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Chaos in a Particular Nonlinear Oscillator

By

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With 4 Figures

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Summary

This paper presents an analysis of new dynamical phenomena using the example of the simple nonsymmetrical anharmonic oscillator. Strange attractors are detected near the critical values of parameters obtained earlier using an approximate analytical method. Long transitional chaotic phenomena, sudden qualitative changes in chaotic dynamics with evolution of chaotic attractors are discussed and illustrated.

1. Introduction

A minimum of three dimensions is necessary to observe chaotic behaviour. To these simple systems belong various periodically forced nonlinear oscillators. Beginning with the pioneering works of Holmes and Ueda, there have been repeated successful discoveries of chaotic behaviour in anharmonic Duffing and van der Pol-Duffing type oscillators [1]—[6]. Often, the transition to chaos in forced oscillators is connected with a sequence of successive bifurcations which precede these irregular motions. Ruelle and Takens were the first to suggest that strange attractors could arise after a finite sequence of bifurcations and might provide models for complicated irregular motions. The scheme which is given by Ruelle and Takens [7] shows how after three, two or even one Hopf bifurcation, the system can undergo a subsequent transition to chaos. The theory was confirmed by experimental investigations in hydrodynamic systems by Gollub and coworkers [8], [9]. Similar results are obtained in an optical system [10].

In this paper, an anharmonic nonlinear mechanical van der Pol-Duffing type oscillator with an exciting force whose amplitude is proportional to the second power of the exciting frequency and a constant load is analysed. The static load causes nonsymmetry of the system which results in a higher probability of creating higher order resonances and accompanying irregular motion. We will show that chaos appears after the bifurcation of the stationary state with one and two frequencies. Also, based on the example of this simple nonlinear oscillator, new dynamical phenomena are presented.

2. The Analysed System, Bifurcation of the Stationary State with One Frequency and Numerical Results

We consider an oscillator governed by the dimensionless equation

$$\ddot{y} - (\beta - \delta y^2)\,\dot{y} + \alpha y + \mu y^3 = q + \eta^2 \cos \eta t. \tag{1}$$

The motion of the oscillators, where the rotation of an unbalanced disk or wheel becomes the source for the exciting force, can be reduced to Eq. (1).

Assuming that the stationary solution has the form

$$y = Y + A \cos \eta t + B \sin \eta t, \tag{2}$$

we obtain from (1)

$$\begin{aligned} &Y\left(\alpha + \mu\left(Y^2 + \frac{3}{2}P^2\right)\right) - q = 0,\\ &B(\alpha - \eta^2) - A\eta\left(-\beta + \delta\left(Y^2 + \frac{1}{4}P^2\right)\right) + 3\mu B\left(Y^2 + \frac{1}{4}P^2\right) = 0,\\ &A(\alpha - \eta^2) + B\eta\left(-\beta + \delta\left(Y^2 + \frac{1}{4}P^2\right)\right) + 3\mu A\left(Y^2 + \frac{1}{4}P^2\right) - \eta^2 = 0, \end{aligned} \tag{3}$$

where $P = (A^2 + B^2)^{1/2}$ is the amplitude of vibration. The perturbated solution of (2) is

$$y_p = Y + \Delta Y + (A + \Delta A) \cos \eta t + (B + \Delta B) \sin \eta t. \tag{4}$$

Assuming that perturbations $\Delta(\cdot)$ and damping coefficients β and δ are small, we obtain

$$\begin{aligned} -2\eta(\Delta\dot{A}) + T(\Delta A) + U(\Delta B) &= 0,\\ 2\eta(\Delta\dot{B}) + V(\Delta A) + W(\Delta B) &= 0, \end{aligned} \tag{5}$$

where

$$\begin{aligned} T &= \beta\eta - \eta\delta Y^2 - \frac{3}{4}\delta\eta A^2 - \frac{1}{4}\delta\eta B^2 + \frac{3}{2}\mu BA,\\ U &= \alpha - \eta^2 + 3\mu Y^2 + \frac{3}{4}\mu A^2 + \frac{9}{4}\mu B^2 - \frac{1}{2}\delta\eta AB,\\ V &= \alpha - \eta^2 + 3\mu Y^2 + \frac{9}{4}\mu A^2 + \frac{3}{2}\mu B^2 + \frac{1}{2}\delta\eta AB,\\ W &= -\beta\eta + \delta\eta Y^2 + \frac{1}{4}\delta\eta A^2 + \frac{3}{4}\delta\eta B^2 + \frac{3}{2}\mu AB. \end{aligned} \tag{6}$$

At the bifurcation point we have

$$W - T = 0, \tag{7}$$

and

$$UV - TW > 0. \tag{8}$$

Consequently, we obtain the set of Eqs. (3), (7) and the inequality (8). In order to obtain the bifurcation curves $\mu(\eta)$ we solve those equations for arbitrarily chosen parameters α, β, δ, q. Suppose, that we have found the Hopf bifurcation curve $\mu(\eta)$. For the parameters lying on one side of this curve the real parts of the eigenvalues of (5) are negative (positive) whereas for the other side they are positive (negative). For each point belonging to this curve we have a corresponding stationary state with Y_H and amplitudes A_H, B_H. After crossing this curve from negative real parts to positive real parts of the parameter plane with nonzero velocity a Hopf bifurcation appears. Our considerations are valid for the averaged system of the equations after substituting (2) and (4) into (1), and, additionally, we investigate only the local bifurcation of Hopf type (see also [11]). This bifurcation in the averaged system of equations is related to the bifurcation of the solution (2) in Eq. (1). We finally obtain the conditions necessary for a bifurcation of the stationary state with one frequency.

