Journal of Applied Mathematics and Physics (ZAMP)
Vol. 41, September 1990
0044-2275/90/050713-15 $ 1.50 + 0.20

Quasiperiodicity, strange non-chaotic and chaotic attractors in a forced two degrees-of-freedom system

By J. Awrejcewicz* and W.-D. Reinhardt, Institute of Technical Mechanics, Technical University Carlo-Wilhelmina, 3300 Braunschweig, West Germany

1. Introduction

In this note we investigate transitions between quasiperiodic, strange chaotic and strange non-chaotic attractors by means of the example of two externally driven coupled nonlinear oscillators. There are two main reasons for our work. First, the coupled oscillators can model many real physical systems and are more common when compared with simple sinusoidally or almost periodically driven anharmonic oscillators. (The possibly complicated nonlinear dynamics of simple oscillators was, among others, presented in references [1–5]). Second, one can expect much more complex behaviour, which goes beyond the occurrence of strange attractors, in coupled forced oscillators [6].

It is known that chaotic attractors may not be strange attractors. They occupy the full toroidal surface and hence are not strange. However, there is another class of attractors which are called strange non-chaotic attractors. In this case the word strange refers to the complicated geometrical structure of an attractor. Some examples of chaotic attractors which are not strange as well as examples of the strange attractors which are not chaotic are given in reference [7].

2. Analysis

We consider two coupled almost periodically driven oscillators

$$\begin{aligned} &m_1\ddot{x}_1 + (c_3 - c_1)\dot{x}_1 - c_3\dot{x}_2 + c_2x_1^2\dot{x}_1 + (k_1 + k_3)x_1 - k_3x_2 + k_2x_1^3 \\ &\quad = q_1\cos(\omega_1 t + \varphi), \\ &m_2\ddot{x}_2 + (c_3 - c_4)\dot{x}_2 - c_3\dot{x}_1 + c_5x_2^2\dot{x}_2 + (k_3 + k_4)x_2 - k_3x_1 + k_5x_2^3 \\ &\quad = q_2\cos\omega_2 t, \end{aligned} \tag{1}$$

* Permanent address: Institute of Applied Mechanics, Technical University Łódź, B. Stefanowskiego 1/15, 90-924 Łódź, Poland.

where m_1, m_2 are masses of the vibrating bodies, $c_1 - c_5$ and $k_1 - k_5$-damping and stiffness coefficients respectively and q_1, q_2 are the amplitudes of exciting forces with frequencies ω_1 and ω_2. In nondimensional form we have

$$\begin{aligned} &\xi_1'' + (\alpha_3 - \alpha_1)\xi_1' - \alpha_3(KM^{-1})^{1/2}\xi_2' + \gamma_1\xi_1^2\xi_1' + (\kappa_1 + \kappa_3)\xi_1 \\ &\quad - \kappa_3(KM^{-1})^{1/2}\xi_2 + \xi_1^3 = B_1\cos(\tau + \varphi), \\ &\xi_2'' + M(\alpha_3 - \alpha_4)\xi_2' - M\alpha_3(MK^{-1})^{1/2}\xi_1' + \gamma_2 K\xi_2^2\xi_2' + M(\kappa_3 + \kappa_4)\xi_2 \\ &\quad - M\kappa_3(MK^{-1})^{1/2}\xi_1 + \xi_2^3 = M^{3/2}K^{-1/2}B_2\cos\nu\tau. \end{aligned} \tag{2}$$

where

$$\begin{aligned} &\tau = \omega_1 t, \quad \xi_1 = (\omega_1 m_1)^{-1}k_2^{1/2}x_1, \quad \xi_2 = (\omega_1 m_2)^{-1}k_5^{1/2}x_2, \\ &M = m_1 m_2^{-1}, \quad K = k_2 k_5^{-1}, \quad \nu = \omega_2\omega_1^{-1}, \quad B_1 = q_1\omega_1^{-3}m_1^{-3}k_2^{1/2} \\ &B_2 = q_2\omega_1^{-3}m_1^{-3}k_2^{1/2}, \quad \alpha_1 = c_1(m_1\omega_1)^{-1}, \quad \alpha_3 = c_3(m_1\omega_1)^{-1}, \\ &\alpha_4 = c_4(m_1\omega_1)^{-1}, \quad \gamma_1 = \omega_1 c_2 k_2^{-1}, \quad \gamma_2 = \omega_1 c_5 k_2^{-1}, \\ &\kappa_1 = k_1 m_1^{-1}\omega_1^{-2}, \quad \kappa_3 = k_3 m_1^{-1}\omega_1^{-2}, \quad \kappa_4 = k_4 m_1^{-1}\omega_1^{-2}. \end{aligned}$$

In order to characterize the chaotic orbits the maximum one-dimensional Lyapunov exponent λ_{max} has been calculated. In the case of quasiperiodic and strange non-chaotic attractors $\lambda_{max} < 0$, whereas in the case of chaos $\lambda_{max} > 0$. The one-dimensional Lyapunov exponent has been determined by casting (2) into an autonomous system of first order differential equations

$$\begin{aligned} &\xi_1' = \eta_1, \\ &\eta_1' = -(\kappa_1 + \kappa_3 + \xi_1^2)\xi_1 - (\alpha_3 - \alpha_1 + \gamma_1\xi_1^2)\eta_1 + \kappa_3 K^{1/2}M^{-1/2}\xi_2 \\ &\quad + \alpha_3 K^{1/2}M^{-1/2}\eta_2 + B_1\cos(\varphi_1 + \varphi), \\ &\xi_2' = \eta_2, \\ &\eta_2' = \kappa_3 M^{3/2}K^{-1/2}\xi_1 + \alpha_3 M^{3/2}K^{-1/2}\eta_1 - (M(\kappa_3 + \kappa_4) + \xi_2^2)\xi_2 \\ &\quad \times (M(\alpha_3 - \alpha_4) + \gamma_2 K\xi_2^2)\eta_2 + M^{3/2}K^{-1/2}B_2\cos\varphi_2, \\ &\varphi_1' = 1, \\ &\varphi_2' = \nu, \end{aligned} \tag{3}$$

where $\varphi_1(0) = \varphi_2(0) = 0$. We have solved (3) together with its variational system ([8–10]) numerically.

