Analytical Conditions for the Existence of a Two-Parameter Family of Periodic Orbits in Nonautonomous Dynamical Systems

JAN AWREJCEWICZ[1] and TSUNEO SOMEYA[2]

[1]*Technical University, Institute of Applied Mechanics, B. Stefanowskiego 1/15, 90-924 Lodz, Poland;*
[2]*The University of Tokyo, Department of Mechanical Engineering, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113, Japan*

(Received: 11 June 1991; accepted: 31 December 1991)

Abstract. The two-parameter perturbation method, applied to the example of periodic oscillations in periodically driven nonlinear dynamical systems, is presented. The analytical conditions are given for the existence of a two-parameter family of periodic orbits in nonautonomous dynamical systems in both non-resonance and resonance cases.

Key words: Periodic orbits, two-parameter perturbation method, resonance and non-resonance cases, implicit function.

1. Introduction

Approximate analytical methods allow one to find the explicitly given relations between parameters in order to fulfil certain optimization criteria, and/or to define bifurcation points in the considered dynamical systems. This is one of the most important tasks in engineering. Also, in many practical cases, the rough use of such techniques makes it possible to estimate the parameter sets mentioned above. At that point, in order to obtain greater accuracy, numerical methods can be used.

Recently, attention on so-called two-parameter bifurcations [1, 2] has increased. It is expected that after determining a sufficient technique for the analysis of two-parameter bifurcation problems, many unsolved questions will be explained.

The aim of this paper is to present an approximate analytical method based on the application of a two-parameter perturbation, in order to find the analytical condition for the existence of a two-parameter family of periodic orbits in nonautonomous systems. An analysis of the existence of two-parameter periodic solutions in the neighborhood of the starting periodic solution of the fully integrable equation system is performed and their structure is shown. We consider nonautonomous nonlinear systems, and a family of periodic solutions for both resonance and non-resonance cases is demonstrated.

The method is developed in the spirit of the asymptotic techniques described in [3–11]. We discuss the problem of whether the analytically obtained solution converges with the starting solution of the fully integrable system for two independent parameters simultaneously driving at zero. The lack of such an analysis can even lead sometimes to erroneous results, because the range of applicability is not defined and the estimation of the small parameter, for which the solution of the truncated equations correspond to the solutions of the original system, is not given [12]. Our paper contains the illustrative example, in which the estimation of the influence of two independent parameters for the obtained results is outlined.

The presented approach is a continuation of earlier research, where the two-parameter

© 1993 *Kluwer Academic Publishers. Printed in the Netherlands*

perturbation method was used to analyze: (a) periodic oscillations in nonlinear parametrically excited mechanical systems [13–15]; (b) discrete-continuous mechanical systems [16]; and (c) bifurcated oscillations [17–20]. The analysis of the convergence of the two families of periodic solutions with the solution of a fully integrable system is an extension of the Zubov method [21] for two-parameter systems.

2. Non-Resonance Case

Consider the following dynamical system

$$\dot{X} = F(t, X, \epsilon, \mu)\,, \tag{1}$$

where (a) F is continuous and $X \in R^n$; (b) F satisfies the Lipshitz condition in a certain finite hyperspace $P \in R^n$, $|F(t, X) - F(t, y)| \le C_p\ |X - Y|$, $t \in (-\infty, +\infty)$, and C_p is a constant dependent on P; (c) $F(t + 2\pi, X) = F(t, X)$ for $\epsilon = \mu = 0$; (d) the solution of $X = X(t, X^0, 0, 0)$ is determined for every $t \ge 0$, $X^0 \in R^n$, where $X = X^0$ for $t = t_0$; (e) ϵ and μ are certain small parameters, and $\epsilon \in [0, \epsilon_0]$, $\mu \in [0, \mu_0]$. Let us assume that the system (1) for $\epsilon = \mu = 0$ has a family of 2π-periodic solutions

$$X = X^0(t, C_1, \ldots, C_k) \tag{2}$$

which depends on k arbitrary constants. Let us look for a certain new function Y defined by

$$X = X^0 + Y(t, \epsilon, \mu)\,. \tag{3}$$

After introducing (3) in (1) we get

$$\dot{X}_0 + \dot{Y}(t, \epsilon, \mu) = F(t, X^0 + Y(t, \epsilon, \mu), \epsilon, \mu)\,, \tag{4}$$

and further

$$\dot{X}_0 + \dot{Y}(t, \epsilon, \mu) = F(t, X^0 + Y(t, 0, 0), 0, 0) + \epsilon F_1^{(1)}(t, Y, \epsilon, \mu) + \mu F_1^{(2)}(t, Y, \epsilon, \mu)\,, \tag{5}$$

because we consider the norm of $Y(t, 0, 0)$ as a small we omit all terms with the power Y^k, for $k \ge 2$, and from (5) we find

$$\dot{X}_0 + \dot{Y}(t, \epsilon, \mu) = F(t, X^0) + R_1(t)Y + \Phi_1(t) + \epsilon F_1^{(1)}(t, Y, \epsilon, \mu) + \mu F_1^{(2)}(t, Y, \epsilon, \mu)\,. \tag{6}$$

