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ANALYSIS OF SELF-EXCITED VIBRATION DUE TO DRY FRICTION IN A SYSTEM WITH TWO DEGREES OF FREEDOM

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The paper is focused on the original approximate method for solving the equations of motion in the synchronization conditions, where the amplitudes are constant in time and the ratio of vibration frequencies is a rational number. The analyzed equations govern the dynamics of self-excited two degrees of freedom mechanical system with friction.

1. Introduction

The aim of this work is to present an analytical approximate method to formulate the averaged equations, which govern the dynamics of the two degrees of freedom system. There is a very wide literature connected with the averaging methods. Our attention is focused on the developing a method for an analysis of multifrequency vibration and the paper is based on such references as [1-5]. This approach consists of the following steps. First, an example of self-excited mechanical system, where friction is responsible for the occurrence of vibrations, is introduced. Then, after applying the normal coordinates the method is demonstrated, which leads to the averaged equations. Stationary and unstationary solutions are discussed.

2. Model System and Equations of Motion

Fig. 1 presents the diagram of the analyzed system. At certain parameter values of the system, it is possible to abserve undying self–excited vibrations due to friction between a body with the mass m_2 and the belt.

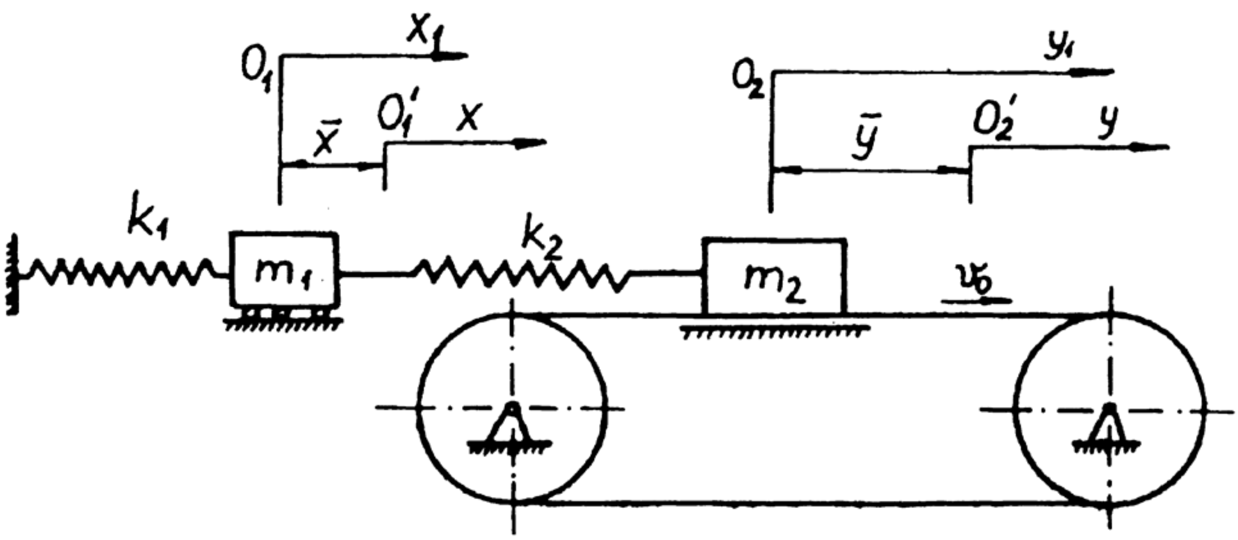

Fig. 1. The analyzed self–excited system with two degrees of freedom

The equations of motion have the form:

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2y_1 = 0,$$

$$m_2\ddot{y}_1 + k_2y_1 - k_2x_1 = m_2g(\mu_o \mathrm{sgn} w - \alpha w + \beta w^3), \quad (1)$$

where: $w = v_o - \dot{y}_1$ – relative velocity of bodies under friction;
μ_o, α, β – coefficients used to circumscribe the friction coefficient by means of polynomials;
m_1, m_2 – masses of rigid bodies;
k_1, k_2 – elastic constants of flexible elements.

