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Gradual and Sudden Transition to Chaos in a Sinusoidally Driven Nonlinear Oscillator

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Two different scenarios leading from periodic orbits to chaotic orbits are traced, based on the shooting method. Starting with the 1/2 resonance solution, the sudden appearance of the strange chaotic attractor with increase of the static load, is discussed and illustrated. For this case the multiplier crosses the unit circle of the complex plane at +1. When the load decreases however the solution first goes through three successive period doubling bifurcations to the fourth subharmonic resonance. This subharmonic resonance becomes particularly sensitive with further decrease in the load and then the motion becomes chaotic.

§1. Introduction

A Duffing-type asymmetric oscillator governed by the equation

$$\ddot{x}+c\dot{x}+x^3=q+F\cos\omega t, \qquad (1)$$

is reconsidered with the use of a numerical technique presented in the next section. This oscillator was investigated by Ueda.[1] He presented strange attractors for $\omega=1$, $F=0.16$, $c=0.15$, $q=0.03$ and $q=0.045$. These results were obtained using the numerical method based on solving the initial value problem. Szemplinska-Stupnicka and Bajkowski[2] examined the behaviour of this oscillator by means of an approximate analytical method. The results obtained by them were then verified by the computer simulation analysis. In some cases, approximate analytical methods allow one to find the parameter space for which chaos appears with reasonable accuracy (see Awrejcewicz[3]). However, when the phase flow is very contracted and the bifurcations appear in a very small interval of the investigated parameter, the use of analytical approach to trace the scenario leading to chaos seems to be impossible. Additionally, the numerical technique based on solving the initial value problem is not only inconvenient, but also does not allow one to trace the behaviour of the system in a systematic way. For this reason this work is concentrated on the numerical technique which by the use of the shooting method and by solving the resulting boundary value problem, allows one to trace bifurcations of the periodic orbits examined.

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§2. Method and Results

In this paper local bifurcations of periodic orbits in the q parameter line are investigated for the following fixed values of parameters: $c=0.15$, $\omega=1.0$, $F=0.16$. The flow of (1) is strictly contracted, which implies only the saddle-node or period doubling local bifurcations exist. Hopf bifurcation are excluded. We assume that the frequency $k^{-1}\omega$ of the periodic orbit we want to find is known. Let $y^{(1)}$ be the initial point of an integration of the considered eq. (1) which is sufficiently close to a periodic orbit. If we integrate numerically over an integration time of $T=k\omega^{-1}$ (shooting), we can consider this procedure as a mapping $G(y^{(1)})=G^{(1)}$. For the periodic orbit a point y_p has the mapping $G(y_p)=y_p$. In other words, y_p is a fixed point for this mapping. As $y^{(1)}$ is only an approximation of y_p, an error $E=y^{(1)}-G^{(1)}$ will occur. Employing a Newton-Raphson procedure one can try to find the zeros of the error function E.

The determination of stability is reduced to the analysis of the variational equations of the non-linear system, which is a system of linear differential equations with periodic coefficients. This is true as long as a non-zero variational (Jacobian) matrix $\boldsymbol{J}$ exists:

$$\Delta p = \boldsymbol{J}(\tau)\Delta p = \frac{\partial y'}{\partial y}\bigg|_{y_p(\tau)} \Delta p, \qquad (2)$$

where $\boldsymbol{J}(\tau + 2\pi k/\omega) = \boldsymbol{J}(\tau)$ and Δp is a perturbation vector of the fixed point y_p. The general solution of (2) is

$$\Delta p = \phi(\tau) \cdot \Delta p(0), \qquad (3)$$

where $\phi(\tau) = \phi(\tau + 2\pi k/\omega)$ is the fundamental (Floquet) matrix. The characteristic equation

$$\chi(\sigma) = \det(\boldsymbol{M}_F - \sigma_1) = 0,$$
$$\boldsymbol{M}_F = \phi(2\pi k/\omega) = \frac{G_F}{y_F}, \qquad (4)$$

yields the characteristic multipliers σ. $\boldsymbol{M}_F$ must be determined by solving the system (2) along a known periodic solution of the non-linear system over T.

For the considered oscillator, in order to investigate the k-subharmonic solution, the integration interval of (1) is transformated to $2\pi k$ length and by the use of shooting it is possible to solve the following boundary value problem and to obtain multipliers from equations

$$M_i^{(m)}(x_1, x_2, q) - x_i = 0,$$
$$\chi[DM^{(m)}(x_1, x_2, q), \delta] = 0,$$
$$x_1 = x,\ x_2 = \dot{x},\ i = 1, 2. \qquad (5)$$

Above, $M_i^{(m)}$ denotes the ith component of the stroboscopic phase portrait (Poincaré map) of (1) with the nondimensional time $\tau = k\omega t$ and χ is the characteristic polynomial whose eigenvalues σ_i are the sought multipliers of the $2k\pi$ periodic Floquet matrix.[4] Calculations were interrupted at the mth step, only if the following norm $\|M_i^{(m)} - x_i\| = \Sigma |M_i^{(m)} - x_i|^2 \leqslant 10^{-6}$. Two multipliers are either real or complex conjugates. We will trace the movement of those multipliers which accompany the change of q.