Solutions of (3) and (7) are found using Newton's method. Sample curves obtained in this way are shown in Fig. 1.

The considered system of algebraic nonlinear equations can possess one, two of three equilibrium paths. It depends on the parameter values. In Fig. 1a we

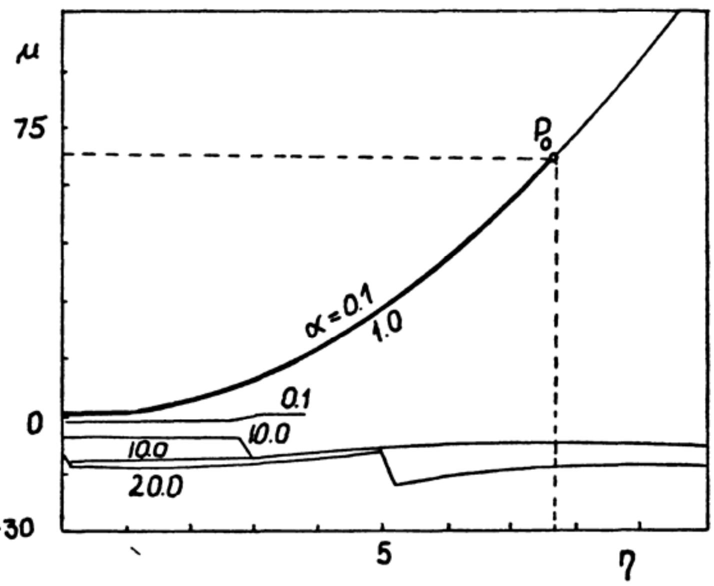

Fig. 1.a)

Fig. 1. Bifurcation curves for the one-frequency solution
a) $\beta = 0.1$, $\delta = 0.1$, $q = 1.0$; b) $\beta = 0.1$, $\alpha = 0.1$, $q = 3.0$

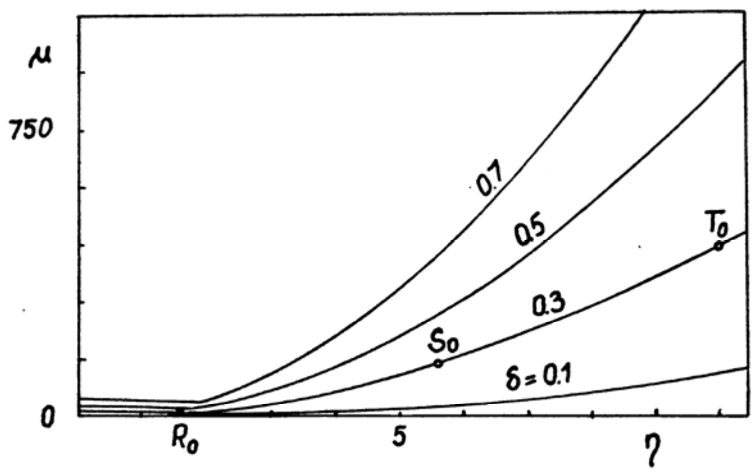

Fig. 1. b)

show that for $\alpha = 0.1$ and $\alpha = 10.0$ we have in each case two bifurcation curves.

For parameters near the critical values, we integrate Eq. (1) numerically by the Runge-Kutta method with the step size $h = 2\pi/50\eta$ and with the initial conditions $y(0) = 0.0$, $\dot{y}(0) = 1.0$. The results are presented in the form of Poincaré maps and frequency spectra. The Poincaré maps consist of 2000 points. These maps are calculated starting with time $t = 100T$ ($T = 2\pi/\eta$), for which the trajectories finally reach the attractor. The Fourier spectra are obtained using Fast Fourier Transform and are presented in three-dimensional form. A decimal scale has been adopted for amplitudes, corresponding frequencies and time. The solutions have been analysed starting from $t = 100T$ and then we have obtained the Fourier components of the motion repeating the analysis of each interval of $300T$. This method of calculation of the Fourier spectra allows us to observe the development of motion and to trace the evolution of the Fourier components with time.

Consider the behaviour of the system near the bifurcation point P_0 in the Fig. 1a. Figure 2 demonstrates sudden changes in nonchaotic and chaotic dynamics as parameter β is varied. For $\beta = 0.08$ we can observe very long, transient oscillations which finally reach the nonchaotic attractor, indicated by the stable fixed point F on the Poincaré map. After a long transient state, a periodic motion with the frequency of the exciting force remains (Fig. 2a). When β is slightly increased (Fig. 2b; $\beta = 0.085$) transient chaos occurs. The development of the power spectrum testifies this. The chaotic motion of the system lasts until $t = 800T$. In the time trajectory the two broad-band regions of frequency near of $\frac{1}{2}\eta$ and $\frac{2}{3}\eta$ are evident. The transient chaotic motion then gradually changes into a regular two-frequency motion with frequencies η and 2η. A clearly dominating component is the frequency of the exciting force. With further increase of β, the transitional phenomena become longer and suddenly, for the critical value of β, the fixed point stability is altered and the