Equations (2) possess the following symmetry under the transformations

$$T\colon (\xi_1, \xi_1', \xi_2, \xi_2', \tau) \to (\xi_1, \xi_1', \xi_2, \xi_2', \tau + 2\pi\nu^{-1}).$$

Therefore, the Poincaré map $M \subset R^4$ defined as the set

$$P\{\xi_1(\tau), \xi_1'(\tau), \xi_2(\tau), \xi_2'(\tau) \mid \tau = 2\pi\nu^{-1}k, \quad k = 1, 2, 3 \ldots\},$$

can be computed, where $\xi_{1,2}(\tau)$ is a solution of equations (2). Finite approximations of P have been calculated numerically making use of the Gear method [11].

We begin with an example of a strange non-chaotic attractor. Following Grebogi et al. ([7]) strange attractors which are not chaotic are much less common in dynamical systems and it is not easy to discover them. We have found such an attractor for the following fixed parameters: $\nu = 0.4933$, $\varphi = 0.0$, $\alpha_1 = \alpha_4 M = -0.2857$, $\alpha_3 = \alpha_3 M = 3$, $\gamma_1 = \gamma_2 K = 0.0$, $\kappa_1 = \kappa_4 M = -0.8163$, $\kappa_3 = \kappa_3 M = 2$, $B_1 = B_2 = 0.25$ (Fig. 1a). The obtained maximum one-dimensional Lyapunov exponent is $\lambda_{max} = -0.018$. Hence the solution is non-chaotic. Numerically calculated (FFT) power spectra presented in Fig. 1b, are shown to be broad-band, which indicates the strangeness of the attractor. Time histories (Fig. 1c) show how both of the oscillators move in a qualitatively similar, but aperiodic manner. Fig. 1a also presents two projections of the Poincaré map indicating that the determined attractor has a strange geometrical structure. We have taken as control parameter $B_1 = B_2$ and we observed that with a small decrease of this parameter the strange non-chaotic attractor persists. However (with a further decrease of $B_1 = B_2$) it undergoes a bifurcation and a new quasiperiodic attractor is born (see the Poincaré map and power spectra shown in Fig. 2).

Consider another example, with the same parameters as in the previous case, with the fixed value $B_1 = B_2 = 0.2$. For $\alpha_3 = \kappa_3 = 0.001$ a strange chaotic attractor has been detected. In this case $\lambda_{max} = 0.017$. The numerical results are shown in Fig. 3. In this figure we observe an interesting situation. In spite of the coupling of the two oscillators, the type of motion of each oscillator is different. The first one (ξ_1) moves chaotically, whereas for the second (ξ_2) a quasiperiodic motion prevails. Chaotic dynamics of the first oscillator (ξ_1) is demonstrated in Fig. 3c. From comparison of time histories with the Fig. 1c the qualitative differences between strange non-chaotic and strange chaotic attractors are clearly visible. Both of the mentioned attractors have the complicated geometrical structure (strange), but only one has a property that nearby orbits diverge exponentially with time.

Increasing α_3 and κ_3 slightly damps the chaotic dynamics of the orbits and the strange chaotic attractor bifurcates into a quasiperiodic one (for example for $\alpha_3 = \kappa_3 = 0.01$ the Lyapunov exponent $\lambda_{max} = -0.0015$ and two projections of Poincaré map are given in Fig. 4). This attractor persists under the further increase of α_3 and for $\alpha_3 = 0.1$ it is shown in Fig. 5.

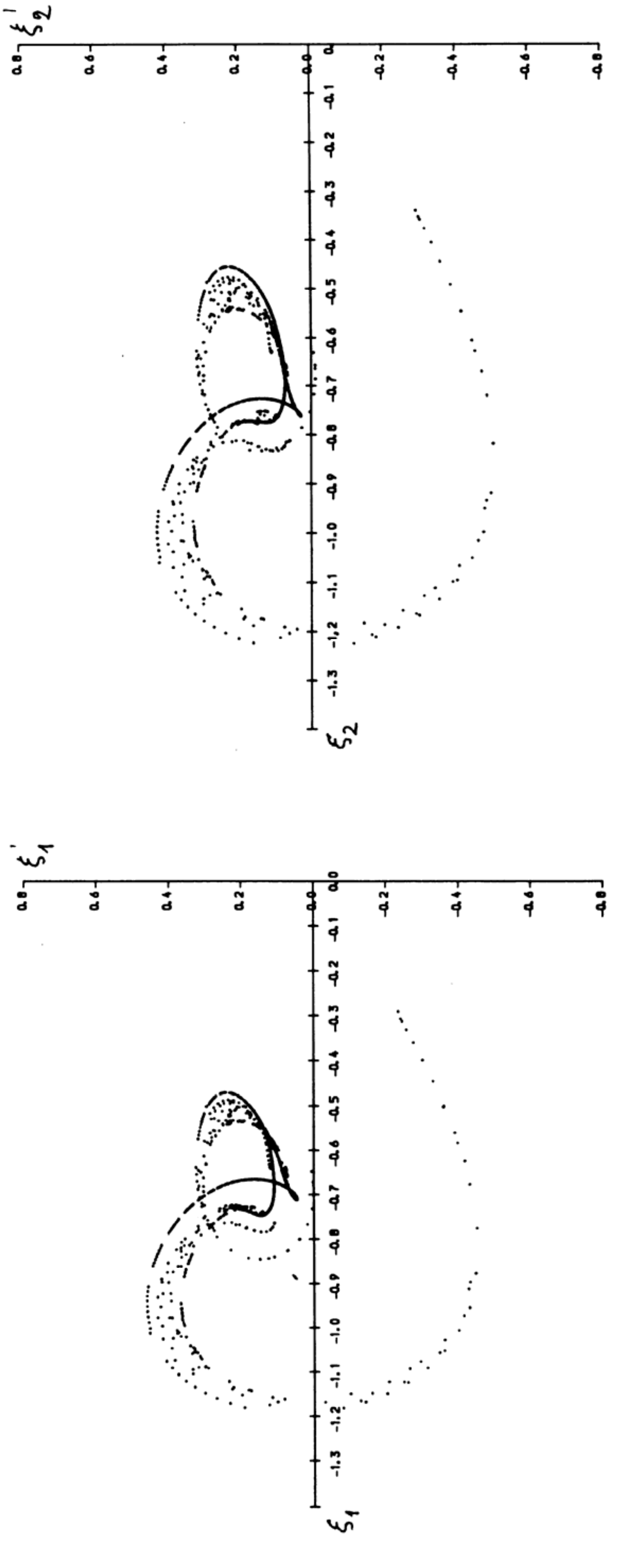

Figure 1: Two projections of the Poincaré map
(a) frequency spectra.

