{30}}{\partial \xi^4} - 2\nu B_{30} a \cos\psi \sin\pi\xi - 2\nu A_{30} \sin\psi \sin\pi\xi =$$
$$= -\left(\frac{dA_{20}}{da} A_{10} + \frac{dA_{10}}{da} A_{20} - 2aB_{10}B_{20}\right) \cos\psi \sin\pi\xi +$$
$$+ \left\{2A_{20}B_{10} + 2A_{10}B_{20} + a\left(\frac{dB_{10}}{da} A_{20} + \frac{dB_{20}}{da} A_{10}\right)\right\} \sin\psi \sin\pi\xi -$$

(D.3) [cont.]

$$- A_{10}^2\frac{\partial W_{10}}{\partial a^2} - 2\frac{\partial^2 W_{10}}{\partial a\partial\psi}(\nu A_{20} + A_{10}B_{10}) - A_{10}\frac{\partial W_{10}}{\partial a}\frac{dA_{10}}{da} -$$
$$-\frac{\partial^2 W_{10}}{\partial\psi^2}(B_{10}^2 + 2\nu B_{20}) - A_{10}\frac{\partial W_{10}}{\partial\psi}\frac{dB_{10}}{da} - 2\nu A_{10}\frac{\partial^2 W_{20}}{\partial a\partial\psi} -$$
$$-2\nu B_{10}\frac{\partial W_{20}^2}{\partial\psi^2} + A_{20}\cos\psi\sin\pi\xi - aB_{20}\sin\psi\sin\pi\xi +$$
$$+A_{10}\frac{\partial W_{10}}{\partial a} + B_{10}\frac{\partial W_{10}}{\partial\psi} + \nu\frac{\partial W_{20}}{\partial\psi} - 2aA_{10}W_{10}\cos^2\psi\sin^2\pi\xi -$$
$$-a^2A_{20}\cos^3\psi\sin^3\pi\xi + W_{10}^2\nu a\sin\psi\sin\pi\xi +$$
$$+2\nu a^2W_{20}\sin\psi\cos\psi\sin^2\pi\xi + 2a^2B_{10}W_{10}\sin\psi\cos\psi\sin^2\pi\xi +$$
$$+a^3B_{20}\sin\psi\cos^2\psi\sin^3\pi\xi - a^2A_{10}\frac{\partial aW_{10}}{da}\cos^2\psi\sin^2\pi\xi -$$
$$-2\nu aW_{10}\frac{\partial W_{10}}{\partial\psi}\cos\psi\sin\pi\xi - a^2B_{10}\frac{\partial W_{10}}{\partial\psi}\cos^2\psi\sin^2\pi\xi -$$
$$-a^2\nu\frac{\partial W_{20}}{\partial\psi}\cos^2\psi\sin^2\pi\xi - AW_{20} + B\eta_{20}\delta(\xi - \overline{\xi}),$$

$$\nu^2\frac{\partial^2\eta_{30}}{\partial\psi^2} + D\eta_{30} = -2A_{10}^2\frac{\partial^2\eta_{10}}{\partial a^2} - 2\frac{\partial^2\eta_{10}}{\partial a\partial\psi}(\nu A_{20} + A_{10}B_{10}) -$$
$$-A_{10}\frac{dA_{10}}{da}\frac{\partial\eta_{10}}{\partial a} - \frac{\partial^2\eta_{10}}{\partial\psi^2}(2\nu B_{20} + B_{10}^2) -$$
$$-A_{10}\frac{dB_{10}}{da}\frac{\partial\eta_{10}}{\partial\psi} - 2\nu A_{10}\frac{\partial^2\eta_{20}}{\partial a\partial\psi} - 2\nu B_{10}\frac{\partial^2\eta_{20}}{\partial\psi^2} -$$
$$-CA_{10}\frac{\partial\eta_{10}}{\partial a} - CB_{10}\frac{\partial\eta_{10}}{\partial\psi} - C\nu\frac{\partial\eta_{20}}{\partial\psi} + FW_{20};$$

(D.4) $\varepsilon^2\mu$:

$$\nu^2\frac{\partial^2 W_{21}}{\partial\psi^2 W_{21}} + \rho^4\frac{\partial^4 W_{30}}{\partial\xi^4} - 2\nu B_{21}a\cos\psi\sin\pi\xi - 2\nu A_{21}\sin\psi\sin\pi\xi =$$
$$= A_{11}\cos\psi\sin\pi\xi - aB_{11}\sin\psi\sin\pi\xi + \nu\frac{\partial W_{11}}{\partial\psi} -$$
$$-a^2A_{11}\cos^3\psi\sin^3\pi\xi + 2\nu a^2W_{11}\sin\psi\cos\psi\sin^2\pi\xi +$$
$$+ a^3B_{11}\sin\psi\cos^2\psi\sin^3\pi\xi - \nu a^2\frac{\partial W_{11}}{\partial\psi}\cos^2\psi\sin^2\pi\xi -$$
$$-AW_{11} + B\eta_{11}\delta(\xi - \overline{\xi}) - B\nu\delta(\xi - \overline{\xi})\frac{\partial\eta_{10}}{\partial\psi},$$

$$\nu^2\frac{\partial^2\eta_{21}}{\partial\psi^2} + D\eta_{21} = -C\nu\frac{\partial\eta_{11}}{\partial\psi} + FW_{11} - FA_{10}\cos\psi\sin\pi\xi +$$
$$+aFB_{10}\sin\psi\sin\pi\xi - F\nu\frac{\partial W_{10}}{\partial\psi};$$

$$\text{(D.5)} \qquad \varepsilon\mu: \qquad \nu^2 \frac{\partial^2 W_{11}}{\partial \psi^2} + \rho^4 \frac{\partial^4 W_{11}}{\partial \xi^4} = 0,$$

$$\nu^2 \frac{\partial^2 \eta_{11}}{\partial \psi^2} + D\eta_{11} = a\nu F \sin\psi \sin\pi\overline{\xi}.$$

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ŁÓDŹ UNIVERSITY OF TECHNOLOGY, ŁÓDŹ

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