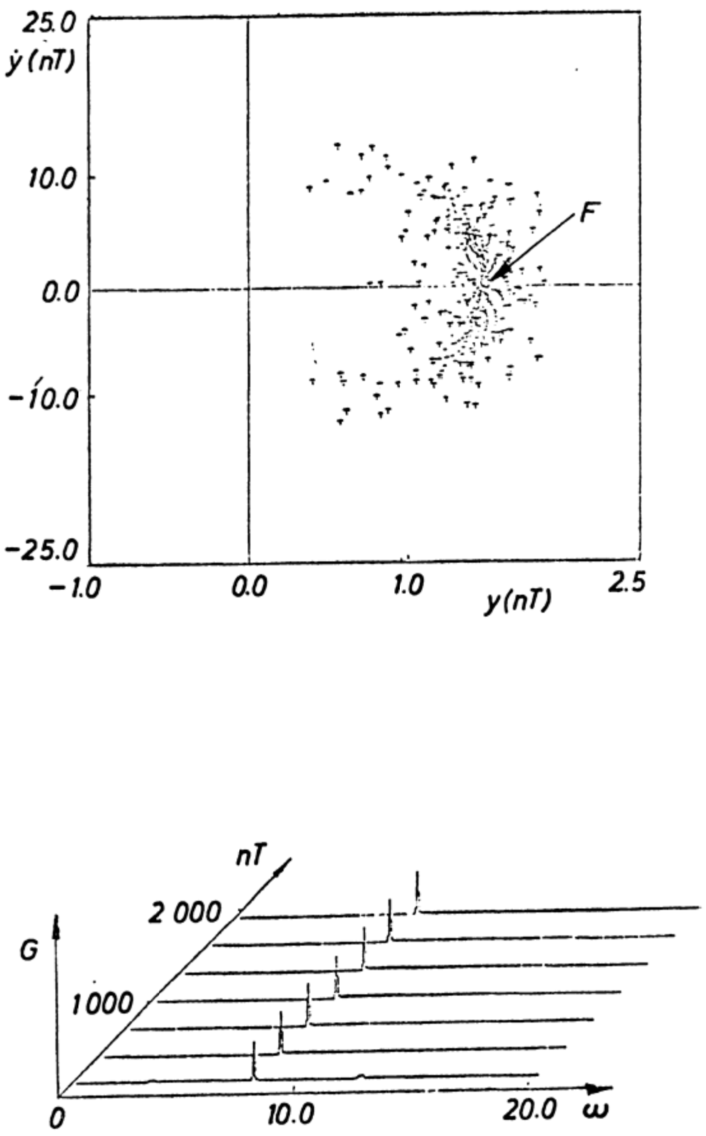

Fig. 2. a)

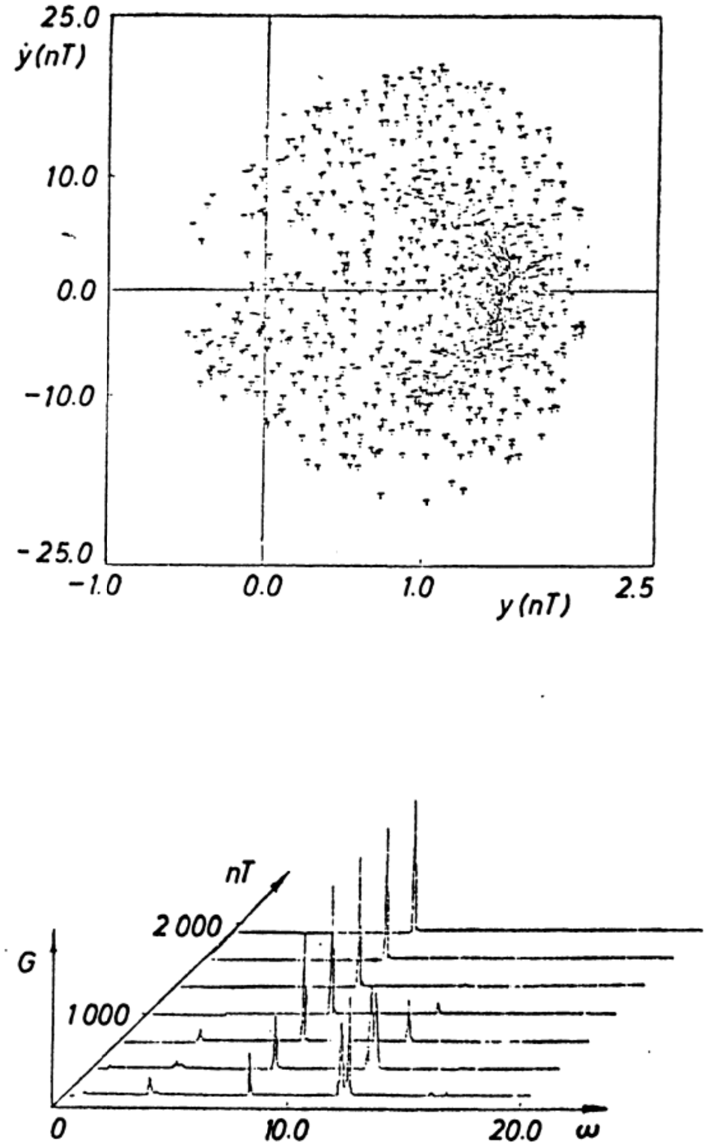

Fig. 2. b)

Fig. 2. Evolution of attractors with change of β and corresponding Fourier spectra: $\alpha = 0.1$; $\delta = 0.1$; $\mu = 69.17$; $\eta = 7.8$; $q = 1.0$: a) $\beta = 0.08$; b) $\beta = 0.085$; c) $\beta = 0.09$; d) $\beta = 0.095$; e) $\beta = 0.098$; f) $\beta = 0.1$