3. Approximate Method

The equation system (1) can be presented in the matrix notation:

$$M\ddot{q} + Kq = Q(\dot{q}) \tag{2}$$

where: M – matrix of the system inertia,
K – matrix of the system rigidity,
q – vector of the generalized coordinates,
$Q(\dot{q})$ – nonlinear exciting force.

The relation between the generalized coordinates q and the normal coordinates ξ is expressed by the relation

$$q = \Psi\,\xi, \tag{3}$$

where: – matrix of eigenvectors.

Having introduced (3) into (4) we get

$$M\Psi\ddot{\xi} + K\Psi\xi = Q(\Psi\xi, \Psi\dot{\xi}). \tag{4}$$

Both sides of (4) are multiplied on the left side by the matrix Ψ^T, and we get:

$$\Psi^T M\Psi\ddot{\xi} + \Psi^T K\Psi\xi = \Psi^T Q(\Psi\xi, \Psi\dot{\xi}). \tag{5}$$

Let us denote

$$P = \Psi^T M\Psi, \qquad U = \Psi^T K\Psi = P\Lambda. \tag{6}$$

Accounting for (6) in (5), we get

$$P\ddot{\xi} + P\Lambda\xi = \Psi^T Q(\Psi\xi, \Psi\dot{\xi}) \tag{7}$$

From(7) we get:

$$\ddot{\xi} + \Lambda_1\xi = \Psi^T Q(\Psi\xi, \Psi\dot{\xi})P^{-1} \tag{8}$$

where:

$$\Lambda_1 = P\Lambda P^{-1} = \mathrm{diag}[\omega_1^2, \dots \omega_n^2] \tag{9}$$

$$Q(\dot{q}) = \begin{bmatrix} 0 \\ f(\dot{y}) \end{bmatrix}, \qquad q = \begin{bmatrix} x \\ y \end{bmatrix},$$

where:

$$f(\dot{y}) = m_2 g\,[\mu_o \mathrm{sgn}(v_o - \dot{y}) - \alpha(v_o - \dot{y}) + \beta(v_o - \dot{y})^3]. \tag{10}$$

We have also assumed

$$\Psi = \begin{bmatrix} \Psi_1^{(1)} & \Psi_1^{(2)} \\ \Psi_2^{(1)} & \Psi_2^{(2)} \end{bmatrix}. \tag{11}$$

On account of the asymmetry of the nonlinear characteristic, the solutions are sought in the form:

$$\begin{aligned} \xi_1 &= \xi_{10} + \xi_1^* = \xi_{10} - A_1(t)\cos\tau_1, \\ \xi_2 &= \xi_{20} + \xi_2^* = \xi_{20} - A_2(t)\cos\tau_2, \end{aligned} \tag{12}$$

while:

$$\begin{aligned} \tau_1 &= \omega_1 t + \Psi_1(t), \\ \tau_2 &= \omega_2 t + \Psi_2(t). \end{aligned} \tag{13}$$

The left sides of (9), after introducing (12) and omitting small quantities of the second order, are

$$\xi_i + \omega_i^2 \xi_i = 2\dot{A}(t)\omega_i \sin\tau_i + 2A_i(t)\omega_i \dot{\Psi}_i \cos\tau_i + + \omega_i^2 \xi_{io}, \qquad i = 1, 2 \tag{14}$$

After accounting for (10), (11), (12) in (3) we obtain

$$\begin{aligned} x &= \Psi_1^{(1)}\xi_{10} + \Psi_1^{(2)}\xi_{20} + \Psi_1^{(1)}\xi_1^* + \Psi_1^{(2)}\xi_2^*, \\ y &= \Psi_2^{(1)}\xi_{10} + \Psi_2^{(2)}\xi_{20} + \Psi_2^{(1)}\xi_1^* + \Psi_2^{(2)}\xi_2^* \\ \dot{y} &= \Psi_2^{(1)}A_1(t)\dot{\tau}_1\sin\tau_1 - \Psi_2^{(1)}\dot{A}_1(t)\cos\tau_1 + \\ &\quad - \Psi_2^{(2)}A_2(t)\dot{\tau}_2\sin\tau_2 + \Psi_2^{(2)}\dot{A}_2(t)\cos\tau_2. \end{aligned} \tag{15}$$