Finally, in accordance with (1), we obtain from (6)

$$\dot{Y} = R_1(t)Y + \phi_1(t) + \epsilon F_1^{(1)}(t, Y, \epsilon, \mu) + \mu F_1^{(2)}(t, Y, \epsilon, \mu)\,, \tag{7}$$

where $F_1^{(1)}(t) = F_1^{(1)}(t + 2\pi)$, $F_1^{(2)}(t) = F_1^{(2)}(t + 2\pi)$, $\phi_1(t) = \phi_1(t + 2\pi)$.

We assume that none of the characteristic exponents of the system (7) (for $\epsilon = \mu = \phi_1 = 0$) is equal to $\pm 2\pi ikN^{-1}$, $k = 0, 1, \ldots$, $i^2 = -1$, where N is a natural number. From (7) for $\epsilon = \mu = 0$

we obtain

$$\dot{Y} = R_1(t)Y + \phi_1(t) , \tag{8}$$

and in this case (5) has one 2π-periodic solution (because $\phi_1(t)$ is periodic). We shall demonstrate that the system (7), and in what follows (1), also has a two-parameter family of periodic solutions with period 2π, approaching the 2π-periodic solution of the system (8) when $\epsilon \to 0$ and $\mu \to 0$.

The system (8) can be reduced, when the solution can be presented as follows

$$Y = S(t)Z , \tag{9}$$

where the matrix $[S]$ has periodic coefficients.

In accordance with the Jerugin's result [22] the system (8) can be reduced if and only if

$$F = S(t)e^{tR} , \tag{10}$$

where R is the matrix with constant coefficients and F is the matrix of the independent fundamental solutions of (8). From (9) and (10) we get

$$Y = Fe^{-tR}Z . \tag{11}$$

Differentiate the left side of (11) and according to (8) we get

$$\dot{Y} = Fe^{-tR}\dot{Z} + R_1(t)Fe^{-tR}Z + \phi_1(t)e^{-tR}Z - Fe^{-tR}RZ = R_1(t)Fe^{-tR}Z + \phi_1(t) , \tag{12}$$

which finally leads to the form

$$\dot{Z} = RZ + \phi(t) , \tag{13}$$

where

$$\phi(t) = \phi_1(t)(I - e^{-tR}Z)S^{-1}(t) , \tag{14}$$

and I is the identity matrix.

Thus, instead of (7) we consider the following system

$$\dot{Z} = RZ + \phi(t) + \epsilon F^{(1)}(t, Z, \epsilon, \mu) + \mu F^{(2)}(t, Z, \epsilon, \mu) . \tag{15}$$

The system (15) for $\epsilon = \mu = 0$ has a 2π-periodic solution

$$Z = e^{Rt}C + \int_0^t e^{R(t-\tau)}\phi(\tau)\,\mathrm{d}\tau . \tag{16}$$

The constant C is found from the condition of periodicity

$$C = (I - e^{2\pi R})^{-1} + \int_0^{2\pi} e^{R(2\pi-\tau)}\phi(\tau)\mathrm{d}\tau . \tag{17}$$

Let us now discuss the periodic solution found near (15). We shall introduce a two-parameter family of vector functions dependent on N and such that

$$\tilde{Y} = \tilde{Y}(N, X, \epsilon, \mu) = X(2\pi N, X, \epsilon, \mu) , \tag{18}$$

where X is the solution of (1). The problem of the existence of a periodic solution of (1) has been reduced to the problem of a fixed point of the mapping (18). Let us assume that (18) has the solution X^0 at $\epsilon = \mu = 0$. To this solution corresponds the $2\pi N$-periodic solution of (1) at $\epsilon = \mu = 0$, with the form

$$X = X_0(t, X^0, 0, 0) . \tag{19}$$

It is necessary to find a periodic solution of (1) which approaches (19) at $\epsilon \to 0$ and $\mu \to 0$. If such a vector function $X^0(\epsilon, \mu)$ is found, then if $X \to X^0$ at ϵ and μ for sufficiently small ϵ and μ, the problem is solved. The question of the existence of periodic solutions of (1) has been reduced to the problem of the existence of the implicit function $X = X^0(\epsilon, \mu)$.