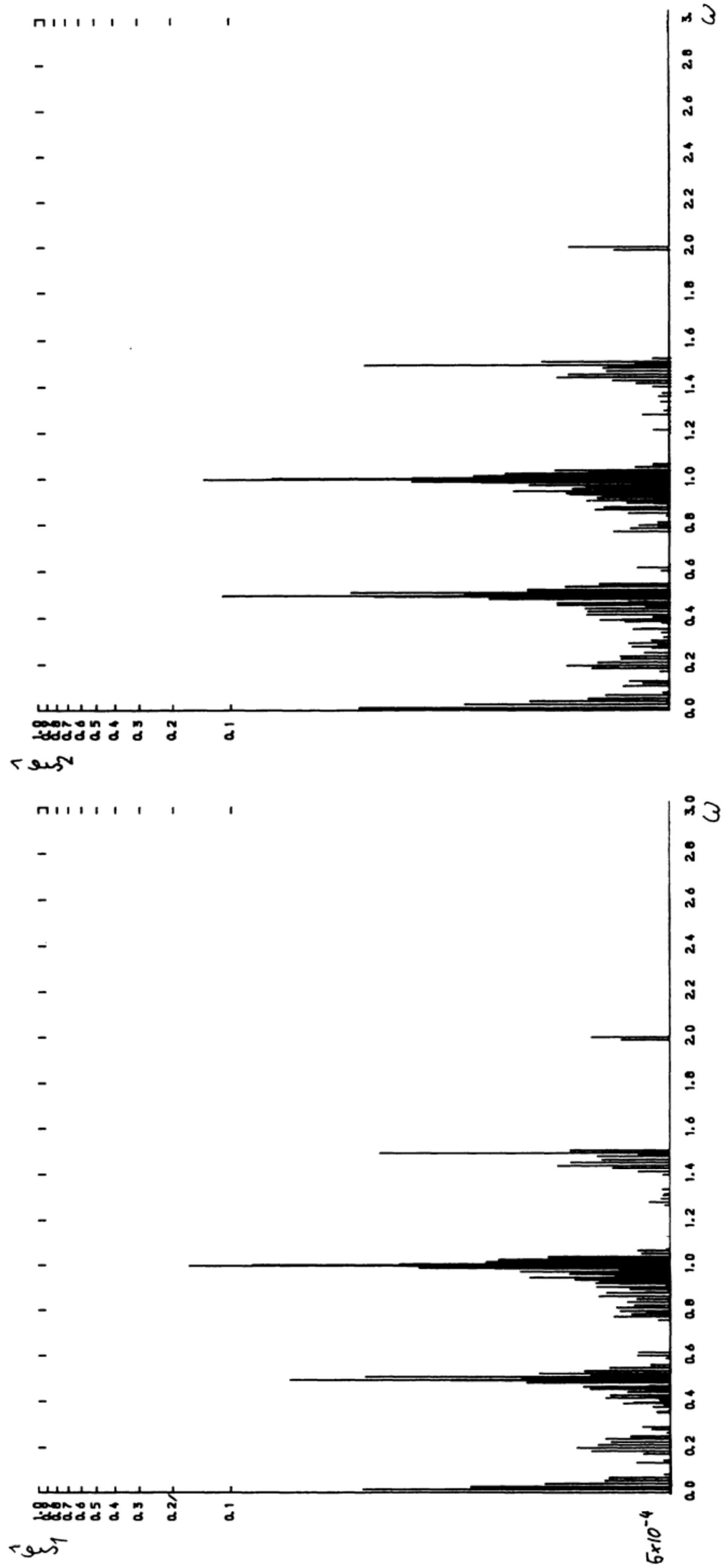

Figure 1: Two projections of the Poincaré map
(b) and time histories.