The main resonance solution is presented in Fig. 1. As is seen from Table I, the increase of q up to $q = 0.5$ has almost no effect on the behaviour of multipliers. However, for the interval of q considered, another solution exists

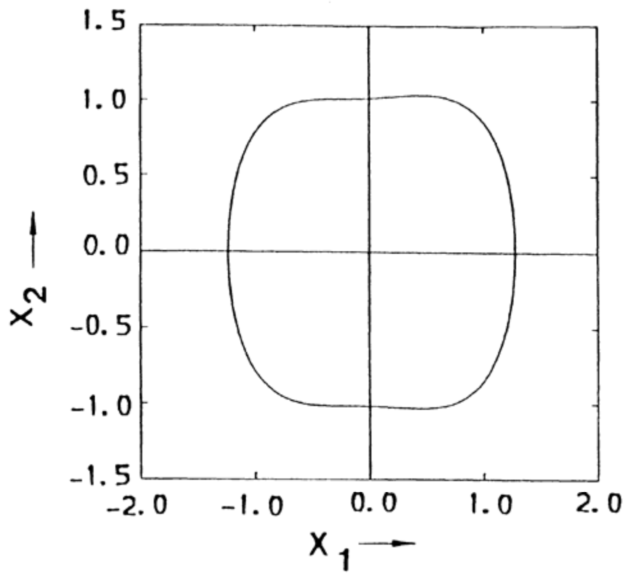

Fig. 1.

Table I.

q	x	$\dot{x}$	Multipliers 1	Multipliers 2
0.03	1.1406	0.65757	0.21; 0.83	0.21; −0.83
0.035	1.1432	0.66083	0.22; 0.82	0.22; −0.82
0.04	1.1458	0.66408	0.21; 0.83	0.21; −0.83
0.045	1.1483	0.66733	0.21; 0.83	0.21; −0.83
0.05	1.1509	0.67057	0.21; 0.83	0.21; −0.83

(see Table II). For $q \in (0.042385; 0.03913)$ this is the $1/2$ subharmonic solution. There are two possible routes to chaos for this solution (small orbit). The violent burst into a strange chaotic attractor appears on raising q more than 0.042385. For this value of q the multiplier crosses the unit circle of the complex plane at $+1$ (a saddle-node bifurcation) and suddenly another complicated structure of the phase flow is born (Fig. 2). With decreasing q, three steps in the subharmonic scenario were observed. In all the three sequences of doubling the period of bifurcations, each new solution is stable over an interval smaller than the interval of stability of the preceding member of sequence. This observation leads to the brief discussion of the compatibility of results presented with the Feigenbaum Universality Constant.[5] Feigenbaum analyzed noninvertible one-dimensional maps $y_{n+1} = F_\lambda(y_n)$, where λ is a parameter. He showed that there is a universal number given by

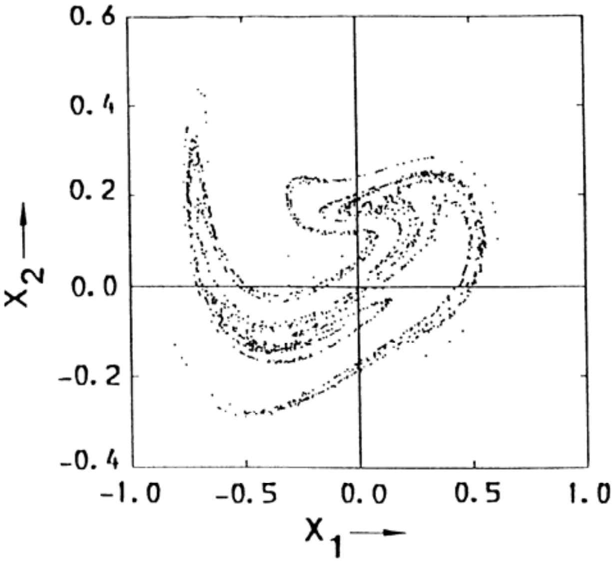

Fig. 2.

$$\delta=\lim_{n\to\infty}\delta_n=\lim_{n\to\infty}\frac{\lambda_n-\lambda_{n-1}}{\lambda_{n+1}-\lambda_n}=4.6692.$$

Based on results of calculation given in Table II the presented sequence of the doubling period does not converge with the Feigenbaum Constant (in our case $\delta=0.5079$). The Feigenbaum universal number has been derived for only one-dimensional maps and although many higher real dimensional physical systems have been found to possess this property, there are also numerical examples showing that in some systems a period-doubling scenario is not compatible with the Feigenbaum one.[6)]

Decreasing a slightly further leads to for $q=0.037$ a strange chaotic attractor (Fig. 3).

