Journal of Sound and Vibration (1990) **137**(1), 159–160

SOME COMMENTS ABOUT STABILITY

Dynamical non-linear systems theory is concerned mainly with stability and bifurcation of fixed points (equilibria or periodic orbits). The aim of this letter is to provide some general descriptions and definitions which allow one also to make satisfactory extensions of the concepts of stability for quasi-periodic or chaotic orbits.

Consider a set of equations

$$dx/dt = F(x), \tag{1}$$

with a continuous right side which satisfies Lipschitz conditions in some part D of n-dimensional Euclidean space, with account taken of these assumptions for $\bigwedge_{x_0 \in D} \bigvee_{x(t,x_0)}$ and $x(0,x_0) = x_0$. This solution of equation (1) is a group because of t: $x(t+\tau, x_0) = x(t, x(\tau. x_0))$. A solution $x(t, x_0)$ for fixed x_0 is a motion. A set $\{x(t, x_0), t \in (-\infty, +\infty)\}$ is a trajectory $x(t, x_0; R)$ of the motion $x(t, x_0)$. A set $x(t, x_0)$, $t \in (0, +\infty)$ is a positive half-trajectory $x(t, x_0; R^+)$, whereas a set $\{x(t, x_0), t \in (-\infty, 0)\}$ is a negative half-trajectory $x(t, x_0; R^-)$. The situation is similar for discrete or continuous phase flow. In this case all definitions given above are true when one replaces x_0 by X_0, where X_0 is a set of initial conditions.

Generally, a set $S \subset D$ is invariant if it contains the trajectories of the system (1) or equivalently $\phi_t(S) = S$, where ϕ is a phase flow [1].

One can define D_1 as a compact set of subsets if it contains a sequence of subsets which converge to a subset A and $A \in D$.

For dissipative dynamical systems, phase flow is contracted. This allows one to take a sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n \to +\infty$, $n \in N$, and $\lim_{n\to\infty} X(t_n, X_0) = X^0 \in D$. The set X^0 which contains $\{X(t, X_0), R^+\}$ is denoted as X^0_+ (a set of attraction) and the set X^0 which contains $\{X(t, X_0), R^-\}$ as X^0_-. Let $O_\varepsilon = O_\varepsilon(X^0_+)$ be ε-values surrounding a set X^0_+ which contain all $X \in E^n$ for which $0 < \rho(X, X^0_+) < \varepsilon$. For $O_\varepsilon(t)$ and $t \to -\infty$ one obtains the domain of attraction of X^0_+.

The defined set X^+_0 is a closed, invariant and indecomposable set, for which $\lim_{t\to+\infty} \rho(X(t, X_0), X^0_+) = 0$. A proof of this theorem is not given here; for more details the reader is recommended to study reference [2].

The set X^0_+ is a limit of a sequence of a compact set of subsets $X(t_n, X_0)$. Note that each subset does not necessarily have to be compact. If a set X^0_+ is not compact and contains orbits which for $t \to +\infty$ diverge from each other at an exponential rate, this is referred to as a strange chaotic attractor. For a chaotic half-trajectory it is easy to give an example showing that it is not compact. It is impossible to take from a sequence $x(t_n, x_0)$ for $n = 1, 2, \ldots$ a subsequence which converges uniformly with $t_n \to +\infty$.

One can define the stability of a set X^0, which can contain all unstable trajectories. A closed invariant set X^0 is stable, if for a given $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that $x(t, O_\delta) \subset (O_\varepsilon \cup X^0)$ for every $t \geqslant 0$. If X^0 is a stable set and additionally $\bigwedge_{x_0 \in O_\delta} \bigvee_{\delta_0} \lim_{t\to+\infty} \rho(x(t, x_0), X^0) = 0$ then X^0 is asymptotically stable. If, additionally, X^0 contains chaotic orbits, in this case one can describe it as an asymptotically stable chaotic attractor.

Let X^0 be a closed invariant set of system (1), and consider small perturbations of this dynamical system. One can use the small parameter ε to denote the small magnitude order of the perturbations. Thus one can write

$$dx/dt = F(x) + F_p(x, t). \tag{2}$$

0022-460X/90/040159+02 $03.00/0

If $F_p(x, t) = F_p(x)$ there is a theorem which defines the behaviour of a new set in the neighbourhood of X^0 (for a proof see reference [2]). This theorem, which is true for a compact set, implies that a perturbation changes an asymptotically stable set X^0 to a new one $X = X(\varepsilon)$, which is closed, asymptotically stable and, for $\varepsilon \to 0$, $X \to X^0$. The same is true for $F_p(x, t)$ if one transforms equations (2) to an autonomous form.

ACKNOWLEDGMENT

This work was sponsored by the Alexander von Humboldt Foundation.

Institute of Technical Mechanics,
Technical University,
Spielmannstrasse 11, 3300 *Braunschweig. West Germany*

J. AWREJCEWICZ

(*Received* 4 *August* 1989)

REFERENCES

1. J. GUCKENHEIMER and P. HOLMES 1983 *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields.* New York: Springer.
2. A. M. SAMOILENKO 1987 *Introduction to a Mathematical Theory of Multifrequency Oscillations.* Moscow: Science. (In Russian.)