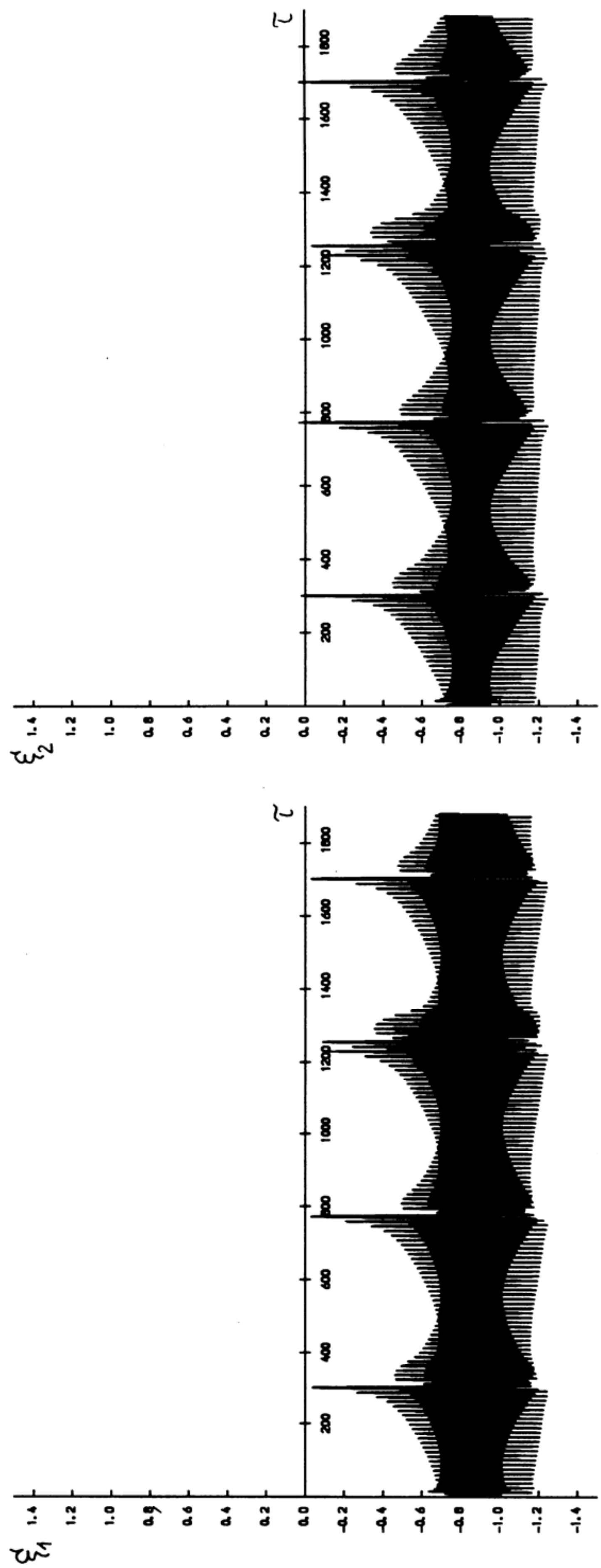

Figure 1: Two projections of the Poincaré map (c) showing the strange non-chaotic attractor.

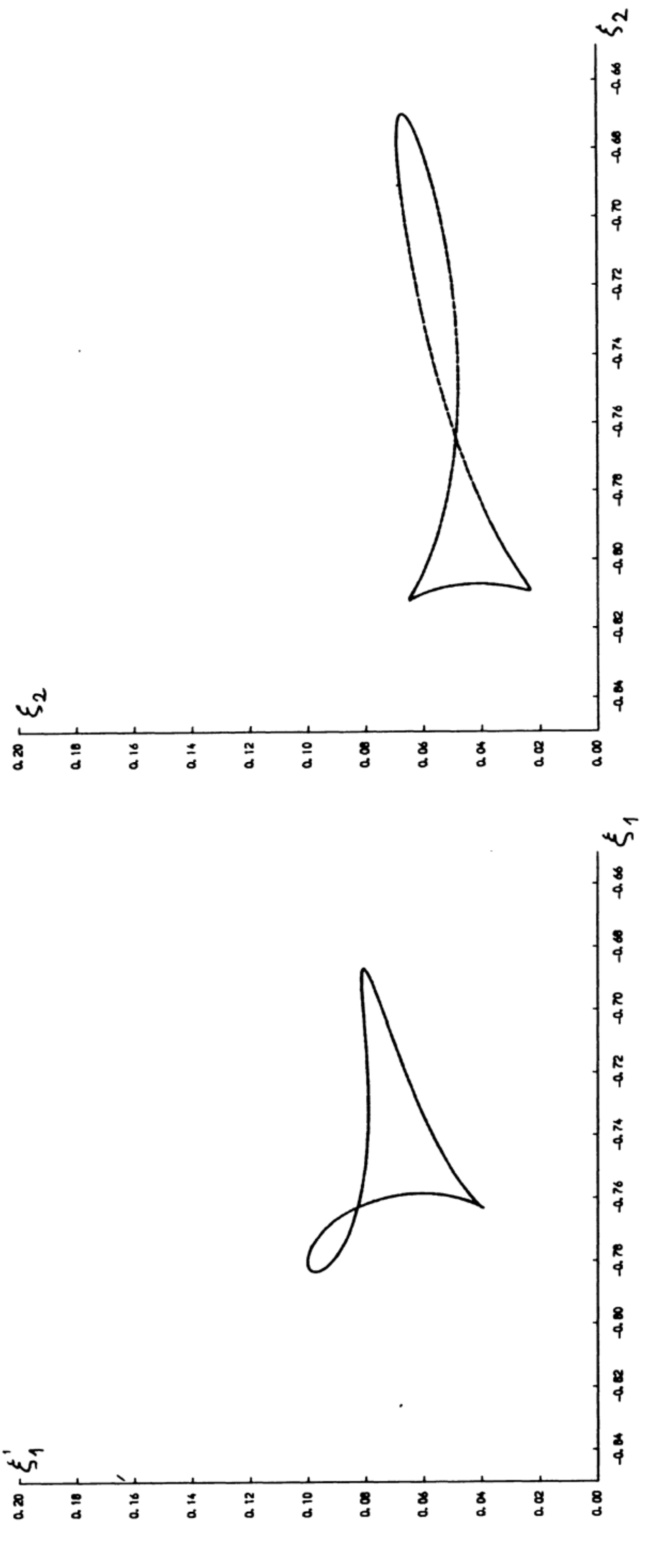

Figure 2: Two projections of the Poincaré map
(a) and frequency spectra.