After such a procedure we have found equations (8) which are coupled only because of the nonlinear terms. The left sides of these equations are substituted by (15) and contain $\acute{A}_i(t)$ and $\Psi_i(t)$ (i = 1,2), i.e. the slow change of the amplitudes and phases. The nonlinear function f depends on two independent variables τ_1 and τ_2 and it is periodic because of each variable. Thus, it can be expanded into a double Fourier series:

$$f = f_{oo} + \sum_{l=1}^{\infty} f^{s}_{ol}\sin l\tau_2 + \sum_{k=1} f^{s}_{ko}\sin k\tau_1 + \sum_{l=1}^{\infty} f^{c}_{ol}\cos l\tau_2 +$$

$$+ \sum_{k=1}^{\infty} f^{c}_{ko}\cos k\tau_1 + \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} (f^{cc}_{kl}\cos k\tau_1\cos l\tau_2 + f^{cs}\cos k\tau_1\sin l\tau_2 +$$

$$+ f^{sc}_{kl}\sin k\tau_1\cos l\tau_2 + f^{ss}_{kl}\sin k\tau_1\sin l\tau_2) \tag{16}$$

where:

$$f_{oo} = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} f\, d\tau_1\, d\tau_2,$$

$$f^{s}_{ol} = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} f\sin l\tau_2\, d\tau_1\, d\tau_2, \quad f^{s}_{ko} = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} f\sin k\tau_1\, d\tau_1\, d\tau_2$$

$$f^{c}_{ol} = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} f\cos l\tau_2 d\tau_1 d\tau_2, \quad f^{c}_{ko} = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} f\cos k\tau_1 d\tau_1 d\tau_2,$$

$$f^{cc}_{kl} = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi} f\cos k\tau_1\cos l\tau_2 d\tau_1 d\tau_2, \tag{17}$$

$$f^{cs}_{kl} = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi} f\cos k\tau_1\sin l\tau_2 d\tau_1 d\tau_2,$$

$$f_{kl}^{ss} = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi} f\sin k\tau_1 \sin l\tau_2 d\tau_1 d\tau_2,$$

$$f_{kl}^{sc} = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi} f\sin k\tau_1 \cos l\tau_2 d\tau_1 d\tau_2.$$

Let us assume that between the frequencies the following relationship exists

$$\omega_2 = \frac{p}{q}\omega_1 + \Omega. \tag{18}$$

where: p, q — the smallest natural numbers describing the frequency relations,
Ω — frequency deviation.

Such an assumption indicates that for $\Omega \cong 0$, ω_2 and ω_1 are in the (p,q) resonance.

Let the following be true

$$\tau_2 = \frac{p}{q}\tau_1 + \varphi(t), \tag{19}$$

where: $\varphi(t)$ — generalized phase difference.

After accounting for (13) in (19) we get

$$\varphi(t) = \omega_2 t + \Psi_2(t) - \frac{p}{q}[\omega_1 t + \Psi_1(t)]. \tag{20}$$

When differentiating (20) against time and accounting for (18) we obtain

$$\dot{\varphi} = \Omega + \dot{\Psi}_2 - \frac{p}{q}\dot{\Psi}_1. \tag{21}$$

After taking into account the formulae in (17) given below

$$2\cos k\tau_1 \cos k\tau_2 = \cos(k\tau_1 + l\tau_2) + \cos(k\tau_1 - l\tau_2),$$

$$2\cos k\tau_1 \sin l\tau_2 = \sin(k\tau_1 + l\tau_2) + \sin(l\tau_2 - k\tau_1), \tag{22}$$

$$2\sin k\tau_1 \cos l\tau_2 = \sin(k\tau_1 + l\tau_2) + \sin(k\tau_1 - l\tau_2),$$