Let us assume that $\tilde{Y} = \tilde{Y}(t, X, \epsilon, \mu)$ is the solution of (1) and it satisfies the initial condition $\tilde{Y} = X$ for $t = 0$. Using the notation in the scalar form, we obtain from (1)

$$\frac{d\tilde{y}_j(\epsilon, \mu)}{dt} = \epsilon F_j^{(1)}(t, \tilde{y}_1, \dots, \tilde{y}_n, \epsilon, \mu) + \mu F_j^{(2)}(t, \tilde{y}_1, \dots, \tilde{y}_n, \epsilon, \mu) ,$$

$$j = 1, \dots, n . \tag{20}$$

We differentiate both sides of the equations (20) consecutively in relation to $x_1, \dots, x_n$. As a result we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \tilde{y}_i}{\partial x_k}\right) &= \sum_{i=1}^{n} \left\{ \epsilon \frac{\partial F_j^{(1)}}{\partial \tilde{y}_i} \frac{\partial \tilde{y}_i}{\partial x_k} + \mu \frac{\partial F_j^{(2)}}{\partial \tilde{y}_i} \frac{\partial \tilde{y}_i}{\partial x_k} \right\} \\ &= \sum_{i=1}^{n} b_{ji} \frac{\partial \tilde{y}_i}{\partial x_k} = \sum_{i=1}^{n} \left\{ \epsilon b_{ji}^{(1)} \frac{\partial \tilde{y}_i}{\partial x_k} + \mu b_{ji}^{(2)} \frac{\partial \tilde{y}_i}{\partial x_k} \right\} , \end{aligned}$$

$$i, k = 1, \dots, n , \tag{21}$$

where according to (21) we have

$$[b_{ji}] = \epsilon b_{ji}^{(1)} + \mu b_{ji}^{(2)} \frac{\partial F_j}{\partial \tilde{y}_i} , \tag{22}$$

calculated for $\tilde{Y} = \tilde{Y}(t, X, \epsilon, \mu)$ and

$$[b_{ji}] = B(t, X, \epsilon, \mu) . \tag{23}$$

From (13), the functions $\partial \tilde{y}_i / \partial x_k$ $(i, k = 1, \dots, n)$ are the solutions of the equation system

$$\frac{\mathrm{d}\tilde{Y}}{\mathrm{d}t} = B(t, X, \epsilon, \mu)\tilde{Y} . \tag{24}$$

Let us assume that the matrix of the solutions of (24) is $[Y^*(t, X, \epsilon, \mu)]$. If we account for the fact that for $\epsilon = \mu = 0$ we have $X = X^0$, then $\tilde{Y} = \tilde{Y}(t, X^0, 0, 0)$ is a $2\pi N$-periodic solution of (1), and in this case the matrix $[B(t, X^0, 0, 0)]$ is also $2\pi N$-periodic. Since $\tilde{Y} = X$ for $t = 0$, then the matrix $[Y^*] = [I]$ for $t = 0$. Thus, the matrix $[Y^*(t, X, \epsilon, \mu)] = \{\epsilon^{(1^*)}(t, X, \epsilon, \mu) + \mu Y^{(2^*)}(t, X, \epsilon, \mu)\}$ is a system of fundamental solutions of (24).

On the other hand, if there is a fixed point $X^{(0)}$ and a number $N > 0$, such that $\tilde{Y}(N, X^{(0)}, \epsilon, \mu) = X^{(0)}$ at $\epsilon = \mu = 0$ and

$$\frac{D(y_1^{(1^*)}, \ldots, y_n^{(1^*)})}{D(x_1^{(1)}, \ldots, x_n^{(1)})} \neq 0 ,$$

$$\frac{D(y_1^{(2^*)}, \ldots, y_n^{(2^*)})}{D(x_1^{(2)}, \ldots, x_n^{(2)})} \neq 0 , \tag{25}$$

where D denotes the determinant obtained for $X = X^0$, where $y_j^{(i^*)} = \tilde{y}_j^{(i)} - x_j^{(i)}$, $i = 1, 2$, then the system (1) for sufficiently small $\epsilon > 0$ and $\mu > 0$ has such a $2\pi N$-periodic solution of $X = X(t, X_0(\epsilon, \mu), \epsilon, \mu)$, that $X = X_0(\epsilon, \mu)$ at $t = 0$, $X_0(\epsilon, \mu) \to X^0$ for $\epsilon \to 0$ and $\mu \to 0$.

