Journal of Sound and Vibration (1993) **160**(3), 566–573

PERIODIC OSCILLATIONS AND TWO-PARAMETER UNFOLDINGS IN NON-LINEAR DISCRETE–CONTINUOUS SYSTEMS WITH DELAY

J. Awrejcewicz†

Research Center for Advanced Science and Technology, The University of Tokyo, 4-6-1 Komaba, Meguro-ku, Tokyo 153, Japan

AND

T. Someya

Musashi Institute of Technology, Department of Mechanical Engineering, 1-28-1 Tamazutsumi, Setagaya-ku, Tokyo 158, Japan

(*Received* 20 *March* 1992)

1. INTRODUCTION

In this letter we discuss the importance of understanding two parameter unfoldings in systems governed by non-linear partial and ordinary equations. The approach is a continuation of some earlier works, in which the two-variable asymptotic expansion technique has been used to analyze periodic oscillations in non-linear parametrically excited mechanical systems [1–3], and bifurcated oscillations [4] as well as oscillations in discrete–continuous systems. The approach used here has been developed from that of reference [5], in which the periodic oscillations were sought in the form of power series of two independent perturbation parameters. The recurrent set of linear differential equations obtained by means of comparing the expressions found at the same powers of two perturbation parameters were then solved by using the harmonic balance method. The technique presented is a generalization of classical asymptotic methods, which have been widely treated in the literature, to the analysis of two-parameter unfoldings of discrete–continuous mechanical systems governed by partial and ordinary non-linear equations with two independent parameters.

The oscillations of non-linear continuous mechanical systems monitored by control (discrete) systems with time delay can be governed by the system of equations under our consideration. Automatic control of furnace heating or some devices of inertial navigation such as string generators and accelerometers can serve as examples of such mixed devices. The most characteristic feature of the above mentioned systems is that the control unit can influence the control system in particular points and the influence is performed with time delay.

2. METHOD

We analyze the system

$$\frac{\partial^2 v(t, x)}{\partial t^2} = L_x^{(2m)}\{v(t, x)\} + \varepsilon f\{x, v(t, x), y(t-\mu)\},$$

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \sum_{p=0}^{P} A_p y(t-\tau_p) + \varepsilon F\{y(t-\mu), v(t-\mu), \xi\}, \qquad (1)$$

† On leave from: Technical University, Division of Dynamics and Control (K-13), B. Stefanowskiego 1/15, 90-924 Lodz, Poland.

0022-460X/93/030566+08 $08.00/0

where $v(t, x)$ satisfies the homogeneous boundary conditions

$$L_x^{(h,j)}\{v(t, x)\}|_{x\in S}=0, \qquad h=1, \ldots, m. \tag{2}$$

(The problem including non-homogeneous boundary conditions can be reduced to one of homogeneous boundary conditions.) The co-ordinate t denotes time and $t\in R$; x is the vector of the co-ordinates and $x\in(G\cup S)$, where S is the limiting set of G; $v(t, x)$ is a certain scalar function determined in the set $R\times G$ and $L_x^{(h,j)}$ are linear operators of order $j\leqslant 2m-1$; $L_x^{(2m)}$ is the linear differential operator of order $2m$ on x; y and F are vectors of an m-dimensional space; A_p are constant matrices of $(m\times m)$ order; f, g_{hj} and components of F are functions of $y(t-\mu)$, $u(t-\mu, \xi)$, $\xi\in(G\cup S)$, while τ_p and μ are time delays. Finally, we assume that ε and μ are small positive parameters. (A similar system was considered in reference [6], in which the first order averaging equations were obtained by using a standard perturbation technique.)

For $\varepsilon=0$ we obtain the following characteristic equations:

$$L_x^{(2m)}\{X(x)\}+\sigma X(x)=0, \qquad L_x^{(h,j)}|_{x\in S}=0, \quad h=1, \ldots, m, \tag{3}$$

$$D(\rho)=\det\left\{\sum_{p=0}^{P} A_p\,\mathrm{e}^{-\tau_p\rho}-E\rho\right\}. \tag{4}$$

We shall consider the case when $\rho=\pm\mathrm{i}\omega$ and assume that equations (3) have neither zero nor $n\mathrm{i}\omega$ $(n\in N)$ roots. Additionally, we shall assume that $\sigma_s\neq\{(p/q)\omega\}$, where p and q are integers. (Here and subsequently, E denotes the unit matrix.)