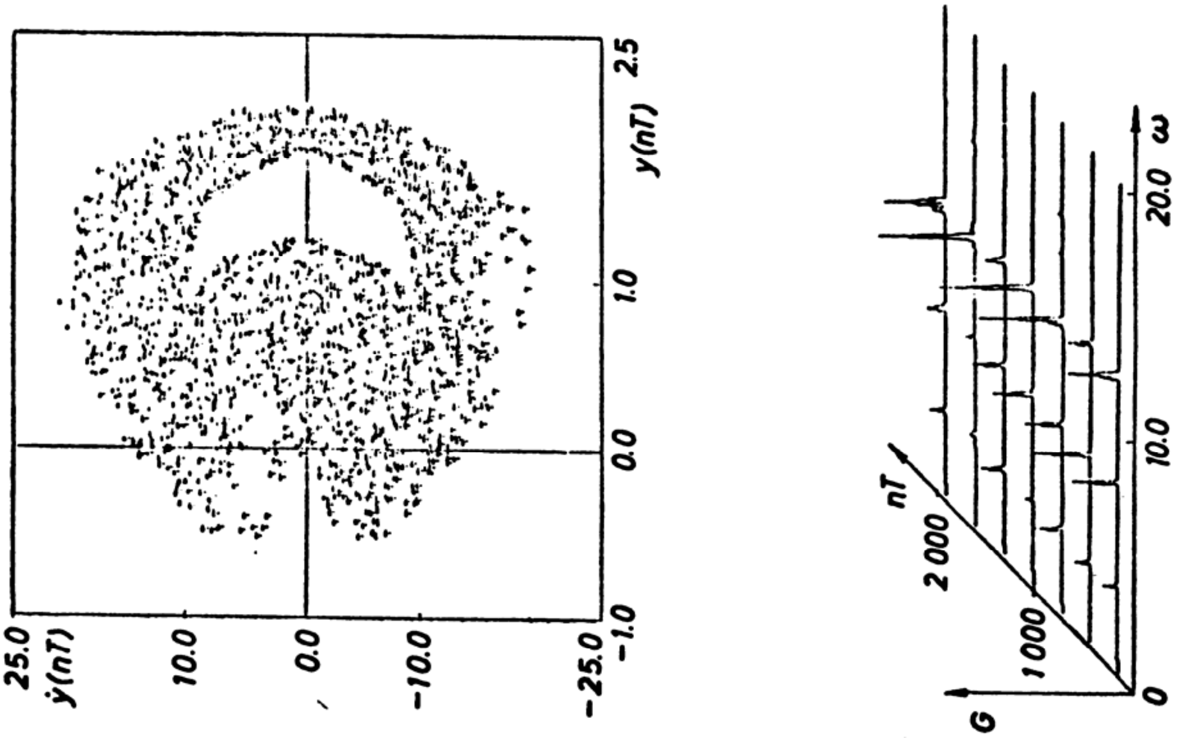
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Fig. 2.c)

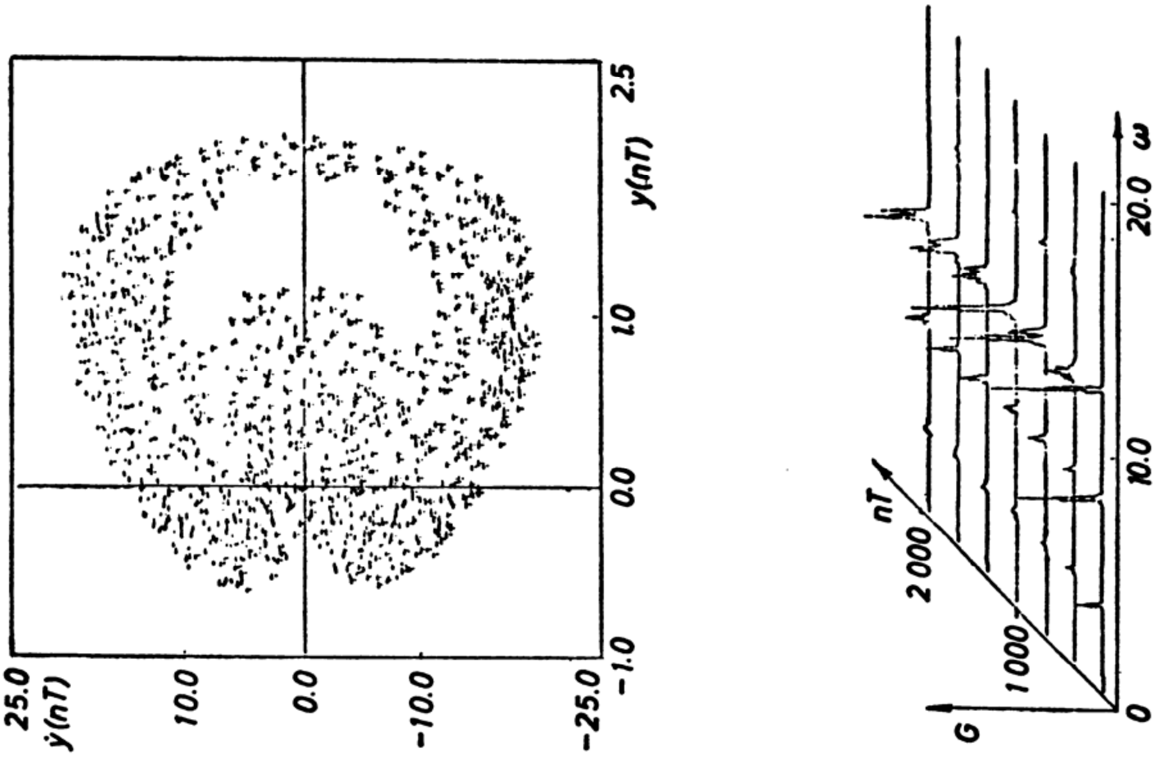
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Fig. 2.d)