$$2\sin k\tau_1 \sin l\tau_2 = \cos(k\tau_1 - l\tau_2) - \cos(k\tau_1 + l\tau_2),$$

we have

$$f = f_{oo} + \sum_{l=1}^{\infty} f_{ol}\sin l\tau_2 + \sum_{k=1}^{\infty} f^{s}_{ko}\sin\tau_1 + \sum_{l=1}^{\infty} f^{c}_{ko}\cos k\tau_1 +$$

$$+ \frac{1}{2}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\Big\{ f^{cc}_{kl}[\cos(k\tau_1 + l\tau_2) + \cos(k\tau_1 - l\tau_2)] +$$

$$+ f^{cs}_{kl}[\sin(k\tau_1 + l\tau_2) + \sin(l\tau_2 - k\tau_1)] + f^{sc}_{kl}[\sin(k\tau_1 + l\tau_2) +$$

$$+ \sin(k\tau_1 - l\tau_2)] + f^{ss}_{kl}[\cos(k\tau_1 - l\tau_2) - \cos(k\tau_1 + l\tau_2)]\Big\}. \tag{23}$$

Having taken (19) into consideration and assumed $n = \frac{k}{p} = \frac{l}{q}$, and introducing all this into (23), we take the terms with τ_1 from the possible combinations, and we have

$$f\tau_1 = f^{s}_{10}\sin\tau_1 + f^{c}_{10}\cos\tau_1 + \frac{1}{2}\sum_{n}\Big\{ f^{cc}_{1-np,nq}\cos(\tau_1 + nq\varphi) +$$

$$+ f^{cc}_{1+np,nq}\cos(\tau_1 - nq\varphi) + f^{cs}_{1-np,nq}\sin(\tau_1 + nq\varphi) +$$

$$+ f^{cs}_{1+np,nq}\sin(\tau_1 + nq\varphi) + f^{sc}_{1-np,nq}\sin(\tau_1 + nq\varphi) +$$

$$+ f^{sc}_{np,nq}\sin(\tau_1 - nq\varphi) + f^{ss}_{1+np,nq}\cos(\tau_1 - nq\varphi) +$$

$$- f^{ss}_{1-np,nq}\cos(\tau_1 + nq\varphi)\Big\}. \tag{24}$$

The expression (24), after transformation, will have the form

$$f\tau_1 = f^{s}\tau_1\sin\tau_1 + f^{c}\tau_1\cos\tau_1, \tag{25}$$

where:

$$f^{s}\tau_1 = f^{s}_{10} + \frac{1}{2}\sum_{n}\Big\{ -f^{cc}_{1-np,nq}\sin nq\varphi + f^{cc}_{1+np,nq}\sin nq\varphi + \tag{26}$$

$$+ f^{cs}_{1-np,nq}\cos nq\varphi + f^{cs}_{1+np,nq}\cos nq\varphi + f^{sc}_{1-np,nq}\cos nq\varphi +$$

$$+ f^{sc}_{1+np,nq}\cos nq\varphi + f^{ss}_{1+np,nq}\sin nq\varphi + f^{ss}_{1-np,nq}\sin nq\varphi\},$$

$$f^{c}\tau_1 = f^{c}_{10} + \frac{1}{2}\sum_n\Big\{f^{cc}_{1-np,nq}\cos nq\varphi + f^{cc}_{1+np,nq}\cos nq\varphi + \tag{27}$$

$$+ f^{cs}_{1-np,nq}\sin nq\varphi + f^{sc}_{1-np,nq}\sin nq\varphi +$$

$$- f_{1+np,nq}\sin nq\varphi + f^{ss}_{1+np,nq}\cos nq\varphi - f^{ss}_{1-np,nq}\cos nq\varphi\Big\}.$$