The periodic solution is sought in the form

$$v(t, x)=\sum_{k=1}^{K}\sum_{l=0}^{L}\varepsilon^k\mu^l V_{kl}\{x, a(t), \psi(t)\},$$

$$y(t)=a(t)\{\alpha\,\mathrm{e}^{\mathrm{i}\psi(t)}+\bar{\alpha}\,\mathrm{e}^{-\mathrm{i}\psi(t)}\}+\sum_{k=1}^{K}\sum_{l=0}^{L}\varepsilon^k\mu^l y_{kl}\{a(t), \psi(t)\}, \tag{5}$$

where

$$\frac{\mathrm{d}a}{\mathrm{d}t}=\sum_{k=1}^{K}\sum_{l=0}^{L}\varepsilon^k\mu^l A_{kl}\{a(t)\}, \qquad \frac{\mathrm{d}\psi}{\mathrm{d}t}=\omega+\sum_{k=1}^{K}\sum_{l=0}^{L}\varepsilon^k\mu^l B_{kl}\{a(t)\}. \tag{6}$$

α and $\bar{\alpha}$ are determined from the equations

$$\sum_{p=0}^{P}(A_p\,\mathrm{e}^{-\tau_p\omega\mathrm{i}}-E\omega\mathrm{i})\alpha=0, \qquad \sum_{p=0}^{P}(A_p\,\mathrm{e}^{\tau_p\omega\mathrm{i}}+E\omega\mathrm{i})\bar{\alpha}=0. \tag{7}$$

The eigenvectors β and $\bar{\beta}$ of the adjoint set of equations are obtained from

$$\sum_{p=0}^{P}(A_p^*\,\mathrm{e}^{\tau_p\omega\mathrm{i}}+E\omega\mathrm{i})\beta=0, \qquad \sum_{p=0}^{P}(A_p^*\,\mathrm{e}^{-\tau_p\omega\mathrm{i}}-E\omega\mathrm{i})\bar{\beta}=0, \tag{8}$$

where A_p^* are the matrices conjugate to the A_p matrices. From the first of equations (5) we have

$$\frac{\partial v}{\partial t}=\sum_{k=1}^{K}\sum_{l=0}^{L}\varepsilon^k\mu^l\left\{\frac{\partial V_{kl}}{\partial a}\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)+\frac{\partial V_{kl}}{\partial\psi}\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)\right\},$$

$$\frac{\partial^2 v}{\partial t^2}=\sum_{k=1}^{K}\sum_{l=0}^{L}\varepsilon^k\mu^l\left\{\frac{\partial^2 V_{kl}}{\partial a^2}\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2+2\frac{\partial^2 V_{kl}}{\partial a\,\partial\psi}\frac{\mathrm{d}\psi}{\mathrm{d}t}\frac{\mathrm{d}a}{\mathrm{d}t}+\frac{\partial V_{kl}}{\partial a}\left(\frac{\mathrm{d}^2 a}{\mathrm{d}t^2}\right)\right.$$
$$\left.+\frac{\partial^2 V_{kl}}{\partial a^2}\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)^2+\frac{\partial V_{kl}}{\partial\psi}\left(\frac{\mathrm{d}^2\psi}{\mathrm{d}t^2}\right)\right\}. \tag{9}$$

From the first of equations (1) we obtain

$$\frac{\partial^2 v}{\partial t^2}-L_x^{(2m)}\{v(t,x)\}=\sum_{k=1}^{K}\sum_{l=0}^{L}\varepsilon^k\mu^l\left\{\frac{\partial^2 V_{kl}}{\partial a^2}\frac{\mathrm{d}a^2}{\mathrm{d}t}+2\frac{\partial^2 V_{kl}}{\partial a\,\partial\psi}\frac{\mathrm{d}\psi}{\mathrm{d}t}\frac{\mathrm{d}a}{\mathrm{d}t}\right.$$
$$\left.+\frac{\partial V_{kl}}{\partial a}\left(\frac{\mathrm{d}^2 a}{\mathrm{d}t^2}\right)+\frac{\partial V_{kl}^2}{\partial a^2}\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)^2+\frac{\partial V_{kl}}{\partial\psi}\left(\frac{\mathrm{d}^2\psi}{\mathrm{d}t^2}\right)-L_x^{(2m)}\{V_{kl}\}\right\}, \tag{10}$$

while from the second we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\mathrm{d}a}{\mathrm{d}t}(\alpha\,\mathrm{e}^{\mathrm{i}\psi(t)}+\bar{\alpha}\,\mathrm{e}^{-\mathrm{i}\psi(t)})+\mathrm{i}a(t)\frac{\mathrm{d}\psi}{\mathrm{d}t}(a\,\mathrm{e}^{\mathrm{i}\psi(t)}-\bar{\alpha}\,\mathrm{e}^{-\mathrm{i}\psi(t)})$$
$$+\sum_{k=1}^{K}\sum_{l=0}^{L}\varepsilon^k\mu^l\left\{\frac{\partial y_{kl}}{\partial a}\frac{\mathrm{d}a}{\mathrm{d}t}+\frac{\partial y_{kl}}{\partial\psi}\frac{\mathrm{d}\psi}{\mathrm{d}t}\right\}. \tag{11}$$