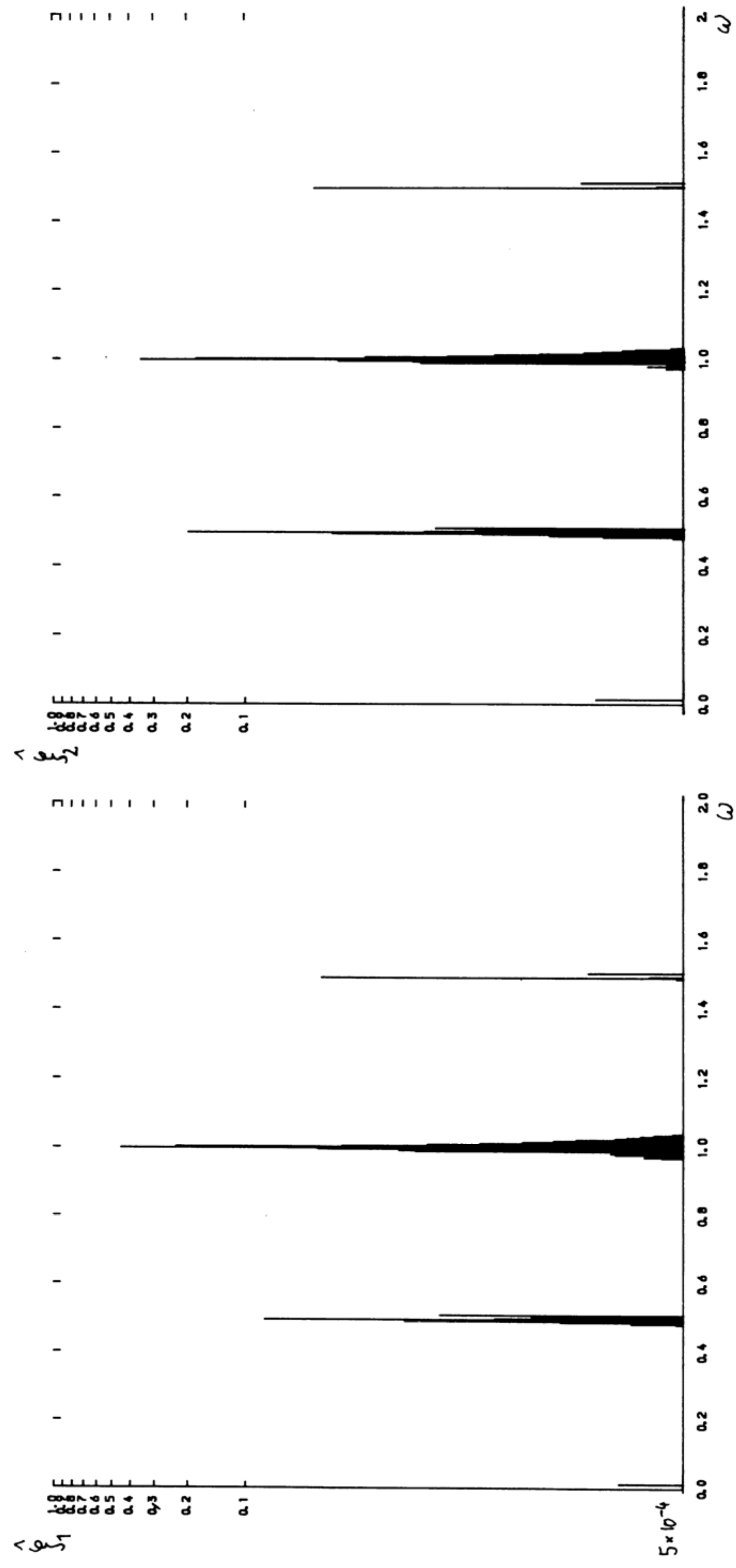

Figure 2: Two projections of the Poincaré map (b) of the quasiperiodic attractor ($B_1 = B_2 = 0.2$).

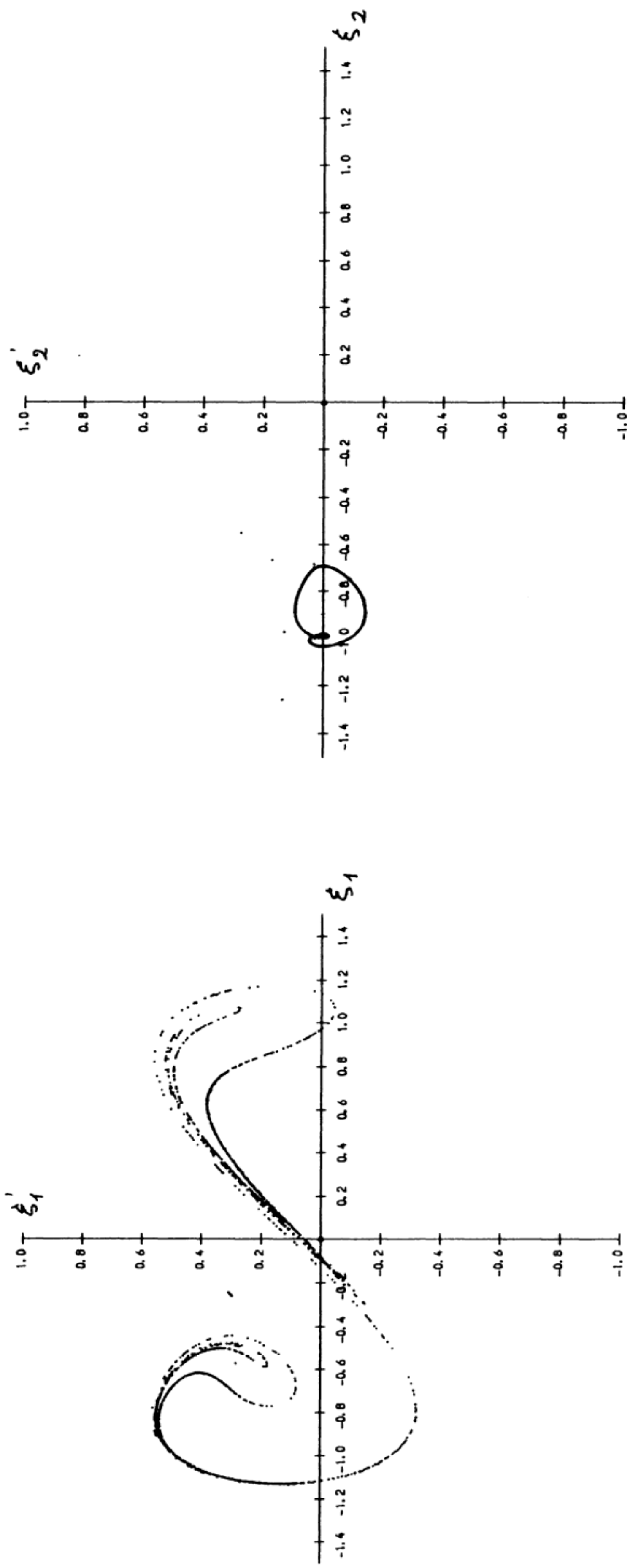

Figure 3: Two projections of the Poincaré map
(a) frequency spectra.