3. Resonance Case

We assume that simple elementary divisors [21] correspond to the resonance multiplied eigenvalues, and that there are k eigenvalues equal to zero and 21 others with the form $\pm iMN^{-1}$ from among the whole set of $\lambda_1, \ldots, \lambda_n$ eigenvalues. The imaginary values will be denoted as $\nu_1, \ldots, \nu_i, -\nu_1, \ldots, \nu_i$. In this case it is always possible to find such a linear transformation of the vector Y with real and constant coefficients so that the system (4) can be reduced to the form

$$\frac{du_s}{dt} = \phi_s(t) + \epsilon F_s^{(1)}(t, U, X, Y, Z, \epsilon, \mu) + \mu F_s^{(2)}(t, U, X, Y, Z, \epsilon, \mu) ,$$

$$\frac{dx_p}{dt} = -\nu_p y_p + \alpha_p(t) + \epsilon A_p^{(1)}(t, U, X, Y, Z, \epsilon, \mu) + \mu A_p^{(2)}(t, U, X, Y, Z, \epsilon, \mu) ,$$

$$\frac{dy_p}{dt} = \nu_p x_p + \eta_p(t) + \epsilon E_p^{(1)}(t, U, X, Y, Z, \epsilon, \mu) + \mu E_p^{(2)}(t, U, X, Y, Z, \epsilon, \mu) ,$$

$$\frac{dz_\tau}{dt} = \sum_{t=1}^{m} p_{ri} z_i + \vartheta_r(t) + \epsilon \Gamma_r^{(1)}(t, U, X, Y, Z, \epsilon, \mu) + \mu \Gamma_r^{(2)}(t, U, X, Y, Z, \epsilon, \mu) ,$$

$$s = 1, \ldots, k , \quad p = 1, \ldots, l , \quad r = 1, \ldots, m , \quad k + 2l + m = n ,$$

$$U = (u_1, \ldots, u_k) , \qquad X = (x_1, \ldots, x_l) ,$$

$$Y = (y_1, \ldots, y_k) , \qquad Z = (z_1, \ldots, z_m) , \tag{26}$$

and the matrix $[p_{ri}]$ has no resonance values. Let us perform the following change of variables in equation (26):

$$x_p = \overline{x_p} \cos \nu_p t + \overline{y_p} \sin \nu_p t ,$$

$$y_p = \overline{y_p} \sin \nu_p t - \overline{y_p} \cos \nu_p t , \tag{27}$$

where $p = 1, \ldots, l$. From (26) we obtain

$$\frac{\mathrm{d}v_s}{\mathrm{d}t} = w_s(t) + \epsilon F_s^{(1)}(t, V, Z, \epsilon, \mu) + \mu F_s^{(2)}(t, V, Z, \epsilon, \mu) ,$$

$$\frac{\mathrm{d}z_r}{\mathrm{d}t} = \sum_{i=1}^{m} p_{ri} z_i + \vartheta_r(t) + \epsilon \Gamma_r^{(1)}(t, V, Z, \epsilon, \mu) + \mu \Gamma_r^{(2)}(t, V, Z, \mu) . \tag{28}$$

where the following denotation is assumed: $s = 1, \ldots, k + 21$, $r = 1, \ldots, m$, and $v_s = u_s$, $v_{k+p} = \overline{x_p}$, $v_{k+l+p} = \overline{y_p}$ for $s = 1, \ldots, k$ and $p = 1, \ldots, l$. Consider the equations (28) at $\epsilon = \mu = 0$. They have a family of periodic solutions with period $2\pi N$ dependent on $(k + 2l)$ constants if the following conditions are satisfied:

$$\int_0^{2\pi N} \omega_s(\tau)\, \mathrm{d}\tau = 0 , \quad s = 1, \ldots, k + 2l . \tag{29}$$

Assume that equations (29) are satisfied. In such a case the family of periodic solutions can be presented as follows:

$$V_s = c_s + \int_0^k w_s(\tau)\, \mathrm{d}\tau ,$$

$$Z = Ce^{P^t}t + \int_0^t e^{P(t-\tau)} \theta(\tau)\, \mathrm{d}\tau , \tag{30}$$

where

$$\theta(t) = (v_1(t), \ldots, v_m(t)) ,$$

$$P = p_{ri} ,$$

$$C = (I - e^{2\pi NP})^{-1} \int_0^{2\pi N} e^{P(2\pi N - \tau)} \theta(\tau)\, \mathrm{d}\tau . \tag{31}$$

We seek a general solution of the system (28) for the following initial conditions

$$V_s(\epsilon, \mu) = c_s + \epsilon d_s^{(1)}(\epsilon, \mu) + \mu d_s^{(2)}(\epsilon, \mu) ,$$

$$Z(\epsilon, \mu) = C + \epsilon G^{(1)}(\epsilon, \mu) + \mu G^{(2)}(\epsilon, \mu) , \tag{32}$$

for $t = 0$, $s = 1, \ldots, k + 2l$. According to (28) we have

$$V_s(t, \epsilon, \mu) = c_s + \epsilon d_s^{(1)}(\epsilon, \mu) + \mu d_s^{(2)}(\epsilon, \mu) + \int_0^t w_s(\tau)\, \mathrm{d}\tau + \epsilon \int_0^t F_s^{(1)}(\tau, V, Z, \epsilon, \mu)\, \mathrm{d}\tau$$

