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LETTERS TO THE EDITOR

SOME COMMENTS ABOUT QUASI-PERIODIC ATTRACTORS

The occurrence of a quasi-periodic torus in non-conservative dynamical systems is of great fundamental importance. After bifurcation of such an attractor, strange chaotic or periodic orbits can appear. These situations are well documented [1]. In this letter we analyze a less common case. The question is "Can the quasi-periodic attractors persist under perturbations?".

Peixoto has shown that a two-frequency quasi-periodic attractor is structurally unstable (see also reference [1]). This means that it is easy to destroy a two-torus by small changes in the system. In the case of N-frequency quasi-periodicity (where $N \geqslant 3$) Newhouse, Ruelle and Takens [2] suggested that arbitrarily small perturbations of such a N-torus convert it to a strange one which is structurally stable.

In this brief report we will investigate two-dimensional tori which sit in four-space, and we will show that in our real mechanical system the investigated tori are structurally stable under relatively large changes of the control parameter.

We consider a non-linear system with two degreees of freedom and almost periodic external excitation:

$$m_1\ddot{x}_1+(c_3-c_1)\dot{x}_1-c_3\dot{x}_2+c_2x_1^2\dot{x}_1+(k_1+k_3)x_1-k_3x_2+k_2x_1^2=q_1\cos(\omega_1 t+\varphi),$$
$$m_2\ddot{x}_2+(c_3-c_4)\dot{x}_2-c_3\dot{x}_1+c_5x_2^2\dot{x}_2+(k_3+k_4)x_2-k_3x_1+k_5x_2^3=q_2\cos\omega_2 t, \quad (1)$$

where m_1 and m_2 are the masses of the vibrating bodies, c_1–c_5 (k_1–k_5) are damping (stiffness) coefficients and q_1 and q_2 are the amplitudes of the excitation forces with frequencies ω_1 and ω_2 respectively.

In non-dimensional form one has

$$\xi_1''+(\alpha_3-\alpha_1)\xi_1'-\alpha_3(K/M)^{1/2}\xi_2'+\gamma_1\xi_1^2\xi_1'+(\kappa_1+\kappa_3)\xi_1-\kappa_3(K/M)^{1/2}\xi_2+\xi_1^3 = B_1\cos(\tau+\varphi),$$
$$\xi_2''+M(\alpha_3-\alpha_4)\xi_2'-M\alpha_3(M/K)^{1/2}\xi_1'+\gamma_2K\xi_2^2\xi_2'+M(\kappa_3+\kappa_4)\xi_2-M\kappa_3(M/K)^{1/2}\xi_1+\xi_2^3 = M^{3/2}K^{-1/2}B_2\cos\nu\tau, \quad (2)$$

where

$$\tau=\omega_1 t,\qquad \xi_1=(\omega_1 m_1)^{-1}k_2^{1/2}x_1,\qquad \xi_2=(\omega_1 m_2)^{-1}k_5^{1}x_2,$$
$$M=m_1m_2^{-1},\qquad K=k_2k_5^{-1},\qquad \nu=\omega_2\omega_1^{-1},\qquad B_1=q_1\omega_1^{-3}m_1^{-3}k_2^{1/2},$$
$$B_2=q_2\omega_1^{-3}m_1^{-3}k_2^{1/2},\qquad \alpha_1=c_1(m_1\omega_1)^{-1},\qquad \alpha_3=c_3(m_1\omega_1)^{-1},$$
$$\alpha_4=c_4(m_1\omega_1)^{-1},\qquad \gamma_1=\omega_1c_2k_2^{-1},\qquad \gamma_2=\omega_1c_4k_2^{-1},\qquad \kappa_1=k_1m_1^{-1}\omega_1^{-2},$$
$$\kappa_3=k_3m_1^{-1}\omega_1^{-2},\qquad \kappa_4=k_4m_1^{-1}\omega_1^{-2}.$$

As equations (2) have the following symmetry under the transformations,

$$T:(\xi_1,\xi_1',\xi_2,\xi_2',\tau)\to(\xi_1,\xi_1',\xi_2,\xi_2',\tau+2\pi\nu^{-1}),$$

we have computed the Poincaré map $M\subset R^4$, defined as the following set:

$$P=\{\xi_1(\tau),\xi_1'(\tau),\xi_2(\tau),\xi_2'(\tau)\,|\,\tau=2\pi\nu^{-1}k,\quad k=1,2,3,\ldots\}.$$

Here $\xi_{1,2}(\tau)$ is a solution of equations (2). Finite approximations of P have been calculated numerically by the use of the Gear method.