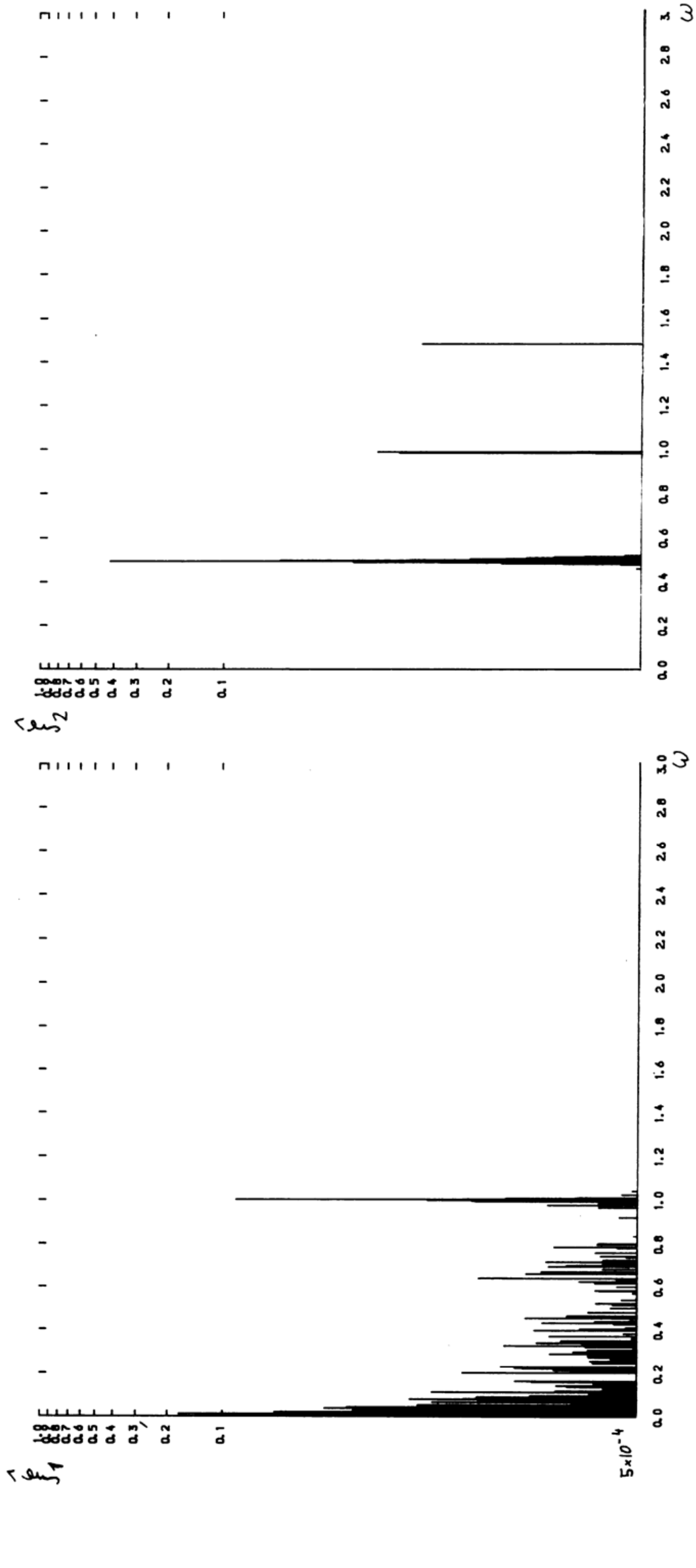

Figure 3: Two projections of the Poincaré map (b) and time histories.

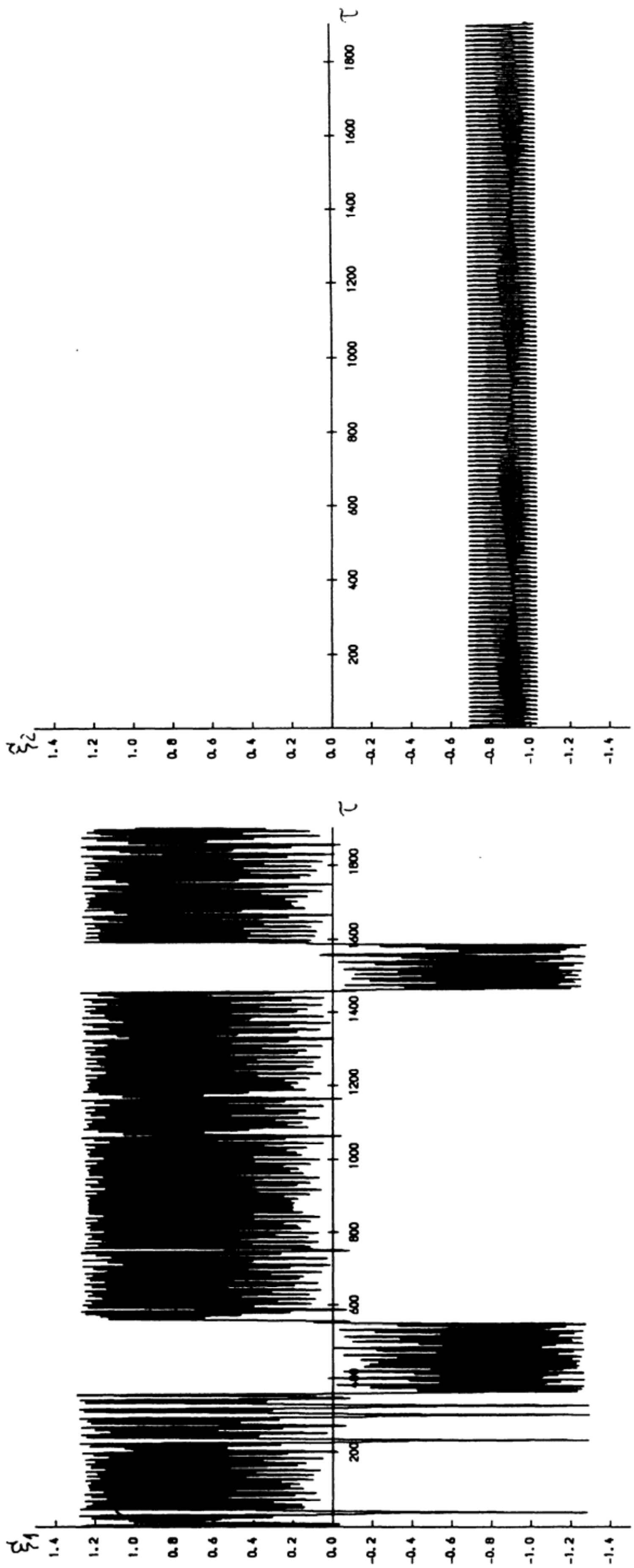

Figure 3: Two projections of the Poincaré map
(c) of the strange chaotic attractor.