$$+ \mu \int_0^t F_s^{(2)}(t, V, Z, \epsilon, \mu)\, \mathrm{d}\tau ,$$

$$Z(t, \epsilon, \mu) = e^{Pt}(C + \epsilon G^{(1)}(\epsilon, \mu) + \mu G^{(2)}(\epsilon, \mu)) + \int_0^t e^{P(t-\tau)}\theta(\tau)\,\mathrm{d}\tau$$
$$+ \epsilon \int_0^t e^{P(t-\tau)}\Gamma^{(1)}(\tau, V, Z, \epsilon, \mu)\,\mathrm{d}\tau + \mu \int_0^t e^{P(t-\tau)}\Gamma^{(2)}(\tau, V, Z, \epsilon, \mu)\,\mathrm{d}\tau\,. \tag{33}$$

We look for the periodic solutions, i.e.,

$$V_s(t, \epsilon, \mu) = V_s(t + 2\pi N, \epsilon, \mu)\,,$$
$$Z(t, \epsilon, \mu) = Z(t + 2\pi N, \epsilon, \mu)\,,$$
$$s = 1, \ldots, k + 2l\,. \tag{34}$$

Taking into account (34) in (33), we obtain

$$\epsilon R_s^{(1)} + \mu R_s^{(2)} = \epsilon \int_0^{2\pi N} F_s^{(1)}(\tau, V, Z, \epsilon, \mu)\,\mathrm{d}\tau$$
$$+ \mu \int_0^{2\pi N} F_S^{(2)}(\tau, V, Z, \epsilon, \mu)\,\mathrm{d}\tau = 0\,,$$
$$(e^{2\pi NP} - I)(\epsilon G^{(1)}(\epsilon, \mu) + \mu G^{(2)}(\epsilon, \mu)) + \epsilon \int_0^{2\pi N} e^{P(2\pi N-\tau)}\Gamma^{(1)}(\tau, V, Z, \epsilon, \mu)\,\mathrm{d}\tau$$
$$+ \mu \int_0^{2\pi N} e^{P(2\pi N-\tau)}\Gamma^{(2)}(\tau, V, Z, \epsilon, \tau)\,\mathrm{d}\tau = 0\,. \tag{35}$$

From (35) we find

$$R_s^{(1)} = \int_0^{2\pi N} F_s^{(1)}(\tau, V, Z, \epsilon, \mu)\,\mathrm{d}\tau\,;$$
$$G^{(1)} = (e^{2\pi NP} - I)^{-1} \int_0^{2\pi N} e^{(2\pi N-\tau)}\Gamma^{(1)}(\tau, V, Z, \epsilon, \mu)\,\mathrm{d}\tau\,;$$
$$R_s^{(2)} = \int_0^{2\pi H} F_s^{(2)}(\tau, V, Z, \epsilon, \mu)\,\mathrm{d}\tau\,;$$
$$G^{(2)} = (e^{2\pi NP} - I)^{-1} \int_0^{2\pi N} e^{(2\pi N-\tau)}\Gamma^{(2)}(\tau, V, Z, \epsilon, \mu)\,\mathrm{d}\tau\,. \tag{36}$$

After calculating $G^{(1)}$ and $G^{(2)}$ and then V and Z from (33), we can determine the unknown set of $d_1^{(i)}, \ldots, d_{k+2l}^{(i)}$, $i = 1, 2$, from the following equations:

$$R_s^{(1)}(d_1^{(1)}, \ldots, d_{k+2l}^{(1)}) = \int_0^{2\pi N} F_s^{(1)}(\tau, V, Z, \epsilon, \mu)\,\mathrm{d}\tau = 0\,,$$
$$R_s^{(2)}(d_1^{(2)}, \ldots, d_{k+2l}^{(2)}) = \int_0^{2\pi N} F_s^{(2)}(\tau, V, Z, \epsilon, \mu)\,\mathrm{d}\tau = 0\,.$$
$$s = 1, \ldots, k + 2l\,. \tag{37}$$

It seems evident, then, that not every periodic solution can be a limiting solution for the periodic solution of the nonlinear system (20) for $\epsilon \to 0$ and $\mu \to 0$. The limiting ones are the solutions which satisfy (37). If the condition (37) and the determinants

$$\frac{D(R_1^{(1)}, \ldots, R_{k+2l}^{(1)})}{D(d_1^{(1)}, \ldots, d_{k+2l}^{(1)})} \neq 0 ,$$

$$\frac{D(R_1^{(2)}, \ldots, R_{k+2l}^{(2)})}{D(d_1^{(2)}, \ldots, d_{k+2l}^{(2)})} \neq 0 , \qquad (38)$$

are satisfied, the equation system has $2\pi N$-periodic solutions for all sufficiently small $\epsilon > 0$ and $\mu > 0$, unrestrictedly approaching the $2\pi N$-periodic solution of the nonlinear system (20) at $\epsilon \to 0$ and $\mu \to 0$.