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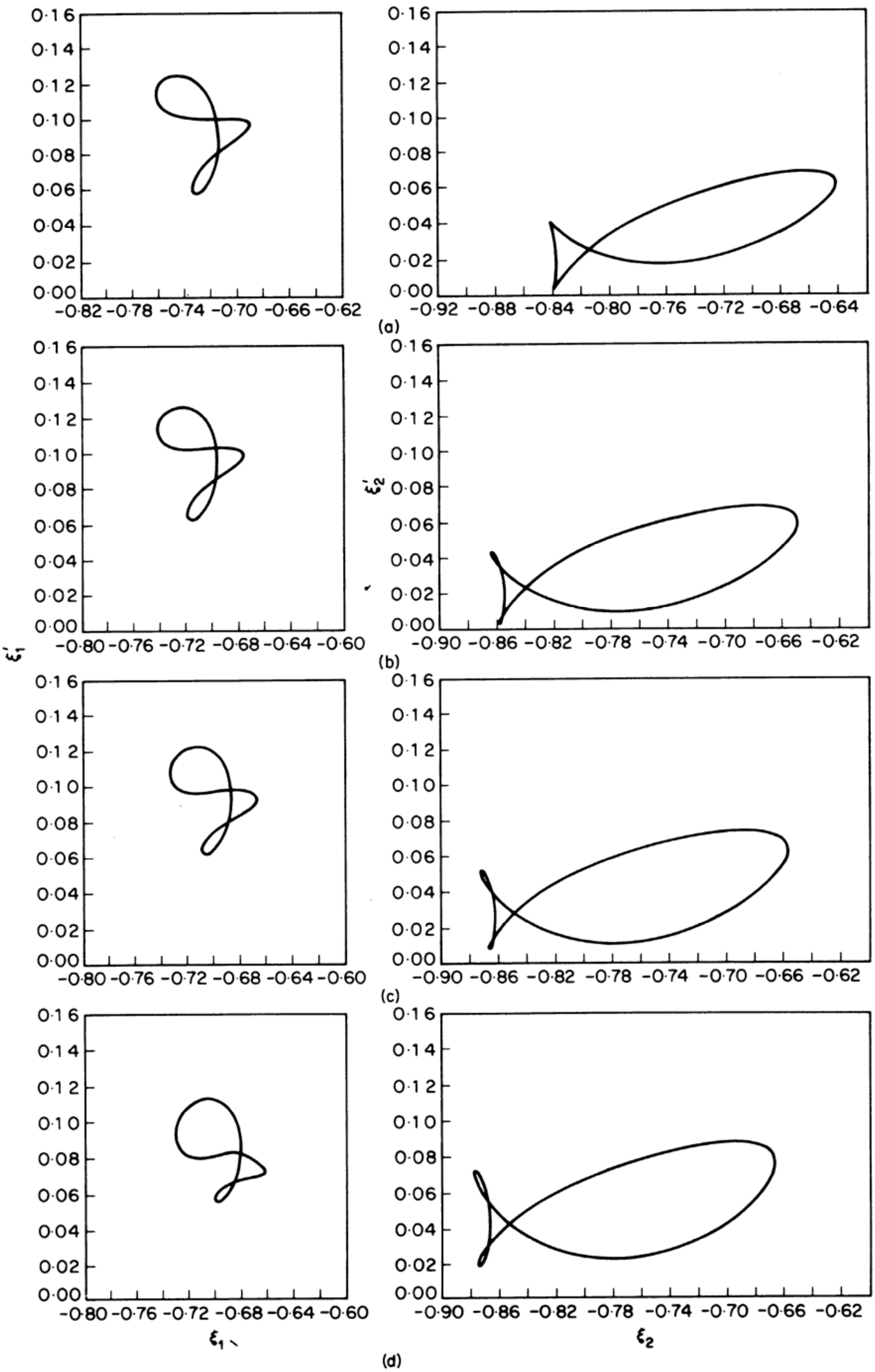

Figure 1. Poincaré maps for $\kappa_3 = 0{\cdot}5$; (a) $\alpha_3 = 1{\cdot}0$; (b) $\alpha_3 = 0{\cdot}5$; (c) $\alpha_3 = 0{\cdot}3$; (d) $\alpha_3 = 0{\cdot}1$.

Consider the behaviour of the system for the following fixed parameters: $B_1 = B_2 = 0{\cdot}2$, $\kappa_1 = \kappa_4 M = -0{\cdot}8163$, $\kappa_3 = \kappa_3 M = 0{\cdot}5$, $\alpha_1 M = 1{\cdot}0$, $\varphi = \gamma_1 = \gamma_2 K = 0$, $\alpha_1 = \alpha_4 M = -0{\cdot}2857$, $\nu = 0{\cdot}4933$. As a control parameter we have chosen α_3 (see Figure 1). For $\alpha_3 = 1{\cdot}0$ two projections of the Poincaré map create two closed curves that are densely covered by points. We have decreased α_3 to the value of 0·1 but the situation has not changed qualitatively. The quasi-periodic torus remains structurally stable.

