VERZWEIGUNG UND CHAOS

ZAMM · Z. angew. Math. Mech. 70 (1990) 4, T 100 – T 101 Akademie-Verlag Berlin

AWREJCEWICZ, J.

Bifurcations of the Oscillations of the Vocal Cords

A mechanical model of the vocal cord was established by CRONJAEGER [1]. This model consists of a nonlinear oscillator with two degrees of freedom with an elastically supported point mass (Fig. 1), and is governed by the set of nonlinear

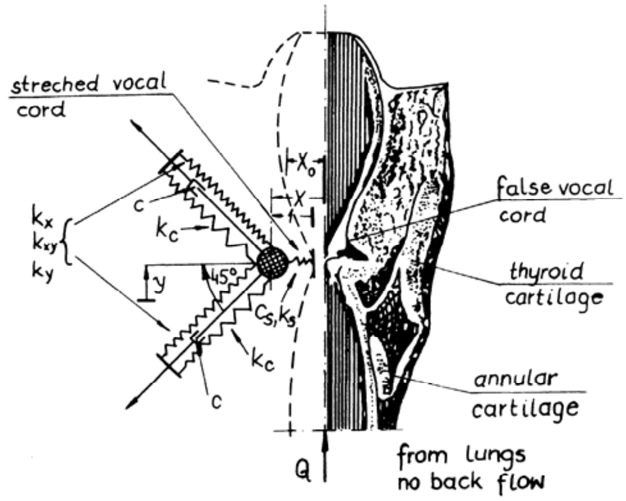

Fig. 1. Mechanical model of the vocal cord

ordinary differential equations which can be presented in the following dimensionless form:

$$\ddot{x} + c\dot{x} + (k_x + k_c((x - X_0)^2 + y^2))(x - X_0) - k_{xy}y - k_s x^{-4}(1 - c_s\dot{x}) = Ep\,,$$
$$\ddot{y} + c\dot{y} + (k_y + k_c((x - X_0)^2 + y^2))\,y - k_{xy}(x - X_0) = Ep\,, \tag{1}$$
$$\dot{p} = Q - \begin{cases}(x-1)\,p^{1/2} & \text{for } x > 1\,,\\ 0 & \text{for } x \leqq 1\,,\end{cases}$$

where c is damping of the vocal cord, k_x horizontal stiffness, k_y vertical stiffness, k_{xy} is a stiffness of the couplings between two directions of motion, k_c cubic type stiffness, k_s hyperbolic type stiffness, c_s damping, X_0 unloaded equilibrium position ($Q = 0$), E average pressure, Q air flow.

Based on equations (1) the systematical approach to trace the behaviour of the system by varying one parameter (in this case c) is presented. First the Hopf bifurcation points are obtained. Then periodic orbits which emanate from Hopf bifurcation points are calculated. By tracing the evolution of the characteristic multipliers it is possible to observe the changes of stability of these solutions and possibly further branching to resonance or quasiperiodic motion. This technique is based on solving a boundary value problem using a shooting method (see BROMMUNDT [2], SEYDEL [3], STABEN [4]).

Calculations were made for the following fixed parameters: $k_x = 1$; $k_c = 0.001$; $X_0 = 0.4$; $k_{xy} = 0.3$; $k_s = 0.001$; $c_s = 0.5$; $E = 0.4$; $k_y = 0.3$; $Q = 10$. We abbreviate the equations of motion to

$$\boldsymbol{h}(z', z, \boldsymbol{\eta}) = \boldsymbol{0}\,, \tag{2}$$

where: $z = (x, x, y, y, p)$ *and* $\boldsymbol{\eta}$ is the vector of parameters. For the constant solution one obtains

$$\boldsymbol{h}(\boldsymbol{0}, z, \boldsymbol{\eta}) = \boldsymbol{0}\,. \tag{3}$$

Equation (3) was solved numerically by Newton's method. In order to investigate their stability the differential equations

$$\Delta z' = \boldsymbol{H}\,\Delta z \tag{4}$$

should be solved (Δz is a vector of small perturbations of the investigated constant solutions). When a pair of complex eigenvalues of $\boldsymbol{H}$ crosses the imaginary axis with nonzero velocity Hopf bifurcation occurs and the new branch of periodic solution is further calculated. Numerical integration yields the fundamental matrix the eigenvalues of which are the characteristic multipliers.

During the calculations the frequency ω ($\tau = \omega t$) enters the equations as a parameter to be kept fixed when we look at a special solution. Because the system of equations (1) is autonomous a phase condition $\dot{x} = 0$ is prescribed in order to fix the unknown frequency ω. During numerical integration because of the nonlinear term x^{-4}, the standard methods based on the Runge-Kutta algorithm are not sufficiently accurate for integration of the considered stiff equations system. A variable-order, variable-step Gear method was used.