During further calculation one can construct the recurrent set of equations obtained by comparing the expressions by $\epsilon^k \mu^l$ (for $k > l$), and the corresponding set obtained for $\mu^l \epsilon^k$ ($l > k$). The last set of recurrent equations is determined after comparing of the expressions by $(\epsilon, \mu)^m$, where $m = k = l$.

4. Example

The example presented below has three important parts: (a) we demonstrate the use of the two perturbation method applied to nonautonomous nonlinear oscillator with delay, (b) the frequency $\omega(\tau - \mu)$ of the linear part of the governing equation depends slowly on time $\tau = \epsilon t$ and additionally on delay the μ, (c) the majority functions are given and the analysis of the influence of parameters for the validity of results is given.

Consider the following oscillator

$$\frac{d^2x}{dt^2} + \omega^2(\tau - \mu)x = -\epsilon x^2 + \mu P \cos t , \qquad (39)$$

where ϵ and μ are two independent small parameters and in order to avoid tedious calculations we assume $\mu \approx \epsilon^2$ (the similar but autonomous system without delay has been considered in [24]).

We put

$$\omega^2(\tau - \mu) = \omega^2(\tau) - 2\mu\omega(\tau) , \qquad (40)$$

to (35) and obtain

$$\frac{dx}{dt} = y ,$$

$$\frac{dy}{dt} = -\omega^2(\tau)x - \epsilon x^2 + 2\mu\omega(\tau)x + \mu P \cos t , \qquad (41)$$

The solution of (41) are sought in the form

$$x = x_0(\tau, t) + \epsilon x_{10}(\tau, t) + \mu x_{01}(t) + \cdots ,$$

$$y = y_0(\tau, t) + \epsilon y_{10}(\tau, t) + \mu y_{01}(t) + \cdots . \tag{42}$$

After introducing (42) to (41) and comparing the terms standing by ϵ and μ we get

$$\frac{dx_{10}}{dt} \equiv \frac{\partial x_{10}}{\partial \tau}\,\epsilon + \frac{\partial x_{10}}{\partial t} = y_{10} + f_{10} ,$$

$$\frac{dy_{10}}{dt} \equiv \frac{\partial y_{10}}{\partial \tau}\,\epsilon + \frac{\partial y_{10}}{\partial t} = -\omega^2 x_{10} + f_{20} , \tag{43}$$

$$\frac{dx_{10}}{dt} = y_{01} ,$$

$$\frac{dy_{10}}{dt} = -\omega^2 x_{01} + 2\omega x_0 + P\cos t , \tag{44}$$

where:

$$f_{10} = -\frac{\partial x_0}{\partial \tau} ,$$

$$f_{20} = -\frac{\partial y_0}{\partial \tau} - x_0^2 - 2\epsilon x_0 x_{10} - \epsilon^2 x_{10}^2 . \tag{45}$$

Suppose that x_{10}, y_{10} are two independent solutions of the following homogeneous system obtained from (43)

$$\frac{dx_{10}}{dt} = y_{01} ,$$

$$\frac{dy_{10}}{dt} = -\omega^2 x_{01} , \tag{46}$$

and additionally

$$x_{10}^{(1)}(0) = y_{10}^{(2)} = 1 ,$$

$$x_{10}^{(2)}(0) = y_{10}^{(1)} = 0 , \tag{47}$$

as well as

$$\Delta = \begin{pmatrix} x_{10}^{(1)} & y_{10}^{(1)} \\ x_{10}^{(2)} & y_{10}^{(2)} \end{pmatrix} = 1 . \tag{48}$$

The following is a general solution of (46)

$$x_{10} = C_1 x_{10}^{(1)} + C_2 x_{10}^{(2)} ,$$

$$y_{10} = C_1 y_{10}^{(1)} + C_2 y_{10}^{(2)} . \tag{49}$$

We assume (for further considerations) that $|x_{10}^{(k)}| \leq \alpha$ and $|y_{10}^{(k)}| \leq \alpha$, where $k = 1, 2$.