A similar situation was met when we analyzed the behaviour of system (2) for $\kappa_3 = 0{\cdot}1$ (other parameters are the same as in the previous example). In the interval considered of $\alpha_3 \in [0{\cdot}05; 0{\cdot}1]$ quasi-periodicity persists and the Poincaré map changes gradually with the change of α_3 (see Figure 2).

Tracing the evolution of the power spectra we have discovered another interesting phenomenon. In Figure 3 are shown the amplitudes of the Fourier components against frequency for $\xi_1(t)$ and $\xi_2(t)$. In the first example presented (Figure 3(a)) both variables include the same frequency components—only the magnitudes of the amplitudes are different. The second example (Figure 3(b)) shows that there are frequency components in one of the variables which do not exist in the second one.

To summarize the results demonstrated above, we find that in our real physical system the two-dimensional tori with N-frequency ($N > 3$) are likely to be encountered and cannot be destroyed by small changes in the system. Additionally, we have observed that each of the two coupled oscillators can move with different frequency components.

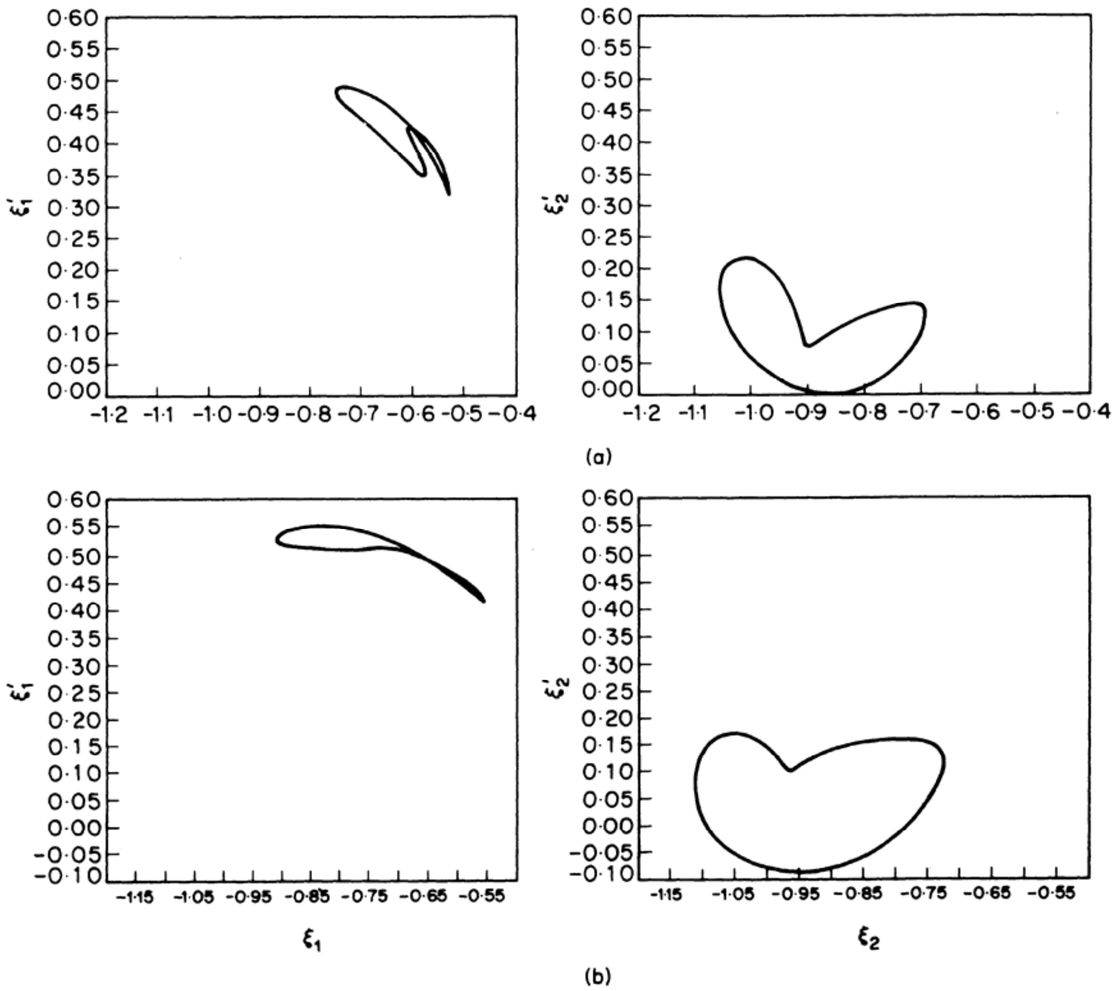

Figure 2. Poincaré maps for $\kappa_3 = 0{\cdot}1$: (a) $\alpha_3 = 0{\cdot}1$; (b) $\alpha_3 = 0{\cdot}05$.