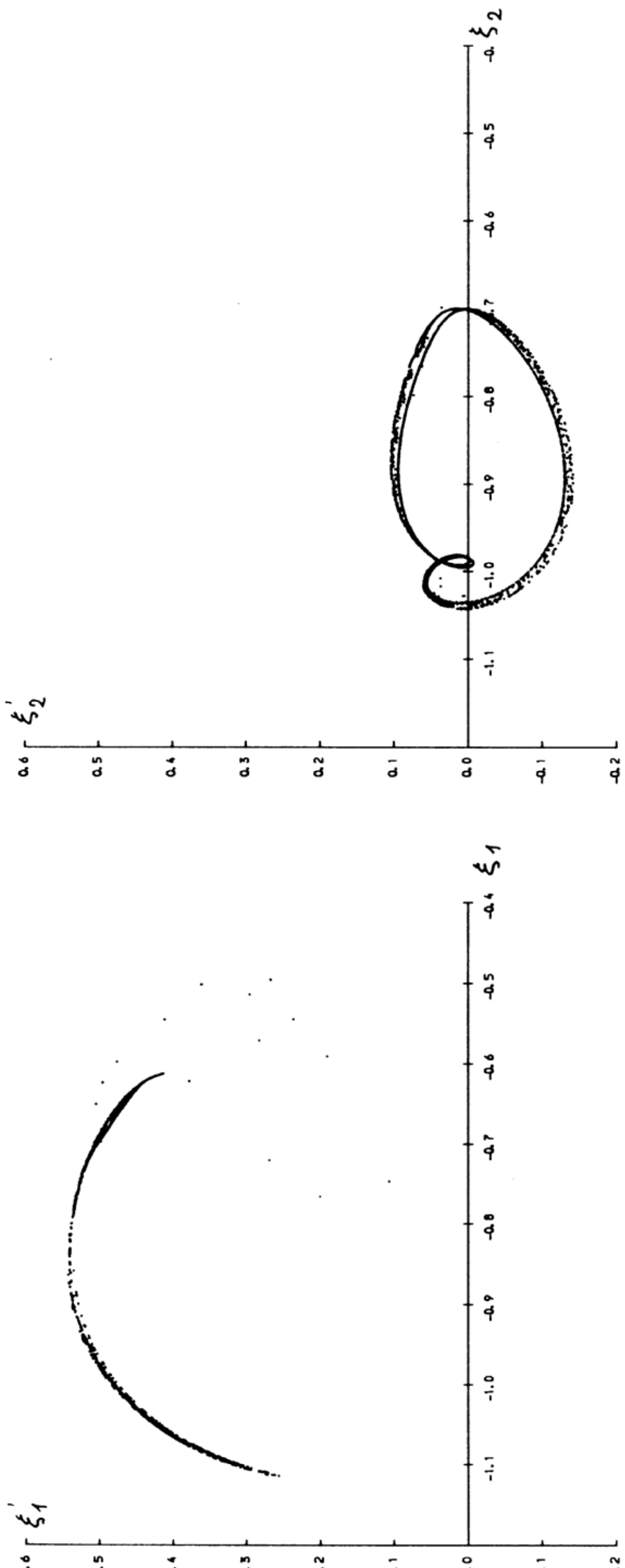

Figure 4
Two projections of the Poincaré map of the quasiperiodic attractor for $\alpha_3 = \kappa_3 = 0.01$.

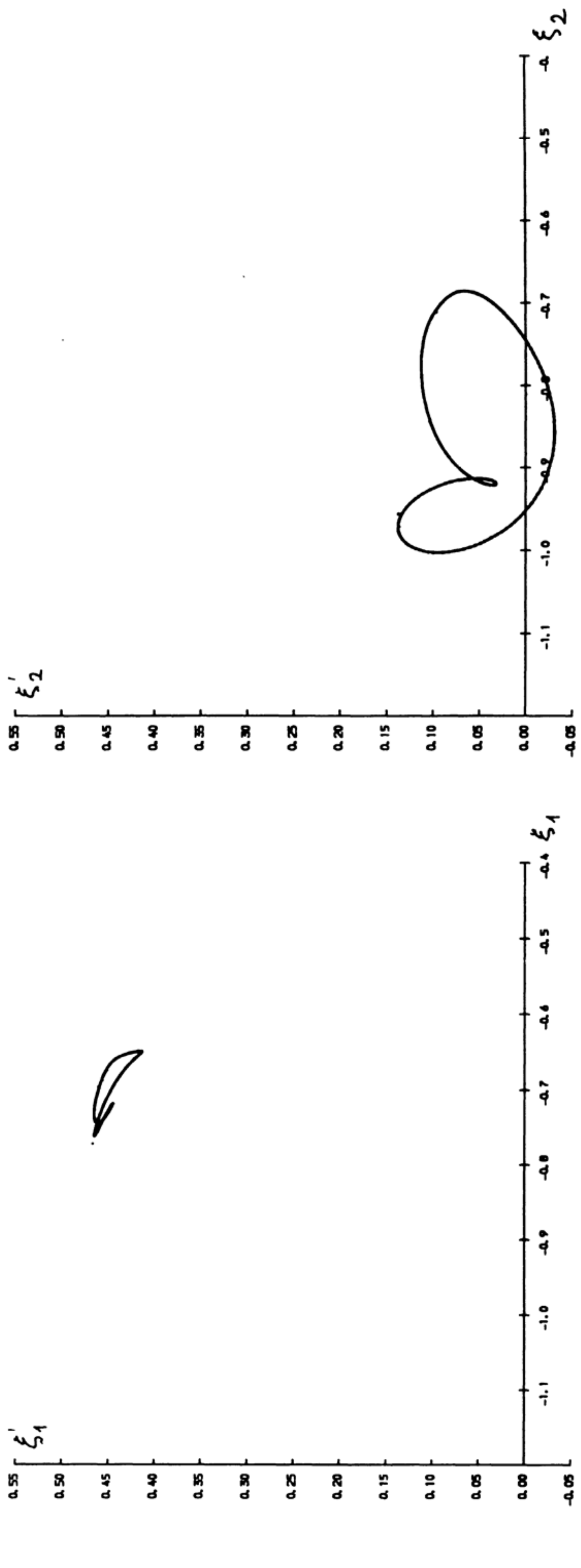

Figure 5: Two projections of the Poincaré map
(a) and frequency spectra.