We substitute (45) to (43) (where now $C_1 = C_1(t)$ and $C_2 = C_2(t)$) and we obtain

$$C_1 = \int_0^t (f_{10} y_{10}^{(2)} - f_{20} x_{10}^{(2)})\, \mathrm{d}t ,$$

$$C_2 = \int_0^t (f_{20} x_{10}^{(1)} - f_{10} y_{10}^{(1)})\, \mathrm{d}t . \tag{50}$$

From (49) we have

$$x_{10} = x_{10}^{(1)} \int_0^t \left\{ -y_{10}^{(2)} \frac{\partial x_0}{\partial \tau} - x_{10}^{(2)} \left(\frac{-\partial y_0}{\partial \tau} - x_0^{(2)} - 2\epsilon x_0 x_{10} - \epsilon^2 x_{10}^{(2)} \right) \right\} \mathrm{d}t$$

$$+ x_{10}^{(2)}(t) \int_0^t \left\{ x_{10}^{(1)} \left(\frac{-\partial y_0}{\partial \tau} - x_0^{(2)} - 2\epsilon x_0 x_{10} - \epsilon^2 x_{10}^{(2)} \right) + y_{10}^{(1)} \frac{\partial x_0}{\partial \tau} \right\} \mathrm{d}t$$

$$y_{10} = y_{10}^{(1)} \int_0^t \left\{ -y_{10}^{(2)} \frac{\partial x_0}{\partial \tau} - x_{10}^{(2)} \left(\frac{-\partial y_0}{\partial \tau} - x_0^{(2)} - 2\epsilon x_0 x_{10} - \epsilon^2 x_{10}^{(2)} \right) \right\} \mathrm{d}t$$

$$+ y_{10}^{(2)}(t) \int_0^t \left\{ x_{10}^{(1)} \left(\frac{-\partial y_0}{\partial \tau} - x_0^{(2)} - 2\epsilon x_0 x_{10} - \epsilon^2 x_{10}^{(2)} \right) + y_{10}^{(1)} \frac{\partial x_0}{\partial \tau} \right\} \mathrm{d}\tau \tag{51}$$

Supposing that

$$|-y_{10}^{(2)} \frac{\partial x_0}{\partial \tau} + x_{10}^{(2)} \frac{\partial y_0}{\partial \tau} + x_{10}^{(2)} x_0^2| \leq b ,$$

$$|y_{10}^{(1)} \frac{\partial x_0}{\partial \tau} - x_{10}^{(1)} \frac{\partial y_0}{\partial \tau} - x_{10}^{(1)} x_0^2| \leq b ,$$

$$|x_0| \leq A , \tag{52}$$

we get from (51) the inequality

$$|z| \leq 2a \int_0^t (b + 2\epsilon\alpha A|z| + \epsilon^2 \alpha z^2)\, \mathrm{d}t , \tag{53}$$

where

$$z(t) = x_{10}(t) = y_{10}(t) .$$

After solving it is possible to show that $x \approx x_0$, $y \approx y_0$ are obtained with the error depending on ϵ smaller than

$$\Delta_{10} = z\epsilon , \tag{54}$$

where

$$|z| \geq |x_{10}| , \quad |z| \geq |y_{10}| , \quad 0 \leq t \leq l , \quad 0 \leq \tau \leq T . \tag{55}$$

Now we consider the influence of the error depending on μ. We take

$$w(t) = x_{10}(t) = y_{01}(t) ,$$

$$|I| \leq B , \quad |\omega_{01}^2| \leq B \quad \text{and} \quad |f_{01}| \leq D , \quad |f_{02}| \leq D , \tag{56}$$

and from (44) we obtain

$$\frac{dw}{dt} = Bw + D . \tag{57}$$

The solution of (57) is

$$w \leq \frac{D}{B}(e^{Bt} - 1) . \tag{58}$$

Finally, the influence of the error depending on ϵ and μ is smaller than

$$\Delta = z\epsilon + \frac{D}{B}(e^{Bt} - I)\mu , \tag{59}$$

where;

$$|z| \geq |x_{10}| , \quad |z| \geq |y_{10}| , \quad |w| \geq |x_{01}| , \quad |w| \geq |y_{01}| , \quad 0 \leq t \leq l , \quad 0 \leq \tau \leq T . \tag{60}$$

5. Concluding Remarks

This paper presents an original method of determining two-parameter periodic solutions in nonlinear discrete dynamical systems that are periodically excited. The method is an extension of the classical perturbation technique for systems with two small independent parameters. The problem of the existence of periodic solutions has been reduced to the analysis of the existence of the fixed points of one-to-one continuous maps. It has been shown that the obtained vector function is a periodic one implicitly dependent on two independent perturbation parameters. If the parameters approach zero, the analytical solution approaches the solution of a fully integrable differential equation system. Initially, the non-resonance case was considered, and the conditions for the existence of periodic solutions in such a case were formulated. Next, the resonance case was considered, where from among the eigenvalues $\lambda_1, \ldots, \lambda_n$ there are k zero ones and 21 with the form $\pm iMN^{-1}$. On the basis of the theory of one-to-one mapping, the necessary condition for the existence of periodic solutions dependent on two small parameters has also been formulated.