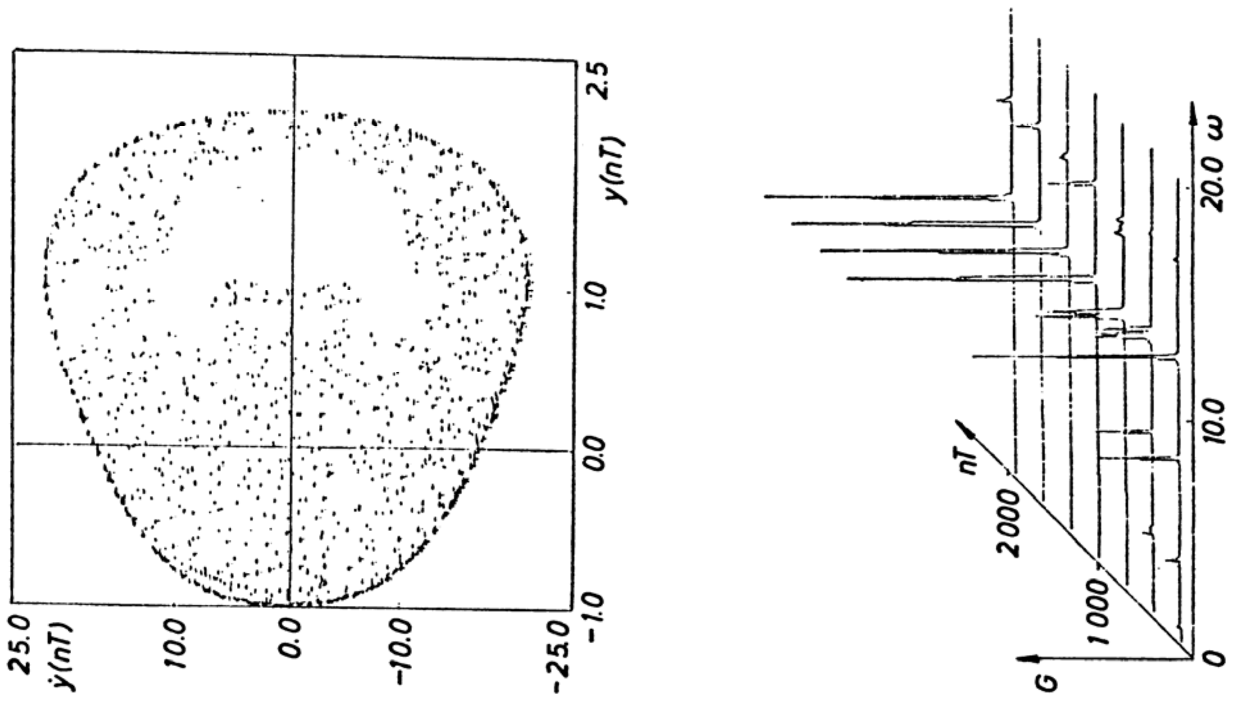

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Fig. 2.e)

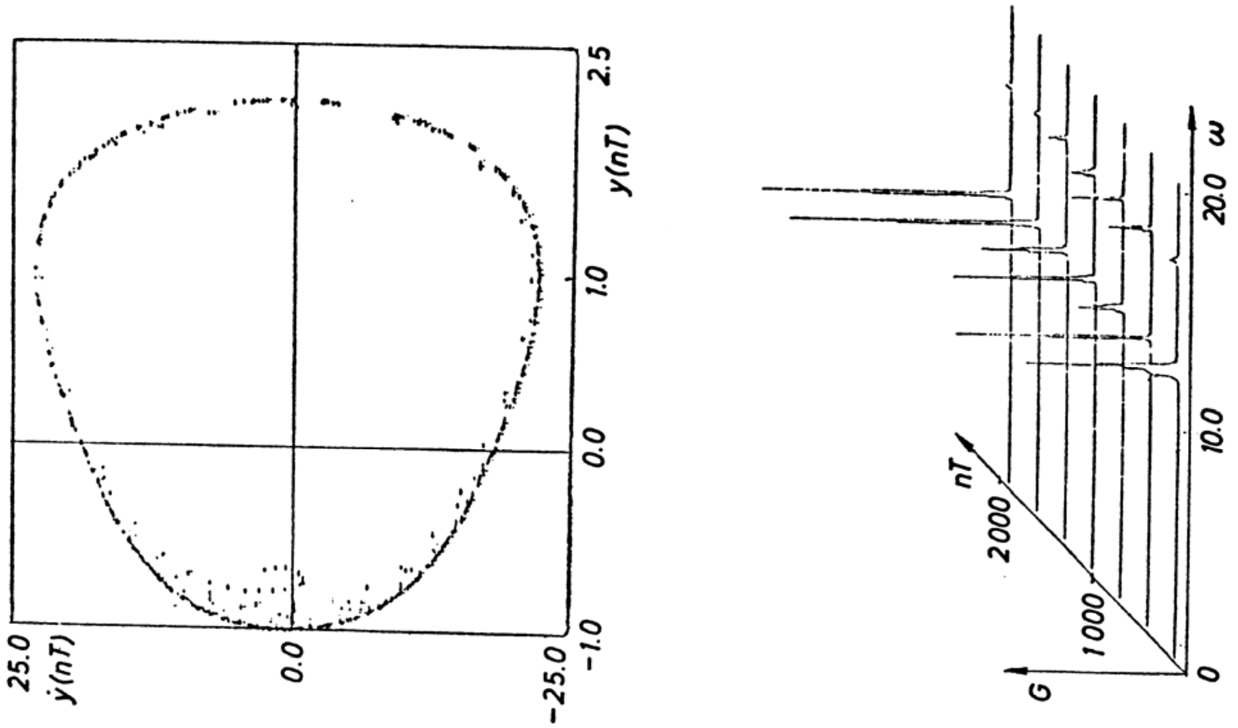

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Fig. 2.f)