From equations (6) it follows that

$$\frac{\mathrm{d}^2 a}{\mathrm{d}t^2}=\varepsilon^2 A_{10}\frac{\mathrm{d}A_{10}}{\mathrm{d}a}+\varepsilon^2\mu\left(\frac{\mathrm{d}A_{10}}{\mathrm{d}a}A_{11}+\frac{\mathrm{d}A_{11}}{\mathrm{d}a}A_{10}\right)$$
$$+\varepsilon^3\left(\frac{\mathrm{d}A_{20}}{\mathrm{d}a}A_{10}+\frac{\mathrm{d}A_{10}}{\mathrm{d}a}A_{20}\right)+O(\varepsilon^k\mu^l;k+l=4), \tag{12}$$

$$\frac{\mathrm{d}a}{\mathrm{d}t}\frac{\mathrm{d}\psi}{\mathrm{d}t}=\varepsilon\omega A_{10}+\varepsilon^2(\omega A_{20}+A_{10}B_{10})+\varepsilon\mu\omega A_{11}+\varepsilon^2\mu(\omega A_{21}+A_{11}B_{10}+A_{10}B_{11})$$
$$+\varepsilon\mu^2 A_{12}\omega+\varepsilon^3(A_{30}\omega+A_{20}B_{10}+A_{10}B_{20}\}+O(\varepsilon^k\mu^l;k+l=4), \tag{13}$$

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}t^2}=\varepsilon^2\frac{\mathrm{d}B_{10}}{\mathrm{d}a}A_{10}+\varepsilon^2\mu\left\{\frac{\mathrm{d}B_{10}}{\mathrm{d}a}A_{11}+\frac{\mathrm{d}B_{11}}{\mathrm{d}a}A_{10}\right\}$$
$$+\varepsilon^3\left\{\frac{\mathrm{d}B_{10}}{\mathrm{d}a}A_{20}+\frac{\mathrm{d}B_{20}}{\mathrm{d}a}A_{10}\right\}+O(\varepsilon^k\mu^l;k+l=4), \tag{14}$$

$$\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2=\varepsilon^2 A_{10}^2+2\varepsilon^2\mu A_{10}A_{11}+2\varepsilon^3 A_{20}A_{10}+O(\varepsilon^k\mu^l;k+l=4), \tag{15}$$

$$\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)^2=\omega^2+2\varepsilon\omega B_{10}+2\varepsilon\mu B_{11}\omega+\varepsilon^2(2\omega B_{20}+B_{10}^2)+2\varepsilon\mu^2 B_{12}\omega$$
$$+2\varepsilon^2\mu(\omega B_{21}+B_{11}B_{10})+2\varepsilon^3(\omega B_{30}+B_{20}B_{10})+O(\varepsilon^k\mu^l;k+l=4). \tag{16}$$

Since y and v can be expressed as power series,

$$y(t-\mu)=\sum_{n=0}^{N}\frac{1}{n!}\frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n}(-\mu)^n,\qquad v(t-\mu,\xi)=\sum_{n=0}^{N}\frac{1}{n!}\frac{\mathrm{d}^n v(t,\xi)}{\mathrm{d}t^n}(-\mu)^n, \tag{17}$$

then the functions εf and εF can be expanded in a power series of small parameters μ and ε. Further calculations were carried out for $n=1$ ($\dot{y}_1=-\mu(\mathrm{d}y/\mathrm{d}t)$ and $\dot{v}_1=-\mu(\mathrm{d}v/\mathrm{d}t)$) and under the assumption that

$$\dot{v}_1=-\mu\sum_{k=1}^{K}\sum_{l=0}^{L}\varepsilon^k\mu^l\left\{\frac{\partial V_{kl}}{\partial a}\frac{\mathrm{d}a}{\mathrm{d}t}+\frac{\partial V_{kl}}{\partial\psi}\frac{\mathrm{d}\psi}{\mathrm{d}t}\right\},$$

$$\dot{y}_1=-\mu\left\{\frac{\mathrm{d}a}{\mathrm{d}t}(\alpha\,\mathrm{e}^{\mathrm{i}\psi(t)}+\bar{\alpha}\,\mathrm{e}^{-\mathrm{i}\psi(t)})-\mu a(t)\left\{\frac{\mathrm{d}\psi}{\mathrm{d}t}\mathrm{i}(\alpha\,\mathrm{e}^{\mathrm{i}\psi(t)}-\bar{\alpha}\,\mathrm{e}^{-\mathrm{i}\psi(t)})\right.\right.$$
$$\left.\left.-\mu\sum_{k=1}^{K}\sum_{l=0}^{L}\varepsilon^k\mu^l\left(\frac{\partial y_{kl}}{\partial a}\frac{\mathrm{d}a}{\mathrm{d}t}+\frac{\partial y_{kl}}{\partial\psi}\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)\right\}\right\}. \quad (18)$$