§3. Conclusions

Two scenarios leading to chaos in the Duffing type oscillator are investigated with respect to static load by solving the boundary value problem. For the considered interval of q there exist two independent (main resonance and 1/2 subharmonic) solutions. The 1/2 resonance burts suddenly into chaos if, with increase of the dimensionless static load q, one of the multiplier passes through the unit circle at $+1$ (a saddle-node bifurcation).

Increasing q draws successive subharmonic bifurcations. However, starting with 32π period the system is extremely sensitive to very small changes of q, making further visualisation very hard. Already for $q=0.037$ chaos has been found (note, that for $q=0.038029$ a 32π periodic orbit is born).

To summarize the results presented, it is shown how in two different ways the 1/2 subharmonic solution (small orbit) can reach chaos with changing q. With increase of q, the

Table II.

q	x	$\dot{x}$	Multipliers 1		Multipliers 2		Period
*) 0.042385	−0.26362	−0.13127	1.0 ;	0.0	0.54;	0.0	
0.042382	−0.26156	−0.13080	0.78;	0.0	0.09;	0.0	
0.042380	−0.26071	−0.13061	0.72;	0.13	0.72;	−0.13	
0.042000	−0.23492	−0.12428	0.27;	0.67	0.27;	−0.67	
0.041500	−0.22024	−0.12033	0.00;	0.73	0.00;	−0.73	4π
0.040000	−0.19277	−0.11227	−0.54;	0.49	−0.54;	−0.49	
0.039300	−0.18314	−0.10923	−0.65;	0.00	−0.83;	0.0	
*) 0.039190	−0.18173	−0.10877	−0.54;	0.00	−1.0 ;	0.0	
0.039150	−0.18966	−0.10992	0.33;	0.0	0.87;	0.0	
0.039110	−0.19487	−0.11066	0.52;	0.10	0.52;	−0.10	
0.039000	−0.20149	−0.11156	0.37;	0.38	0.37;	−0.38	8π
0.038300	−0.22495	−0.11436	−0.70;	0.0	−0.40;	0.0	
*) 0.03822	−0.226810	−0.11455	−1.0 ;	0.0	−0.27;	0.0	
0.03815	−0.235540	−0.11602	−0.01;	0.24	−0.01;	−0.24	
0.03808	−0.214580	−0.11210	−0.28;	0.32	−0.28;	−0.32	
0.03803	−0.21257	−0.11165	−0.23;	0.0	−0.88;	0.0	16π
*) 0.038029	−0.21253	−0.11164	−0.08;	0.0	−1.0 ;	0.0	

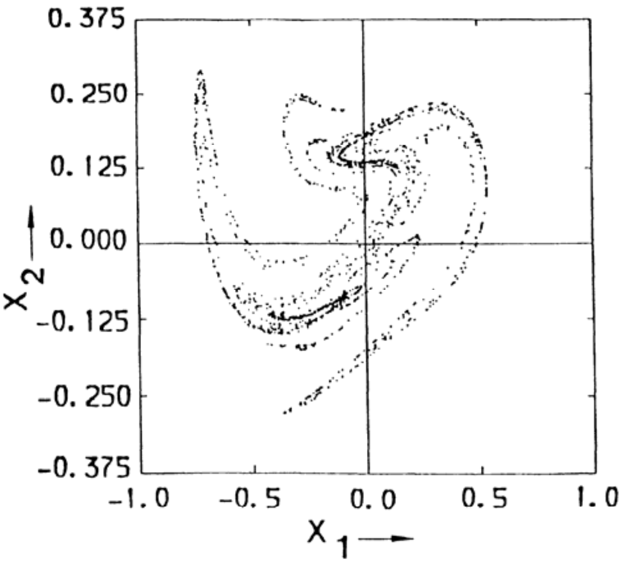

Fig. 3.

strange attractor appeared suddenly and the previous periodic orbit disappeared. Contrary to this scenario, another accompanying decrease in q leads gradually to chaotic motion. Additionally, all of the subharmonic solutions obtained exist further only their stability has changed.

The main resonance (large orbit) has no influence on the behaviour of the small orbit and is stable in the interval of q considered. The transition between the main resonance and the strange chaotic attractor is only possible with a sudden change in initial conditions—"jump" phenomenon. The paper was supported by the Alexander von Humboldt Foundation.

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