trajectories shift away from the domain of the previous attractor (Figs. 2c, d). Additionally, inside the area of this strange attractor there exists a space which does not contain any points. This space was previously covered by the points of the Poincaré map presented in Fig. 2a. For $\beta = 0.098$ we observe again the long chaotic transitional phenomena and then for $t > 1000T$ the points of the Poincaré maps are lying on the closed curves. Because the corresponding frequency spectra are discrete (with two components) the corresponding attractor is quasiperiodic. With very small increase of β ($\beta = 0.1$) the chaotic transitional phenomena do not appear and a quasiperiodic attractor is clearly visible. As can be seen in Fig. 2f, we have two frequency components $\omega_1 = 12.3$, $\omega_2 = 16.7$. For $\beta > 0.1$ (more precisely $\beta = 0.11$, $\beta = 0.12$) no changes have been found in the behaviour of the system.

Comparison of this results with the "crisis phenomena" suggested by Grebogi and Ott [12] provides an interesting insight. The authors define a crisis as a collision between a chaotic attractor and a coexisting unstable fixed point. They distinguish two types of "crisis", the "boundary" and the "interior" crisis. The first leads to sudden destruction of the chaotic attractor and its basin of attraction, while the second can cause sudden changes in the size of the chaotic attractor. In our case however one can observe another phenomenon. First we have obtained a fixed point on the Poincaré map and then a very long chaotic transitional phenomenon has appeared which for the further increase of β changes into a strange attractor. This provides the evidence for another new unique "crisis" type different from those analysed by Grebogi and Ott, where the changes of stability of the previously stable fixed point cause the shift to irregular motion. Now we can consider the new attractor of mixed type containing both a co-existing, chaotic attractor and an unstable fixed point. With further increase of β, another crisis occurs and leads to both the destruction of a strange attractor and creation of a quasiperiodic attractor.

We have analysed the evolution of Poincaré maps with the change of the nonlinear rigidity coefficient μ, the other parameters are: $\alpha = \beta = \delta = 0.1$, $\eta = 7.8$, $q = 1.0$. For $\mu = 40.0$ we have discovered regular motion with one frequency equal to η. With further increase in μ the transitional phenomenon becomes longer and longer. Increase of μ to the value of $\mu = 65.0$ results in chaos. Then for $\mu = 68.5$, a two frequency motion is observed.

Evolution of the motion with change in δ has also been analysed ($\alpha = \beta = 0.1$, $\mu = 69.17$, $\eta = 7.8$, $q = 1$). For $\delta = 0.12$ we have found the stable fixed point on the Poincaré map. With a slight decrease of δ ($\delta = 0.115$) the previous stable point became unstable and chaos appeared. Inside the area of the strange attractor there was a domain without points. This domain was previously covered by points of the attractor for $\delta = 0.12$. With further decrease of δ (for $\delta = 0.11$) we have obtained a "weak" chaotic attractor. Then for $\delta = 0.09$ the motion becomes regular and a quasiperiodic attractor with two frequencies appeares. For the results of the numerical analysis (for the parameters near point P_0)

we have marked in Fig. 3 the regions of chaotic motion. It can be seen that chaotic orbits appear in the neighborhood of 2η, $\eta/2$, $\frac{3}{2}\eta$.

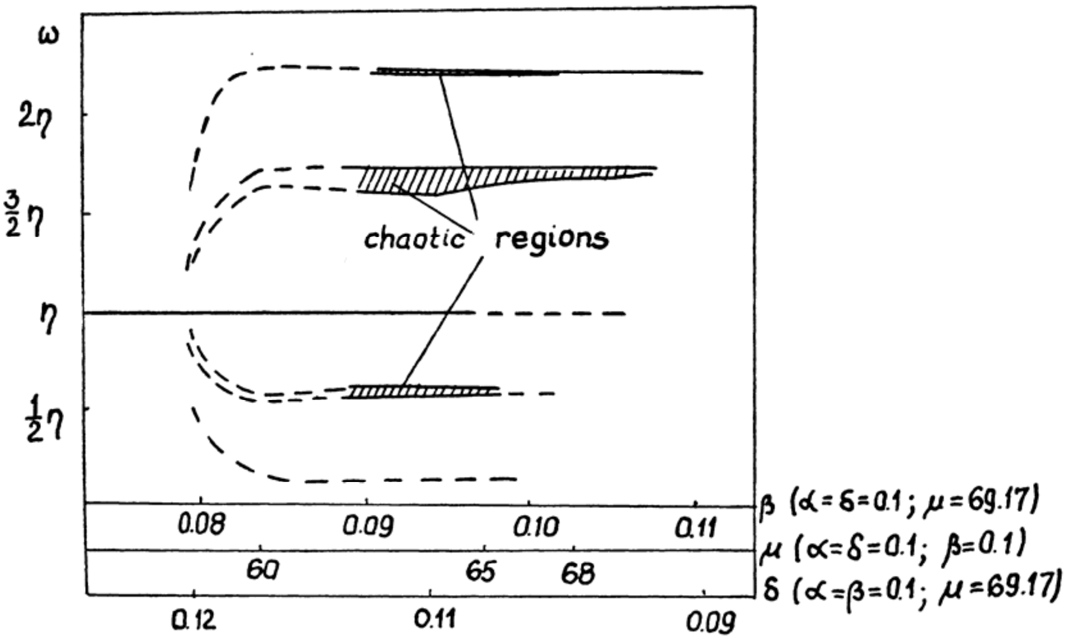

Fig. 3. Regions of chaotic motion for parameters near the point P_0

Finally, we have investigated the route to chaos for parameters of the system placed near the bifurcation points R_0, S_0, T_0. For parameters near the point R_0 ($\alpha = 1.0$, $\beta = 0.1$, $\delta = 0.3$, $\eta = 1.6$, $q = 3$) the evolution from regular motion ($\mu = 14.3$) through chaos ($\mu = 14.35$—14.45) to quasiperiodic motion ($\mu = 15.0$) was observed.