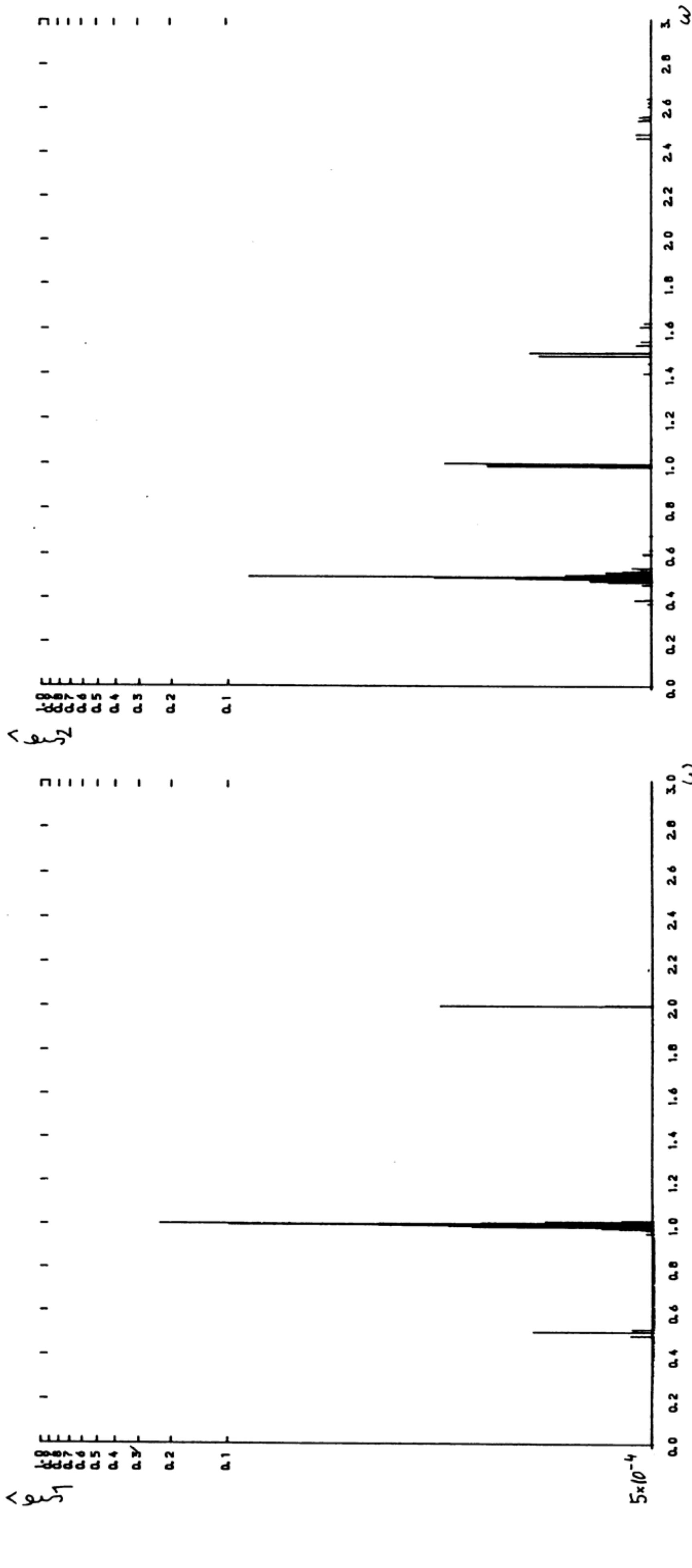

Figure 5: Two projections of the Poincaré map
(b) of the quasiperiodic attractor for $\alpha_3 = 0.1$ and $\kappa_3 = 0.01$.

3. Conclusions

For the investigated system we have discovered strange non-chaotic and strange chaotic attractors as well as quasiperiodic orbits. In the first example we have examined the persistence of the strange non-chaotic attractors for perturbations in the amplitudes of two exciting forces with incommensurable frequencies. For a small decrease of $B_1 = B_2$ the system preserves a strange non-chaotic attractor, which for a further decrease of these parameters is destroyed and is replaced by a quasiperiodic attractor.

In the second example we began with a strange chaotic attractor. We have illustrated that in one of the two coupled oscillators chaotic motion, whereas in the other quasiperiodic motion, prevails. Further we have demonstrated that a transition to a quasiperiodic attractor (with the increase of α_3) can occur.

Acknowledgement

This report was supported by the Alexander von Humboldt Foundation.

References

[1] H. Troger, *Chaotic behaviour in simple mechanical systems*. ZAMM *62*, 18–27 (1982).
[2] S. Nowak and R. G. Fredlich, *Transition to chaos in the Duffing oscillator*. Phys. Rev. *A27*, 3660–3663 (1982).
[3] G. Schmidt, *Onset of chaos and global analytical solutions for Duffing's oscillator*. ZAMM *66*, 129–140 (1986).
[4] J. Awrejcewicz, *Two kinds of evolution of strange attractors for the example of a particular non-linear oscillator*. ZAMP, *40*, 375–386 (1989).
[5] T. Kapitaniak, J. Awrejcewicz and W.-H. Steeb, *Chaotic behaviour of an anharmonic oscillator with almost periodic excitation*. J. Phys. A: Math. Gen. *20*, L355–358 (1987).
[6] O. E. Rössler (private communication).
[7] C. Grebogi, E. Ott, S. Pelikan and J. A. Yorke, *Strange attractors that are not chaotic*. Physica *13D*, 261–268 (1984).
[8] A. Wolf, J. B. Swift, H. L. Swinney and J. A. Vastano, *Determining Lyapunov exponents from a time series*. Physica *16D*, 285–317 (1985).
[9] J. Wright, *Method for calculating a Lyapunov exponent*. Phys. Rev. *A29*, 2924–2927 (1984).
[10] C. Froeschle, *The Lyapunov characteristic exponents and applications*. J. Theor. Appl. Mech. (Numero special), 101–132 (1984).
[11] G. Hall and J. M. Watt, *Modern Numerical Methods for Ordinary Differential Equations*, Clarendon Press, London 1976.

Abstract

Strange non-chaotic, strange chaotic and quasiperiodic attractors are demonstrated to exist for a system of two non-linear coupled oscillators with almost periodic excitations. For same parameter values a transition from a strange non-chaotic to a quasiperiodic attractor is presented, whereas for other parameter values a shift from the strange chaotic attractor to a quasiperiodic one is found.

(Received: July 5, 1989; revised: September 18, 1989)