The necessary derivatives of the functions f and F were calculated at the point $\mu=\varepsilon=0$ and $v_0=0$, $y_0(t)=a(t)\{\alpha\,\mathrm{e}^{\mathrm{i}\psi}+\bar{\alpha}\,\mathrm{e}^{-\mathrm{i}\psi}\}$. The sequence of recurrent linear differential equations obtained is of the following form:

ε:
$$\omega^2\,\partial^2V_{10}(x,a,\psi)/\partial\psi^2=L_x^{(2m)}\{V_{10}\}+f_\varepsilon,$$
$$\omega\frac{\partial y_{10}(a,\psi)}{\partial\psi}=\sum_{p=0}^{P}A_py_{10}(a,\psi-\tau_p\omega)-A_{10}(\alpha\,\mathrm{e}^{\mathrm{i}\psi}+\bar{\alpha}\,\mathrm{e}^{-\mathrm{i}\psi})-\mathrm{i}aB_{10}(\alpha\,\mathrm{e}^{\mathrm{i}\psi}-\bar{\alpha}\,\mathrm{e}^{-\mathrm{i}\psi})$$
$$-\sum_{p=1}^{P}\tau_pA_p\{A_{10}(\alpha\,\mathrm{e}^{\mathrm{i}(\psi-\tau_p\omega)}+\bar{\alpha}\,\mathrm{e}^{-\mathrm{i}(\psi-\tau_p\omega)})$$
$$-\mathrm{i}aB_{10}(\alpha\,\mathrm{e}^{\mathrm{i}(\psi-\tau_p\omega)}-\bar{\alpha}\,\mathrm{e}^{-\mathrm{i}(\psi-\tau_p\omega)})\}+F_\varepsilon; \quad (19)$$

ε^2:
$$\omega^2\,\partial^2V_{20}(x,a,\psi)/\partial\psi^2=L_x^{(2m)}\{V_{20}\}+f_{\varepsilon\varepsilon},$$
$$\omega\frac{\partial y_{20}(a,\psi)}{\partial\psi}=\sum_{p=0}^{P}A_py_{20}(a,\psi-\tau_p\omega)-A_{20}(\alpha\,\mathrm{e}^{\mathrm{i}\psi}+\bar{\alpha}\,\mathrm{e}^{-\mathrm{i}\psi})-\mathrm{i}a\beta_{20}(\alpha\,\mathrm{e}^{\mathrm{i}\psi}-\bar{\alpha}\,\mathrm{e}^{-\mathrm{i}\psi})$$
$$-\sum_{p=1}^{P}A_p\{\tau_pA_{20}(\alpha\,\mathrm{e}^{\mathrm{i}(\psi-\tau_p\omega)}+\bar{\alpha}\,\mathrm{e}^{-\mathrm{i}(\psi-\tau_p\omega)})$$
$$+\mathrm{i}aB_{20}(\alpha\,\mathrm{e}^{\mathrm{i}(\psi-\tau_p\omega)}-\bar{\alpha}\,\mathrm{e}^{-\mathrm{i}(\psi-\tau_p\omega)})\}+F_{\varepsilon\varepsilon}; \quad (20)$$

$\varepsilon\mu$:
$$\omega^2\,\partial^2V_{11}(x,a,\psi)/\partial\psi^2=L_x^{(2m)}\{V_{11}\}+f_{\varepsilon\mu},$$
$$\omega\frac{\partial y_{11}}{\partial\psi}=\sum_{p=0}^{P}A_py_{11}(a,\psi-\tau_p\omega)-A_{11}(\alpha\,\mathrm{e}^{\mathrm{i}\psi}+\bar{\alpha}\,\mathrm{e}^{-\mathrm{i}\psi})-\mathrm{i}aB_{11}(\alpha\,\mathrm{e}^{\mathrm{i}\psi}-\bar{\alpha}\,\mathrm{e}^{-\mathrm{i}\psi})$$
$$-\sum_{p=1}^{P}\tau_pA_p\{A_{11}(\alpha\,\mathrm{e}^{\mathrm{i}(\psi-\tau_p\omega)}+\bar{\alpha}\,\mathrm{e}^{-\mathrm{i}(\psi-\tau_p\omega)})$$
$$-\mathrm{i}aB_{11}(\alpha\,\mathrm{e}^{\mathrm{i}(\psi-\tau_p\omega)}-\bar{\alpha}\,\mathrm{e}^{-\mathrm{i}(\psi-\tau_p\omega)})\}+F_{\varepsilon\mu}; \quad (21)$$

