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AN ANALYTICAL METHOD FOR DETECTING HOPF BIFURCATION SOLUTIONS IN NON-STATIONARY NON-LINEAR SYSTEMS

1. INTRODUCTION

An analytical method is presented for determining the parameter–frequency relations and the dependency of the bifurcation parameter on the other parameters for non-stationary non-linear systems, including also the case when the system is excited by a periodic force. This method is based on the combination of the classical harmonic balancing method and the perturbation method. The perturbation technique has been described, for example, by Malkin [1] and by Nayfeh [2, 3] and the particular perturbation techniques for analyzing non-stationary non-linear systems with and without external periodic force have been presented in references [4, 5]. A similar technique was applied by the author [6] to determine the post-critical family of solutions after Hopf bifurcation in non-linear non-autonomous oscillators with one bifurcation parameter, it being assumed that the amplitude of the exciting force was small. An analytical approach to the bifurcation phenomena seems to be important for two reasons. The bifurcations are the door to chaotic behavior and full knowledge of these phenomena allows one to know more about the occurrence of chaotic orbits. Secondly, and in any case during numerical integration of the ordinary non-linear differential equations, sometimes the successful calculations cannot be continued through a singularity which is often caused by bifurcations. In this case an analytic treatment must be considered.

2. EXAMPLE 1 (WITHOUT EXTERNAL FORCE)

Consider a vibrating mechanical system with one degree of freedom—a rotor with unequal moments of inertia of its cross-section, with its mass concentrated in its center. The equation of motion of the system has the form

$$m\ddot{y}+\tfrac{1}{2}(k_1+k_2+(k_1-k_2)\cos 2\omega t)y+k_0y^3=0, \tag{1}$$

where m is the concentrated mass, k_0 is the non-linear rigidity, k_1, k_2 are the rotor rigidities, and ω is the rotation frequency of the rotor. After a change of variable, a dimensionless form of the equation is obtained,

$$\mathrm{d}^2x/\mathrm{d}\tau^2+(\delta^2+\mu\cos 2\tau)x+\xi x^3=0, \tag{2}$$

where

$$x=y(k_1/k_0)^{-1/2}, \qquad \tau=\omega t, \qquad \xi=k_1/(m\omega^2),$$

$$\delta^2=\tfrac{1}{2}(k_1+k_2)/m\omega^2, \qquad \mu=\tfrac{1}{2}(k_1-k_2)/(k_1+k_2)\omega^2.$$

Equation (2) can be expressed as a system of two differential equations of the first order,

$$\dot{x}_1=x_2, \qquad \dot{x}_2=-\delta^2x_1-\mu x_1\cos 2\tau-\xi x_1^3, \tag{3}$$

where $x_1=y$.

The bifurcation solutions are sought in the form of a series in the perturbation parameter ε connected with the vibration amplitude, of the form

$$x_i=x_i^0+\varepsilon x_i'+\tfrac{1}{2}\varepsilon^2x_i''+\tfrac{1}{6}x_i'''\varepsilon^3+\cdots, \tag{4}$$

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and the frequency δ^2 and the parameter μ also can be developed into power series in the perturbation parameter,

$$\delta^2 = \delta_c^2 + \varepsilon\delta' + \tfrac{1}{2}\varepsilon^2\delta'' + \tfrac{1}{6}\varepsilon^3\delta''' + \cdots, \tag{5}$$

$$\mu = \mu_c + \varepsilon\mu' + \tfrac{1}{2}\varepsilon^2\mu'' + \tfrac{1}{2}\varepsilon^3\mu''' + \cdots, \tag{6}$$

where $\delta_c^2 = n^2$ and $\mu_c = 0$. After substituting equations (4)-(6) into equations (3) and developing $x_i^{(*)}$ into Fourier series, harmonic balancing is performed at ε and $\sin n\tau$, $\cos n\tau$, and the following is obtained:

$$p_{2n0}^{s'} = -np_{1n0}^{c'}, \qquad p_{2n0}^{c'} = np_{1n0}^{s'}. \tag{7}$$

Then

$$x_1^3 = \varepsilon^3((p_{1n0}^{c'})^3(\tfrac{3}{4}\cos n\tau + \tfrac{1}{4}\cos 3n\tau) + \tfrac{3}{4}(p_{1n0}^{c'})^2 p_{1n0}^{s'}(\sin 3n\tau + \sin n\tau)$$
$$+ \tfrac{3}{4}(p_{1n0}^{s'})^2 p_{1n0}^{c'}(\cos n\tau - \cos 3n\tau) + (p_{1n0}^{s'})^3(\tfrac{3}{4}\sin n\tau - \tfrac{1}{4}\sin 3n\tau)). \tag{8}$$