For parameters $\alpha = 1.0$, $\beta = 0.1$, $\delta = 0.3$, $\eta = 5.6$, $q = 3$ (near the point S_0) and $\mu = 135$ there are four frequencies in the motion, but the magnitude of the amplitudes corresponding to these frequencies evolve with time and the Poincaré map has a complicated structure. Frequency spectra are more broad-band and the corresponding amplitudes vary in time for $\mu = 137$. With a further increase of μ ($\mu = 139$) a quasiperiodic attractor appeared.

For a set of parameters near the point T_0 ($\alpha = 1.0, \beta = 0.1, \delta = 0.3, \eta = 10.0$, $q = 3.0$) we have observed transitional chaotic phenomena at $\mu = 440.0$, and then, for $\mu = 442.0$ and $\mu = 443.0$, appearance of the strange chaotic attractor. For $\mu = 446.0$ chaotically transitional phenomena occured for $t < 900T$. For $t > 900T$ we obtained the quasiperiodic attractor with three frequencies.

It is interesting that all strange chaotic attractors discovered possess a horizontal axis of symmetry.

3. Bifurcation of the Stationary State with Two Frequencies and the Numerical Results

We assume that the stationary solution has the form

$$y = Y + A\cos\eta t + B\sin\eta t + R\cos\Omega t \tag{9}$$

where Ω is the dimensionless frequency of self excited vibrations. From Eq. (1), taking into account (9), one can obtain the nonlinear algebraic equation set, as described in Section 2.

Considering the perturbated solution of (9)

$$\begin{aligned} y_p = {} & Y + \Delta Y + (A + \Delta A)\cos\eta t + (B + \Delta B)\sin\eta t \\ & + (R + \Delta R)\cos\Omega t + \Delta x\sin\Omega t, \end{aligned} \tag{10}$$

and assuming that the values of the coefficients β, δ, $\Delta\cdot$ are small, one can obtain the linear set of differential perturbation equations. The characteristic equation has the form

$$\begin{aligned} & 16\eta^2\Omega^2\lambda^4 + 8\Omega\eta\big((c_{44} - c_{33})\,\eta + \Omega(c_{22} - c_{11})\big)\,\lambda^3 \\ & + 4\Big(\eta\Omega(c_{11} - c_{22})\,c_{33} + \Omega^2(c_{12}c_{21} - c_{11}c_{22}) \\ & \quad + \eta\Omega(c_{32}c_{23} - c_{13}c_{31}) + \eta^2c_{43}c_{34} - \eta c_{44}\big(\eta c_{33} + \Omega(c_{11} - c_{22})\big)\Big)\,\lambda^2 \\ & + 2\Big(\Omega(c_{13}c_{21}c_{32} + c_{12}c_{23}c_{31} - c_{13}c_{22}c_{31} + c_{11}c_{22}c_{33} - c_{12}c_{21}c_{33} - c_{32}c_{23}c_{11}) \\ & \quad + \eta c_{34}(c_{22}c_{43} + c_{13}c_{41} - c_{11}c_{43} - c_{23}c_{42}) \\ & \quad + c_{44}\big(\eta(c_{11} - c_{22})\,c_{33} + \Omega(c_{12}c_{21} - c_{11}c_{22}) + \eta(c_{32}c_{23} - c_{13}c_{31})\big)\Big)\,\lambda \\ & \quad + c_{34}(c_{12}c_{21}c_{43} + c_{11}c_{23}c_{42} + c_{13}c_{22}c_{41} - c_{13}c_{21}c_{42} - c_{12}c_{23}c_{41} - c_{11}c_{22}c_{43}) \\ & \quad + c_{44}(c_{13}c_{21}c_{32} + c_{12}c_{23}c_{31} + c_{11}c_{22}c_{33} - c_{12}c_{21}c_{33} - c_{32}c_{23}c_{11} - c_{13}c_{22}c_{31}) = 0, \end{aligned} \tag{11}$$

where c_{ij} $(i, j = 1, \ldots, 4)$ depend on the parameters and Y, A, B, R, Ω.

The necessary conditions of the existence of a Hopf bifurcation point is the existence in Eq. (11) of two purely imaginary eigenvalues, whereas all other eigenvalues have negative real parts. The full system of bifurcation equations in this case we obtain by assigning the expressions for λ^3, λ^2 and λ to zero while the last expression ought to be greater than zero. Unfortunately, using this technique we are unable to find bifurcation curves, but we have found two isolated solutions. In the first case the critical parameters are: $\alpha = 0.05$, $\beta = 0.019$, $q = 0.00253$, $\mu = 6.274$, $\eta = 0.1218$, $\delta = 0.4$. Hopf type bifurcation takes place by changing the δ parameter. From numerical experiments we have obtained for $\delta = 0.4$ a periodic motion and, with further increase of this co-