ε^3:
$$\omega^2\,\partial^2V_{30}(x,a,\psi)/\partial\psi^2=L_x^{(2m)}\{V_{30}\}+f_{\varepsilon^3},$$
$$\omega\frac{\partial y_{30}(a,\psi)}{\partial\psi}=\sum_{p=0}^{P}A_py_{30}(a,\psi-\tau_p\omega)-A_{30}(\alpha\,\mathrm{e}^{\mathrm{i}\psi}+\bar{\alpha}\,\mathrm{e}^{-\mathrm{i}\psi})-\mathrm{i}aB_{30}(\alpha\,\mathrm{e}^{\mathrm{i}\psi}-\bar{\alpha}\,\mathrm{e}^{-\mathrm{i}\psi})$$
$$-\sum_{p=1}^{P}\tau_pA_p\{A_{30}(\alpha\,\mathrm{e}^{\mathrm{i}\psi-\tau_p\omega}+\bar{\alpha}\,\mathrm{e}^{-\mathrm{i}(\psi-\tau_p\omega)})$$
$$+\mathrm{i}aB_{12}(\alpha\,\mathrm{e}^{\mathrm{i}(\psi-\tau_p\omega)}-\bar{\alpha}\,\mathrm{e}^{-\mathrm{i}(\psi-\tau_p\omega)})\}+F_{\varepsilon^3}; \quad (22)$$

$\varepsilon\mu^2$: $$\omega^2 \partial^2 V_{12}(x, a, \psi)/\partial\psi^2 = L_x^{(2m)}\{V_{12}\} + f_{\varepsilon\mu^2},$$

$$\omega \frac{\partial y_{12}(a, \psi)}{\partial \psi} = \sum_{p=0}^{P} A_p y_{12}(a, \psi - \tau_p\omega) - A_{12}(\alpha\, e^{i\psi} + \bar{\alpha}\, e^{-i\psi}) - iaB_{12}(\alpha\, e^{i\psi} - \bar{\alpha}\, e^{-i\psi})$$
$$- \sum_{p=1}^{P} \tau_p A_p\{A_{12}(\alpha\, e^{i(\psi-\tau_p\omega)} + \bar{\alpha}\, e^{-i(\psi-\tau_p\omega)})\}$$
$$+ iaB_{12}(\alpha\, e^{i(\psi-\tau_p\omega)} - \bar{\alpha}\, e^{-i(\psi-\tau_p\omega)})\} + F_{\varepsilon\mu^2}; \quad (23)$$

$\varepsilon^2\mu$: $$\omega^2 \partial V_{21}(x, a, \psi)/\partial\psi^2 = L_x^{(2m)}\{V_{21}\} + f_{\varepsilon^2\mu},$$

$$\omega \frac{\partial y_{21}(a, \psi)}{\partial \psi} = \sum_{p=0}^{P} A_p y_{21}(a, \psi - \tau_p\omega) - A_{21}(\alpha\, e^{i\psi} + \bar{\alpha}\, e^{-i\psi}) - iaB_{21}(\alpha\, e^{i\psi} - \bar{\alpha}\, e^{-i\psi})$$
$$- \sum_{p=1}^{P} \tau_p A_p\{A_{21}(\alpha\, e^{i(\psi-\tau_p\omega)} + \bar{\alpha}\, e^{-i(\psi-\tau_p\omega)})$$
$$+ iaB_{12}(\alpha\, e^{i(\psi-\tau_p\omega)} - \bar{\alpha}\, e^{-i(\psi-\tau_p\omega)})\} + F_{\varepsilon^2\mu}. \quad (24)$$

Here

$$f_\varepsilon = f(x, y_0), \qquad F_\varepsilon = F(x, \dot{v}_1, y, \dot{y}_1),$$

$$f_{\varepsilon^2} = -2\omega B_{10}\frac{\partial^2 V_{10}}{\partial\psi^2} - 2\omega A_{10}\frac{\partial^2 V_{10}}{\partial a\,\partial\psi} + \frac{\partial f}{\partial v} V_{10} + \sum_{l=1}^{m}\frac{\partial f}{\partial y_l} y_{(10)l},$$

$$F_{\varepsilon^2} = -\frac{\partial y_{10}}{\partial a}A_{10} - \frac{\partial y_{10}}{\partial\psi}B_{10} - \frac{\partial F}{\partial v}V_{10} + \sum_{l=1}^{m}\frac{\partial F}{\partial y_l}y_{(10)l}, \qquad f_{\varepsilon\mu} = 0,$$

$$F_{\varepsilon\mu} = -\sum_{n=1}^{m}\frac{\partial F}{\partial \dot{y}_{1n}} a\omega i(\alpha\, e^{i\psi} - \bar{\alpha}\, e^{-i\psi}),$$