After harmonic balancing at ε^3, one obtains, for $n \neq 2$,

$$n^2 p_{1n0}^{c'}\delta'' = \tfrac{3}{4}((p_{1n0}^{c'})^3 + (p_{1n0}^{s'})^2 p_{1n0}^{c'}),$$
$$n^2 p_{1n0}^{s'}\delta'' = \tfrac{3}{4}((p_{1n0}^{s'})^3 + (p_{1n0}^{c'})^2 p_{1n0}^{s'}). \tag{9}$$

The following is the solution of equations (9):

$$\mu'' = p_{1n0}^{c'} = 0, \qquad \delta''_{(1)} = \tfrac{3}{4}(1/n^2)\xi(p_{1n0}^{s'})^2; \tag{10}$$

$$\mu'' = p_{1n0}^{s'} = 0, \qquad \delta''_{(2)} = \tfrac{3}{4}(1/n^2)\xi(p_{1n0}^{c'})^2. \tag{11}$$

On the other hand, for $n = 2$ the following is obtained:

$$p_{120}^{c'} = \delta'' = 0, \qquad \mu'' = -\tfrac{3}{16}\xi(p_{120}^{s'})^2; \tag{12}$$

$$p_{120}^{s'} = \delta'' = 0, \qquad \mu'' = -\tfrac{3}{16}\xi(p_{120}^{c'})^2. \tag{13}$$

3. EXAMPLE 2 (INCLUDING EXTERNAL FORCE)

Let now an harmonic force have an effect on the system considered in Example 1. In this case the equation of motion has the form

$$m\ddot{y} + \tfrac{1}{2}(k_1 + k_2 + (k_1 - k_2)\cos 2\omega t)y + k_0 y^3 = P_0 \cos \omega_1 t. \tag{14}$$

Let

$$\tau_* = \omega t, \qquad x = y(k_1/k_0)^{-1/2}, \qquad \lambda_* = (p_0/m\omega^2)(k_0/k_1)^{1/2},$$
$$\xi_* = k_1/(m\omega^2), \qquad \delta_*^2 = (k_1 + k_2)/2m\omega^2, \qquad \mu_* = (k_1 - k_2)/2m\omega^2, \qquad \omega_1/\omega = \omega_*. \tag{15}$$

Then equation (14) will have the form

$$d^2x/d\tau^2 + (\delta_*^2 + \mu_* \cos 2\tau_*)x + \xi_* x^3 = \lambda_* \cos \omega_* \tau_*. \tag{16}$$

The system of two differential equations replacing equation (16) is

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\delta_*^2 x_1 - \xi_* x_1^3 - \mu_* x_1 \cos 2\tau_* + \lambda_* \cos \omega_* \tau_*, \tag{17}$$

where $\lambda_* = \varepsilon\lambda_{1*}$. The Fourier series in this case is double and has the independent variable $n\tau_*$ and $l\omega_*\tau_*$.

Consider first the nonresonance case ($l\omega_* \neq n$), where $n, l \in N$. After harmonic balancing at ε and $\sin n\tau_*$, $\cos n\tau_*$ and $\sin l\omega_*\tau_*$, $\cos l\omega_*\tau_*$ and after solving the algebraic equations set one has

$$p^{s'}_{2n0} = -np^{c'}_{1n0}, \qquad p^{c'}_{2n0} = np^{s'}_{1n0}; \tag{18}$$

$$p^{s'}_{10l} = p^{c'}_{20l} = 0, \qquad p^{c'}_{10l} = \lambda_1/(n^2 - l^2\omega_*^2), \qquad p^{s'}_{20l} = -1\lambda_1/(n^2 - l^2\omega_*^2). \tag{19}$$