The obtained results are very important because, in fact, in many dynamical systems the observed periodic orbits may be governed by more than one independent parameters. The presented method enables the independently generated families of periodic orbits to be controlled. It should be pointed out that such behaviour can not be detected by the use of the single-perturbation technique or numerical observations. This is because in both cases the independent families are identifying a single-variable family of periodic orbits. It can lead to some incorrect results. For example, suppose that one of the independent periodic families vanishes due to a saddle-node connection and chaotic orbits appear (for numerical examples of such behaviour see for instance [23]). The other families of the periodic orbits, however, governed by the other control parameters still exist. In this case we have obtained the explanation for the coexistence of periodic and chaotic orbits, which can not be explained by the classical approach.

As has been mentioned in the Introduction, the lack of the range of applicability of the asymptotic analysis can lead to untrue results. The special example illustrates a general technique to obtain the majority functions, the estimation of time t and two perturbation parameters ϵ and μ which lead to the correct results. Also the estimation of the error, which corresponds to the analytically found solution, is explicitly given.

Acknowledgement

This work was supported by the Japanese Society for the Promotion of Science and was performed during my (J.A.) stay in the Department of Mechanical Engineering of the Tokyo University.

References

1. Guckenheimer, J. and Holmes, P. J., *Nonlinear Oscillations, Dynamical Systems and Bifurcation of Vector Field*, Springer Verlag, New York, 1983.
2. Seydel, R., *From Equilibrium to Chaos*, Elsevier, New York, 1988.
3. Malkin, I. G., *Some Problems in the Theory of Nonlinear Oscillations*, Nauka, Moscow, 1956, in Russian.
4. Giacaglia, G. E., *Perturbation Methods in Non-Linear Systems*, Springer Verlag, New York, 1972.
5. Bogoliubov, N. N. and Mitropolskii, Y. A., *Asymptotic Methods in the Theory of Non-Linear Oscillations*, Hindustan Publ. Corp., Delhi, 1961.
6. Hale, J. K., *Oscillations in Nonlinear Systems*, McGraw Hill, New York, 1963.
7. Nayfeh, A. H. and Mook, D. T., *Nonlinear Oscillations*, Wiley Interscience, New York, 1979.
8. Nayfeh, A. H., *Perturbation Methods*, Wiley Interscience, New York, 1981.
9. Nayfeh, A. H., *Introduction to Perturbation Techniques*, Wiley Interscience, New York, 1981.
10. O'Malley, R. E., *Introduction to Singular Perturbations*, Academic Press, New York and London, 1974.
11. Moiseev, N. N., *Asymptotic Methods in Nonlinear Mechanics*, Nauka, Moscow, 1981, in Russian.
12. Bajaj, A. K. and Johnson, J. M., 'Asymptotic techniques and complex dynamics in weakly nonlinear forced mechanical systems', *International Journal of Non-Linear Mechanics* **25**(2/3), 1990, 211–226.
13. Awrejcewicz, J., 'Vibration system: Rotor with self-excited support', in *Proceedings of the International Conference on Rotordynamics*, Tokyo, Sept. 14–17, 1986, 517–522.
14. Awrejcewicz, J., 'Determination of the limits of the unstable zones of the unstationary nonlinear systems', *International Journal of Non-Linear Mechanics* **23**(1), 1988, 87–94.
15. Awrejcewicz, J., 'Parametric and self-excited vibrations induced by friction in a system with three degrees of freedom', *Korean Society of Mechanical Engineering Journal* **4**(2), 1990, 156–166.
16. Awrejcewicz, J., 'Determination of periodic oscillations in nonlinear autonomous discrete-continuous systems with delay', *International Journal of Solids and Structures* **27**(7), 1991, 825–832.
17. Awrejcewicz, J., 'The analytical method to detect Hopf bifurcation solutions in the unstationary nonlinear systems', *Journal of Sound and Vibration* **129**(1), 1989, 175–178.
18. Awrejcewicz, J., 'Hopf bifurcation in Duffing's oscillator', *PAN American Congress of Applied Mechanics*, Rio de Janeiro, 1989, 640–643.
19. Awrejcewicz, J., *Bifurcation and Chaos in Simple Dynamical Systems*, World Scientific, Singapore, 1989.
20. Awrejcewicz, J. and Someya T., 'Analytical condition for the existence of two-parameter family of periodic orbits in the autonomous system', *Journal of the Physical Society of Japan* **60**(3), 1991, 781–784.
21. Zubov, W. I., *Theóry of Oscillations*, Vyssaia skola, Moscow, 1979, in Russian.
22. Demidowicz, B. I., *Mathematical Theory of Stability*, PWN, Warsaw, 1968, in Polish.
23. Awrejcewicz, J., *Bifurcation and Chaos in Coupled Oscillators*, World Scientific, Singapore, 1991.
24. Jerugin, E. P., *Lectures on the General Theory of Differential Equations*, Nauka and Technika, Minsk, 1979, in Russian.