$$f_{\varepsilon^3} = \frac{\partial^2 f}{\partial v^2}V_{10}^2 + \frac{\partial f}{\partial v}V_{20} + \sum_{l=1}^{m}\frac{\partial^2 f}{\partial y_l^2}y_{(10)l}^2 + \sum_{l=1}^{m}\frac{\partial f}{\partial y_l}y_{(20)l}$$
$$+ 2\sum_{l=1}^{m}\frac{\partial^2 f}{\partial v\,\partial y_l}V_{10}y_{(10)l} - 2\omega B_{10}\frac{\partial^2 V_{20}}{\partial\psi^2} - (2\omega B_{20} + B_{10}^2)\frac{\partial^2 V_{10}}{\partial\psi^2}$$
$$- A_{10}\frac{dB_{10}}{da}\frac{\partial V_{10}}{\partial\psi} - A_{10}\frac{dA_{10}}{da}\frac{\partial V_{10}}{\partial a} - 2\omega A_{10}\frac{\partial^2 V_{20}}{\partial a\,\partial\psi}$$
$$- 2(\omega A_{20} + A_{10}B_{10})\frac{\partial^2 V_{10}}{\partial\psi\,\partial a} - A_{10}^2\frac{\partial^2 V_{10}}{\partial a^2},$$

$$F_{\varepsilon^3} = -\frac{\partial y_{10}}{\partial a}A_{20} - \frac{\partial y_{20}}{\partial a}A_{10} - \frac{\partial y_{20}}{\partial\psi}B_{20} + \frac{\partial^2 F}{\partial v^2}V_{10}^2 + \frac{\partial f}{\partial v}V_{20} + \sum_{l=1}^{m}\frac{\partial^2 F}{\partial y_l^2}y_{(10)l}$$
$$+ \sum_{l=1}^{m}\frac{\partial^2 F}{\partial y_l^2}y_{(10)l} + \sum_{l=1}^{m}\frac{\partial F}{\partial y_l}y_{(10)l}^2 + \sum_{l=1}^{m}\frac{\partial F}{\partial y_l}y_{(20)l} + 2\sum_{l=1}^{m}\frac{\partial^2 F}{\partial v\,\partial y_l}V_{10}y_{(10)l},$$

$$f_{\varepsilon\mu^2} = \sum_{n=1}^{m}\frac{\partial^2 f}{\partial \dot{y}_{1n}^2}a^2\omega^2(\alpha\, e^{i\psi} - \bar{\alpha}\, e^{-i\psi})^2, \qquad F_{\varepsilon\mu^2} = \sum_{n=1}^{M}\frac{\partial^2 F}{\partial \dot{y}_{1n}^2}a^2\omega^2(\alpha\, e^{i\psi} - \bar{\alpha}\, e^{-i\psi})^2,$$

$$f_{\varepsilon^2\mu}=2\frac{\partial f}{\partial v}V_{11}+2\sum_{l=1}^{m}\frac{\partial f}{\partial y_{(11)l}}+2\sum_{n=1}^{m}\frac{\partial f}{\partial \dot{y}_{1n}}\{-A_{10}(\alpha\,e^{i\psi}+\bar{\alpha}\,e^{-i\psi})-aB_{10}i(\alpha\,e^{i\psi}-\bar{\alpha}\,e^{-i\psi})\}$$

$$-\frac{\partial y_{11}}{\partial\psi}\omega-V_{10}a\omega i\sum_{n=1}^{m}\frac{\partial^2 f}{\partial v\,\partial\dot{y}_{1n}}(\alpha\,e^{i\psi}-\bar{\alpha}\,e^{-i\psi})-y_{10}a\omega i(\alpha\,e^{i\psi}-\bar{\alpha}\,e^{-i\psi})\sum_{n=1}^{m}\sum_{l=1}^{m}\frac{\partial^2 f}{\partial y_l\,\partial\dot{y}_{1n}}$$

$$-2\omega B_{10}\frac{\partial^2 V_{11}}{\partial\psi^2}-2\omega B_{11}\frac{\partial^2 V_{10}}{\partial\psi^2}-2\omega A_{10}\frac{\partial^2 V_{11}}{\partial a\,\partial\psi}-2\omega A_{11}\frac{\partial^2 V_{11}}{\partial a\,\partial\psi},$$

$$F_{\varepsilon^2\mu}=-A_{10}\frac{\partial y_{11}}{\partial a}-A_{11}\frac{\partial y_{10}}{\partial a}-B_{10}\frac{\partial y_{11}}{\partial a}-B_{11}\frac{\partial y_{10}}{\partial\psi}$$

$$-\sum_{n=1}^{m}\frac{\partial^2 F}{\partial v\,\partial\dot{y}_{1n}}V_{10}a\omega i(\alpha\,e^{i\psi}+\bar{\alpha}\,e^{-i\psi})$$