From(9) we get

$$\tau_1 = \frac{q}{p}\tau_2 - \frac{q}{p}\varphi. \tag{28}$$

Analogously to the procedure with (23), after accounting for (28), the terms with τ_2 are separated, and the following is obtained:

$$f\tau_2 = f^{s}_{01}\sin\tau_2 + f^{c}_{01}\cos\tau_2 + \frac{1}{2}\sum_n\Big\{f^{cc}_{np,1-nq}\cos(\tau_2 - nq\varphi) +$$

$$+ f^{cc}_{np,nq-1}\cos(\tau_2 - nq\varphi) + f^{cs}_{np,1-nq}\sin(\tau_2 - nq\varphi) + f^{cs}_{np,1+nq}\sin(\tau_2 + nq\varphi)$$

$$+ f^{sc}_{np,1-nq}\sin(\tau_2 - nq\varphi) + f^{sc}_{np,nq-1}\sin(\tau_2 - nq\varphi) +$$

$$+ f^{ss}_{np,nq-1}\cos(\tau_2 - nq\varphi) - f^{ss}_{np,1-nq}\cos(\tau_2 - nq\varphi)\Big\}. \tag{29}$$

The expression (29), after transformation, will adopt the form:

$$f\tau_2 = f^{s}\tau_2\sin\tau_2 + f^{c}\tau_2\cos\tau_2 \tag{30}$$

where:

$$f^{s}\tau_2 = f^{s}_{o1} + \frac{1}{2}\sum_n\Big\{f^{cc}_{np,1-nq}\sin nq\varphi + f^{cc}_{np,nq-1}\sin nq\varphi +$$

$$+ f^{cs}_{np,1-nq}\cos nq\varphi + f^{cs}_{np,1+nq}\cos nq\varphi + f^{sc}_{np,1-nq}\cos nq\varphi +$$

$$+ f^{sc}_{np,nq-1}\cos nq\varphi + f^{ss}_{np,nq-1}\sin nq\varphi - f^{ss}_{np,1-nq}\sin nq\varphi\Big\}, \tag{31}$$

$$f^c\tau_2 = f^c_{o1} + \frac{1}{2}\sum_n\Big\{f^{cc}_{np,1-nq}\cos nq\varphi + f^{cc}_{np,nq-1}\cos nq\varphi +$$

$$- f^{cs}_{np,1-nq}\sin nq\varphi + f^{cs}_{np,1+nq}\sin nq\varphi - f^{sc}_{np,1-nq}\sin nq\varphi +$$

$$- f^{sc}_{np,nq-1}\sin nq\varphi + f^{ss}_{np,nq-1}\cos nq\varphi - f^{ss}_{np,1-nq}\cos nq\varphi\Big\}. \tag{32}$$

Roughly speaking, the connections applied above have supplied all terms with $\sin\tau_i$ and $\cos\tau_i$ ($i = 1, 2$) which of course have important meaning for the results obtained finally.
Having equated the right sides of (8) and (14), and accounted for the terms found at $\sin\tau_1$, $\cos\tau_1$ and the absolute term, we obtain the following differential equations system:

$$\omega_i^2\xi_{io} = p_i\Psi_2^{(i)}f_{oo},$$

$$2\dot{A}_i(t)\omega_i = p_i\Psi_2^{(i)}f^s\tau_i, \tag{33}$$

$$2A_i(t)\omega_i\dot{\varphi}_i(t) = p_i\Psi_2^{(i)}f^c\tau_i, \quad i = 1, 2.$$

Theoretically, from the second equations of (33) we can obtain A_i and then from the third one, $\varphi_i(t)$.
However, the analytical solutions of such equations exist only in very rare cases.
In the case when the steady state is analysed, ie. after assuming $\dot{A}_i(t) = \dot{\varphi}(t) = 0$, the equations are simplified and after accounting for (21) the following is obtained:

$$\omega_i^2\xi_{io} = p_i\Psi_2^{(i)}f_{oo}, \tag{34}$$

$$f^s\tau_i = 0,$$

$$\Omega + \frac{p_2\Psi_2^{(2)}f^c\tau_2}{2A_2\omega_2} - \frac{p}{q}\frac{p_1\Psi_2^{(1)}f^c\tau_1}{2A_1\omega_1} = 0.$$