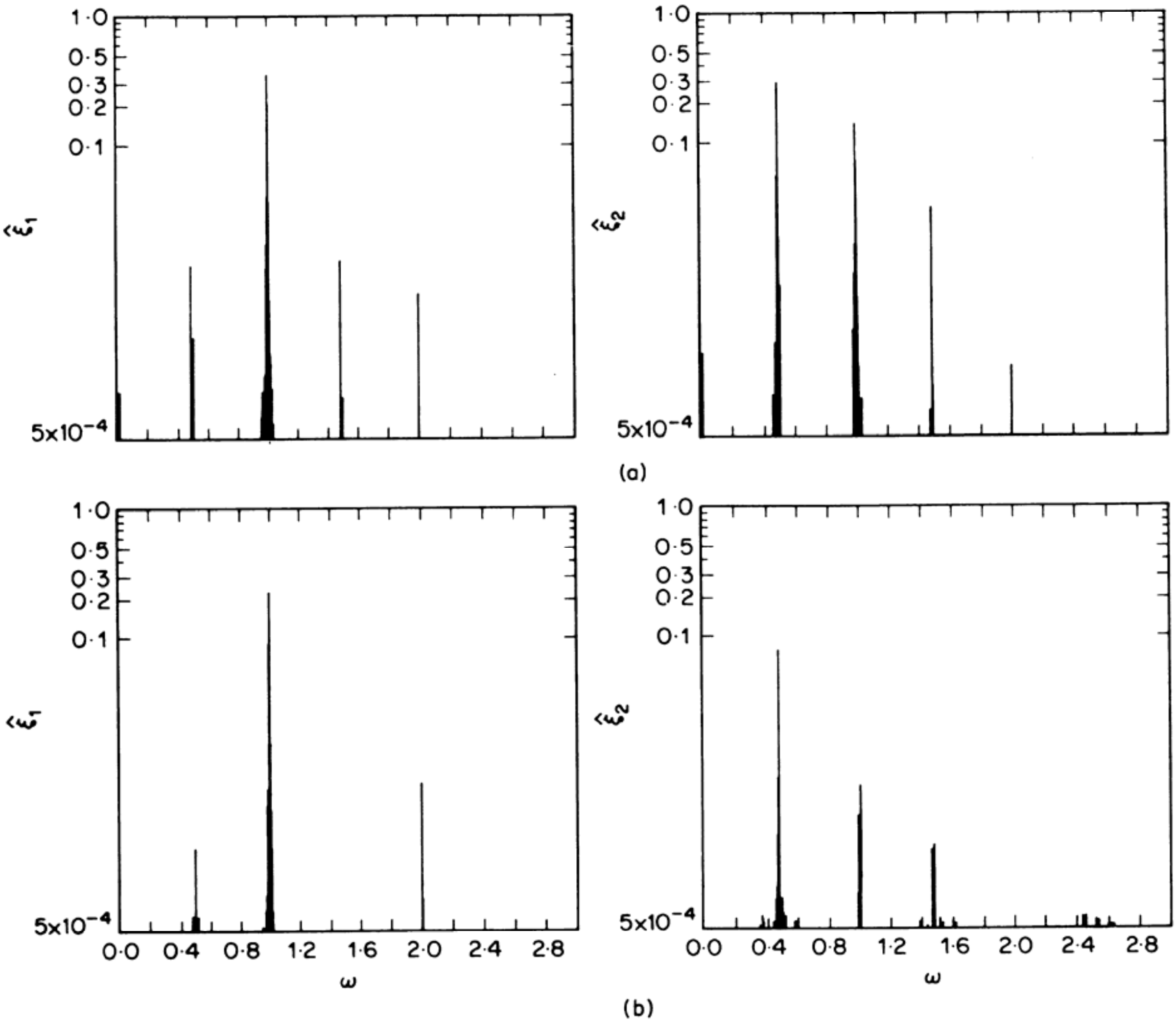

Figure 3. Frequency spectra for the parameters of (a) Figure 1(a) and (b) Figure 2(a).

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