$$-\sum_{l=1}^{m}\sum_{n=1}^{m}\frac{\partial^2 F}{\partial y_l\,\partial\dot{y}_{1n}}y_{10}a\omega i(\alpha\,e^{i\psi}+\bar{\alpha}\,e^{-i\psi})$$

$$+2\sum_{l=1}^{M}\frac{\partial F}{\partial y_l}y_{(11)l}+2\frac{\partial F}{\partial v}V_{11}-\omega\frac{\partial F}{\partial\dot{v}_1}\frac{\partial V_{11}}{\partial\psi}$$

$$+2\sum_{n=1}^{m}\frac{\partial F}{\partial\dot{y}_{1n}}\left\{-A_{10}(\alpha\,e^{i\psi}+\bar{\alpha}\,e^{-i\psi})-aB_{10}(\alpha\,e^{i\psi}+\bar{\alpha}\,e^{-i\psi})-\frac{\partial y_{11}}{\partial\psi}\omega\right\}. \tag{25}$$

To achieve a complete ordering of all of the recurrent equations we take the additional condition that $\varepsilon^{i-1}<\mu^i$, where i is a positive integer. After expanding the function $f_{(\cdot)}$ into a Fourier series one obtains

$$f_{(\cdot)}=\sum_{s=1}^{\infty}\sum_{n=1}^{\infty}\{b_{(\cdot)sn}(a)\cos n\psi+c_{(\cdot)sn}(a)\sin n\psi\}X_s(x), \tag{26}$$

where

$$b_{(\cdot)sn}(a)=\frac{1}{2\pi l}\int_0^l dx\int_0^{2\pi}f_{(\cdot)}(x,a,\psi)X_s(x)\cos n\psi\,d\psi,$$

$$c_{(\cdot)sn}(a)=\frac{1}{2\pi l}\int_0^l dx\int_0^{2\pi}f_{(\cdot)}(x,a,\psi)X_s(x)\sin n\psi\,d\psi. \tag{27}$$

The functions $F_{(\cdot)}$ are expanded into a complex Fourier series

$$F_{(\cdot)}=\sum_{n=-\infty}^{\infty}C_{(\cdot)n}(a)\,e^{in\psi}, \tag{28}$$

where

$$C_{(\cdot)n}(a)=\frac{1}{2\pi}\int_0^{2\pi}F_{(\cdot)}\,e^{in\psi}\,d\psi,\qquad n=\pm1,\pm2,\ldots \tag{29}$$

We describe a procedure of solving the recurrent set of ODE's based on equations (19). In order to avoid terms ascending unrestrictedly in time in these equations the following conditions must be satisfied:

$$-\{A_{10}(a)+\mathrm{i}aB_{10}(a)\}\left\{(\alpha,\beta)+\sum_{p=1}^{P}\tau_p(A_p\alpha,\beta)\,\mathrm{e}^{-\tau_p\omega\mathrm{i}}\right\}+(C_{(10)1}(a),\beta)=0. \quad (30)$$

Here (a, b) denotes the scalar product. By equating to zero the real and the imaginary parts of equation (30), we obtain two equations to determine the quantities $A_{10}(a)$ and $B_{10}(a)$. Then we can find V_{10} and y_{10}, and further successfully solve the recurrent set of equations.

3. TWO-PARAMETER UNFOLDINGS

From equations (6) one obtains

$$\Phi(a)=\mathrm{d}a/\mathrm{d}t=\varepsilon A_{10}+\varepsilon^2A_{20}+\varepsilon^3A_{30}+\varepsilon\mu A_{11}+\varepsilon^2\mu A_{21}+\varepsilon\mu^2A_{12}+O(\varepsilon^k\mu^l;k+l=4),$$
$$\omega(a)=\mathrm{d}\psi/\mathrm{d}t=\omega+\varepsilon B_{10}+\varepsilon^2B_{20}+\varepsilon^3B_{30}+\varepsilon\mu B_{11}+\varepsilon^2\mu B_{21}+\varepsilon\mu^2B_{12}+O(\varepsilon^k\mu^l;k+l=4), \quad (31)$$

at the initial conditions $a(t_0)=a_0$, $\psi(t_0)=\psi_0$. From the first of equations (31) we obtain the dependence $a(t)$, which upon introduction into the latter of equations (31) enables us to determine the dependence $\psi\{a(t)\}$. Thanks to this it is possible to analyze the slow transient processes leading to steady state. The latter are analyzed by assuming that $\mathrm{d}a/\mathrm{d}t=0$, which leads to the algebraic equation

$$\Phi(a,\varepsilon,\mu)=A_{10}+\varepsilon A_{20}+\varepsilon^2A_{30}+\mu A_{11}+\varepsilon\mu A_{21}+\mu^2A_{12}=0, \quad (32)$$

where the A_{kl} are functions of a. Because a branching problem does not change (at least qualitatively) when we change the co-ordinates, we can introduce the following unfolding classification. In what follows, equation (31) is transformed into

$$A_{30}\varepsilon^2+2A'_{21}\varepsilon\mu+A_{12}\mu^2+2A'_{20}\varepsilon+2A'_{11}\mu+A_{10}=0, \quad (33)$$

where

$$A'_{21}=\tfrac{1}{2}A_{21}, \qquad A'_{20}=\tfrac{1}{2}A_{20}, \qquad A'_{11}=\tfrac{1}{2}A_{11}. \quad (34)$$