After calculating the derivative of (20) and accounting for $\dot{\varphi} = 0$, we obtain the following:

$$\omega_2 + \dot{\Psi}_2 = \frac{p}{q}(\omega_1 + \dot{\Psi}_1). \tag{35}$$

Assuming that $\omega_1' = \dot{\Psi}_1 + \omega_1, \omega_2' = \dot{\Psi}_2 + \omega_2$ and taking this into account in (35), we have

$$\omega_2' = \frac{p}{q}\omega_1'. \tag{36}$$

The equations (34) govern the synchronization state, where the viabration amplitudes are constant in time, and the derivatives of $\dot{\Psi}_i$ do not change in time, either, and they are such corrections of the free vibrations frequency of the conservative system that the relationship (36) is fulfilled, i.e. the resonance of the (p,q) order appears. The frequency corrections are determined from the relations

$$2A_1\omega_i\dot{\Psi}_i = p_i\Psi_2^{(i)}f^r\tau_i,\ i = 1, 2. \tag{37}$$

The motion obtained is a periodic one. The non−zero expressions $f^r\tau_i$ are obtained as a result of averaging the noncontinuous function sgn $\beta v_o - \Psi_2^{(1)}\omega_1A_1\sin\tau_1 - \Psi_2^{(2)}\omega_2A_2\sin\tau_2/$ from the formulae of (17), when

$$v_o < \Psi_2^{(1)}A_1\omega_1\sin\tau_1 + \Psi_2^{(2)}A_2\omega_2\tau_2. \tag{38}$$

In the asynchronic state, when the corrections of $\dot{\Psi}_i = 0$ (for i = 1, 2), the vibration frequencies of the weakly nonlinear system overlap the free vibration frequencies of the conservative system.

When solving the algebraic nonlinear equations system (34), we find the sought values of amplitudes A_i, the values of aperiodic deflections ξ_{io}, and the corrections concerning the free vibrations frequency in steady state in synchronic state.

3. Conclusion

This paper presents the analytical approximate approach for finding the periodic and quasiperiodic solutions of the analyzed system. The considerations are limited only to the first harmonics of τ_1 and τ_2 and a special Fourier double series is used to find all these harmonics. Equations leading to the stationary and nonstationary states are formulated and some implications for the self–excited system with two degrees of freedom are discussed.

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Analiza drgań samowzbudnych wywołanych tarciem w układzie o dwóch stopniach swobody

Streszczenie

Celem pracy jest prezentacja analitycznej metody przybliżonej prowadzącej do sformułowania uśrednionych równań różniczkowych nieliniowych pierwszego rzędu. Metodyka postępowania została przedstawiona na przykładzie drgań samowzbudnych wywołanych tarciem w układzie o dwóch stopniach swobody.

Rozpatrzono dwa przypadki, gdy częstości drgań własnych ω_1 i ω_2 są w rezonansie rzędu (p,q) oraz gdy drgania odbywają się z dala od rezonansu. Najpierw wprowadzone zostały współrzędne normalne, dzięki którym lewe strony równań różniczkowych opisujących dynamikę układu zostały rozprzęgnięte. Następnie użycie podwójnego szeregu Fouriera i związków trygonometrycznych (22) pozwoliło na znalezienie wszystkich wyrazów z pierwszymi harmonicznymi $\sin\tau_i$, $\cos\tau_i$ (i = 1, 2). Metoda pozwala na analizę stanów stacjonarnych i niestacjonarnych w analizowanym układzie.

W oparciu o zmodyfikowaną przybliżoną metodę analityczną wyznaczono amplitudy drgań A_i, wartości uchybów aperiodycznych ξ_{io} oraz poprawki na częstość drgań w reżimie synchronicznym. W tym ostatnim przypadku prawdziwy jest związek $\omega'_2 = \frac{p}{q}\omega'_1$, gdzie p i q są liczbami całkowitymi, oraz $\omega'_1 = \omega_1 + \dot{\Psi}_1$, $\omega'_2 = \omega_2 + \dot{\Psi}_2$ ($\dot{\Psi}_1$ i $\dot{\Psi}_2$ są wspomnianymi poprawkami).

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