The branching diagram for damping $0.0 < c < 0.4$ is presented in Fig. 2. The stable solutions are marked with the continuous line, whereas the unstable with the dashed one. The previous stable steady-state solution becomes unstable and the periodic limit cycle emanates from the Hopf bifurcation point H_1. At this point a new periodic solution has a

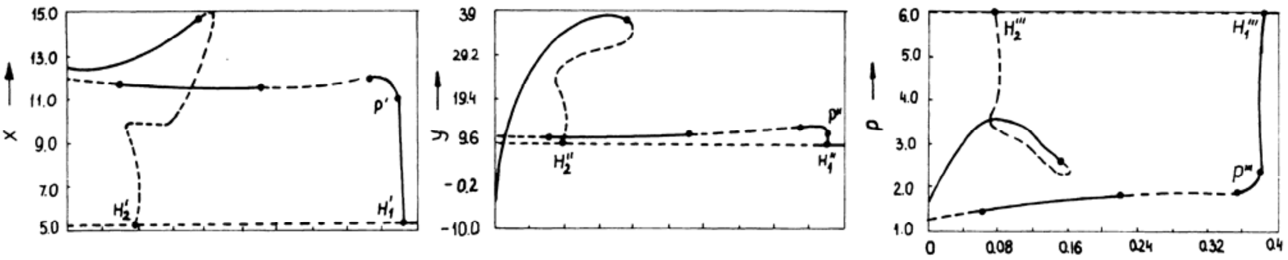

Fig. 2. Bifurcation diagram

frequency which is equal to the imaginary eigenvalue obtained from (4). In the begining this solution is stable and further undergoes the successive changes into subharmonic resonances. All marked bifurcation points are those of period doubling bifurcation (one multiplier crosses the unit circle of the complex plane at -1) except at the point P. At this point the pair of purely imaginary multipliers crosses simultaneously the unit circle of the complex plane.

From the second Hopf bifurcation point H_2 branches the second periodic solution which is unstable. This solution has changed its stability at the period doubling point. Two examples of periodic orbits calculated in a such way are given in Fig. 3.

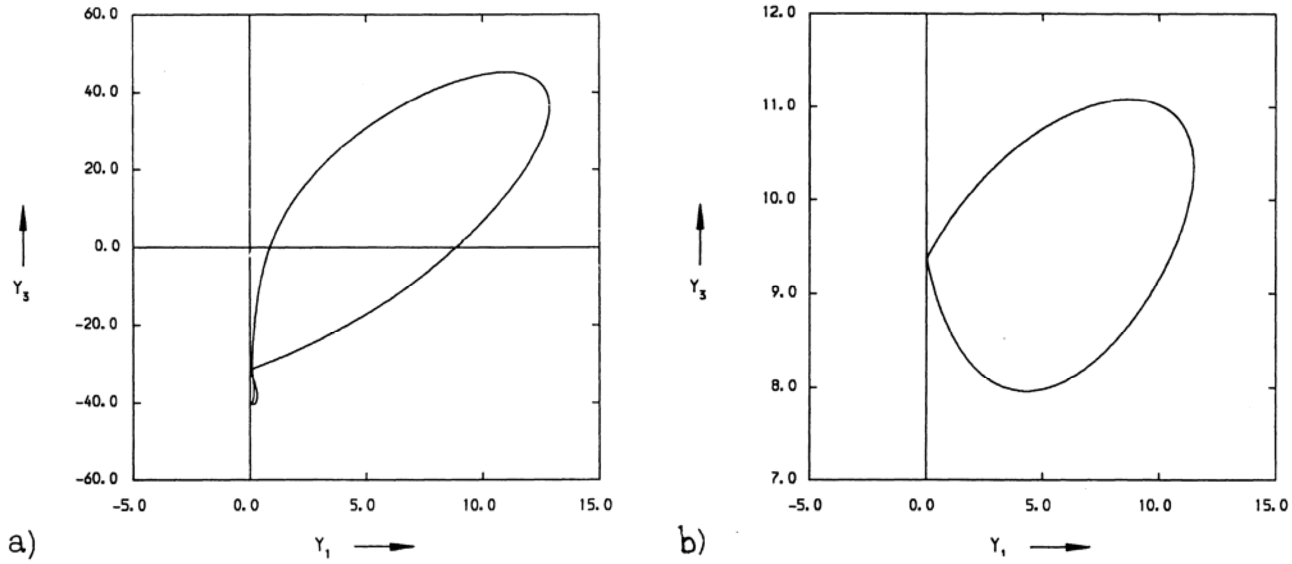

Fig. 3. Examples of periodic oscillations of the vocal cords: a) $c = 0.08$, $\omega = 1.2755$; b) $c = 0.15917$, $\omega = 1.41353$. Note, that orbit (a) has lower frequency compared with (b) and that both possess a cusp near $y_1 = 0$ ($y_1 = x$; $y^3 = y$)

In some narrow interval of damping there exist two independent stable periodic solutions. It is possible to "jump" from one to another.

This work was supported by Alexander von Humboldt Foundation.

References

1 Cronjaeger, J.: Die Entstehung des primären Stimmklangs im menschlichen Kehlkopf – ein Modell. Ph.D. Thesis, Braunschweig 1978.
2 Brommundt, E.: Bifurcation of self-excited rotor vibrations. VII Intern. Conf. Nonl. Vibr., Berlin 1975, Akademie-Verlag, Berlin 1977, pp. 123–134.
3 Seydel, R.: From equilibrium to chaos: Practical bifurcation and stability analysis. Elsevier, New York 1988.
4 Staben, H.: Einfluß des Lagerspiels auf die unwuchterregten Schwingungen eines horizontal gelagerten Rotors. ZAMM **68** (1988) 4, T 122–T 124.

Address: Dr. Jan Awrejcewicz, Technical University, Institute of Applied Mechanics, Stefanowskiego 1/15, 90-924 Łódź, Poland. Presently with: Institut für Technische Mechanik, TU – Braunschweig, Spielmannstr. 11, D-3300 Braunschweig, F.R.G.