Equation (33) presents implicit second order algebraic functions if A_{30}, A'_{21} and A_{12} are not equal to zero at the same time. The form of the function is determined by the expressions

$$W=\det\begin{pmatrix}A_{30} & A'_{21} & A'_{20}\\ A'_{21} & A_{12} & A'_{11}\\ A'_{20} & A'_{11} & A_{10}\end{pmatrix}, \qquad V=\det\begin{pmatrix}A_{30} & A'_{21}\\ A'_{21} & A_{12}\end{pmatrix}, \qquad S=A_{30}+A_{12},$$
$$W_{22}=A_{30}A_{10}-(A'_{20})^2, \qquad W_{11}=A_{12}A_{10}-(A'_{11})^2. \quad (35)$$

By means of shifting the origin of the co-ordinate system and turning the axis, it is possible to obtain the following functional forms (expressions W, V and S are the invariants of such shifts and turns): (1) $V>0$, $AW<0$—curve (33) is the ellipse $\varepsilon^2/A^2+\mu^2/B^2=1$; (2) $V>0$, $W=0$—equation (33) can be transformed to $(\varepsilon^2/A^2)+(\mu^2/B^2)=0$ and the solution is the point (0, 0); (3) $V>0$, $AW>0$—curve (33) is an imaginary ellipse (no real curve exists); (4) $V<0$, $W\neq0$—equation (33) defines the equilateral hyperbola $(\varepsilon^2/a^2)-(\mu^2/B^2)=1$; (5) $V<0$, $W=0$—the solution of equation (33) is a pair of intersecting lines $(\varepsilon^2/$

$A^2)-(\mu^2/B^2)=0$; (6) $V=0$, $W\neq 0$—the curve governed by equation (33) is a parabola $\mu^2=2p\varepsilon$; (7) $V=0$, $W=0$, $W_{11}<0$ or $W_{22}>0$—equation (33) presents a pair of parallel lines $\mu^2-A^2=0$; (8) $V=0$, $W=0$, $W_{11}>0$ or $W_{22}>0$—the solution of equation (30) is the imaginary parallel lines $\mu^2+A^2=0$ (no real curve exists); (9) $V=0$, $W=0$, $W_{11}=0$ or $W_{22}=0$—the solution of equation (30) is a double line $\mu^2=0$.

The coefficients of equation (33) are functions of the amplitude a and their values are determined by the functions $f_{(\cdot)}$. It should be emphasized that the two-parameter family governed by equation (33) can undergo elementary catastrophes. If we have one-parameter family, then the fold catastrophe is possible. Two parameters can give a reason for the occurrence of a cusp. Finally, if a third parameter is introduced, a swallowtail can occur (for details see reference [7]). All of the above mentioned catastrophes are concerned with the static situations. If we consider additionally the dynamics governed by the amplitude equation (the first of equations (31)), and then the dynamics of a flow defined by the full equations (1), a very complicated behaviour is expected.

ACKNOWLEDGMENTS

The first author (J.A.) would like to acknowledge the hospitality of the Research Center for Advanced Science and Technology, The University of Tokyo, during the preparation of this report.

REFERENCES

1. J. Awrejcewicz 1986 *Proceedings of the International Conference on Rotordynamics* (*Tokyo* 14–17 *September*), 517–522. Vibration system: rotor with self excited support.
2. J. Awrejcewicz 1988 *International Journal of Nonlinear Mechanics* **23**(1), 87–94. Determination of the limits of the unstable zones of the unstationary non-linear systems.
3. J. Awrejcewicz 1990 *Korean Society of Mechanical Engineering Journal* **4**(2), 156–166. Parametric and self-excited vibrations induced by friction in a system with three degrees of freedom.
4. J. Awrejcewicz 1989 *Pan-American Congress of Applied Mechanics* (*Rio de Janeiro*). Hopf bifurcation in Duffing's oscillator.
5. J. Awrejcewicz 1991 *International Journal of Solids and Structures* **27**(7), 825–832. Determination of periodic oscillations in non-linear autonomous discrete–continuous systems with delay.
6. W. P. Rubanik 1985 *Vibrations in Complex Quasilinear Systems with Delay*. Minsk: University Press (in Russian).
7. R. Seydel 1988 *From Equilibrium to Chaos*. New York: Elsevier.