Then

$$\begin{aligned}
x_1^3 = {} & (p^{c'}_{1n0})^3(\tfrac{3}{4}\cos n\tau_* + \tfrac{1}{4}\cos 3n\tau_*) + \tfrac{3}{4}(p^{c'}_{1n0})^2 p^{s'}_{1n0}(\sin 3n\tau_* + \sin n\tau_*) \\
& + \tfrac{3}{4}(p^{s'}_{1n0})^2 p^{c'}_{1n0}(\cos n\tau_* - \cos 3n\tau_*) + (p^{s'}_{1n0})^3(\tfrac{3}{4}\sin n\tau_* - \tfrac{1}{4}\sin 3n\tau_*) \\
& + \tfrac{3}{2}(p^{c'}_{1n0})^2 p^{c'}_{10l}\cos l\omega_*\tau_* + \tfrac{3}{4}(p^{c'}_{1n0})^2 p^{c'}_{10l}(\cos(2n + l\omega_*)\tau_* + \cos(2n - l\omega_*)\tau_*) \\
& + \tfrac{3}{2}p^{c'}_{1n0}p^{s'}_{1n0}p^{c'}_{10l}(\sin(2n + l\omega_*)\tau_* + \sin(2n - l\omega_*)\tau_*) + \tfrac{3}{2}(p^{s'}_{1n0})^2 p^{c'}_{10l}\cos l\omega_*\tau_* \\
& - \tfrac{3}{4}(p^{s'}_{1n0})^2 p^{c'}_{10l}(\cos(2n + l\omega_*)\tau_* + \cos(2n - l\omega_*)\tau_*) \\
& + \tfrac{3}{2}p^{c'}_{1n0}(p^{c'}_{10l})^2\cos n\tau_* + \tfrac{3}{4}p^{c'}_{1n0}(p^{c'}_{10l})^2(\cos(n + 2l\omega_*)\tau_* + \cos(n - 2l\omega_*)\tau_*) \\
& + \tfrac{3}{2}p^{s'}_{1n0}(p^{c'}_{10l})^2\sin n\tau_* + \tfrac{3}{2}p^{s'}_{1n0}(p^{c'}_{10l})^2(\sin(n + 2l\omega_*)\tau_* + \sin(n - 2l\omega_*)\tau_*) \\
& + (p^{c'}_{10l})^3(\tfrac{3}{4}\cos l\omega_*\tau_* + \tfrac{1}{4}\cos 3l\omega_*\tau_*).
\end{aligned}$$

When comparing the terms at orders ε^3 and $\sin n\tau_*$, $\cos n\tau_*$, as well as at $\sin l\omega_*\tau_*$ and $\cos l\omega_*\tau_*$ one can obtain μ'' and δ'' and thus finally, for this case,

$$\mu = 0,$$

$$\delta^2 = \tfrac{3}{4}\xi\left(\frac{1}{n^2}(\tfrac{1}{2}(p^{c'}_{1n0}\varepsilon)^2 + \tfrac{1}{2}(p^{s'}_{1n0}\varepsilon)^2 + (p^{c'}_{10l}\varepsilon)^2\right) + l^{-2}\omega_*^{-2}((p^{c'}_{1n0}\varepsilon)^2 + (p^{s'}_{1n0}\varepsilon)^2 + \tfrac{1}{2}(p^{c'}_{10l}\varepsilon)^2)). \tag{21}$$

Let

$$p^{c'}_{1n0}\varepsilon = A\cos\varphi, \qquad p^{s'}_{1n0}\varepsilon = A\sin\varphi. \tag{22}$$

Then the following are obtained from using these expressions (22):

$$\mu = 0,$$

$$\frac{A^2}{2n^2l^2\omega_*^2/(l^2\omega_*^2 + 2n^2)} + \frac{\lambda_*^2}{2(n^2 - l^2\omega_*^2)l^2\omega_*^2n^2/(2l^2\omega_*^2 + n^2)} - \frac{\delta^2}{\tfrac{3}{4}\xi} = 0. \tag{23}$$

The second of equations (23) has a geometric representation as a quadric surface—a cone.

The parametric forcing occurs when either $n \neq 1$ and $n \cong \omega_* l$, $n = 1$ and $n \neq \omega_* l$, or $n = 1 \cong \omega_* l$. Consider the last most complex case of resonance. Let the following connections occur:

$$l^2\omega_*^2 - 1 = a', \qquad \lambda_* = \varepsilon^2\lambda_1, \tag{24}$$

where $a' = \varepsilon^2 a$. Then, by proceeding as in the previous case one obtains

$$\delta_*^2/(\omega_*/2) - \frac{A^2}{(2/3\xi)} = 2(\mu_* - a'). \tag{25}$$

Equation (25) is the equation of hyperbolic paraboloid.

4. CONCLUSIONS

An analysis of the Hopf bifurcation in nonstationary non-linear systems exemplified by the Mathieu-Duffing oscillator has been presented. Considerations have been limited to determining the parameter-frequency relations and the dependence of the bifurcation parameter on the other parameters of the systems. The method of development into series of the perturbation parameter connected with the vibration amplitude (on the assumption that it is a small quantity) and the method of harmonic balancing have been employed. Cases with and without external force have been considered. For the latter case, with the vibration far from resonance, the vibration amplitude A, the amplitude of the excitation force λ_* and the frequency δ, create a quadric surface, which is a cone. On the other hand, for vibrations near resonance, the frequency δ_*, the vibration amplitude A, and the parametric excitation factor μ_* create a surface of the second degree, which is a hyperbolic paraboloid.

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