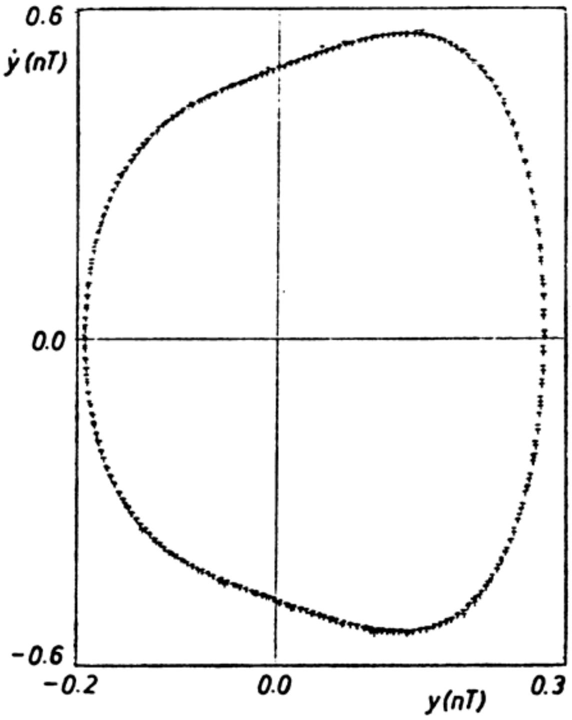

Fig. 4. a)

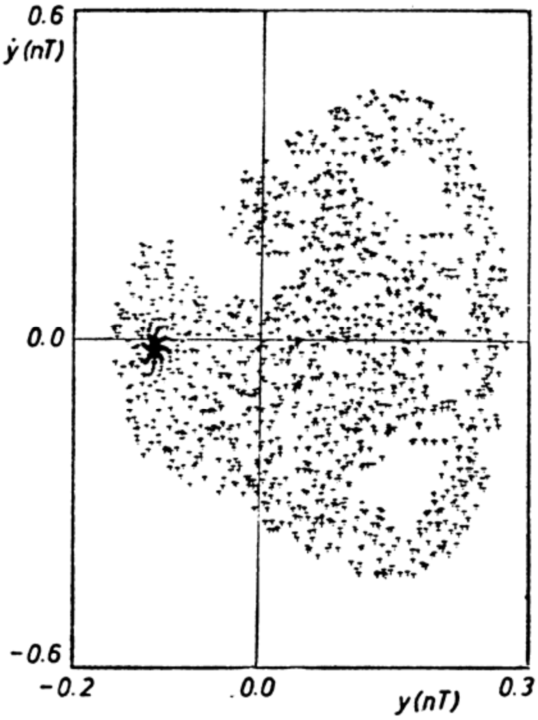

Fig. 4. b)

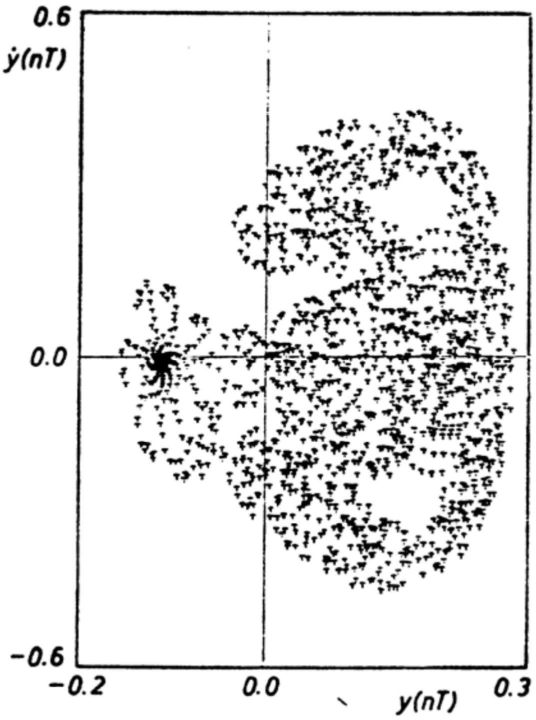

Fig. 4. c)

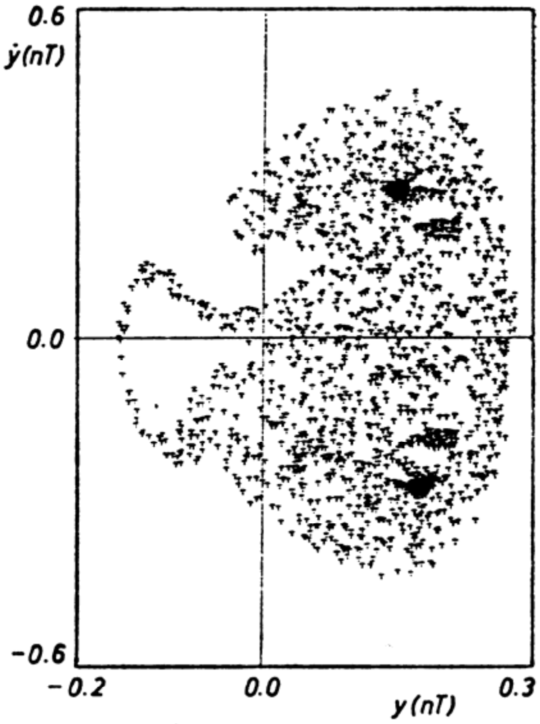

Fig. 4. d)

Fig. 4. Evolution of attractors for the two-frequency solution: $\alpha = \beta = 0.02$; $q = -0.2665$; $\mu = 100.1$; $\eta = 0.724$ a) $\delta = 0.6$; b) $\delta = 0.73$; c) $\delta = 0.76$; d) $\delta = 0.762$; e) $\delta = 0.9$

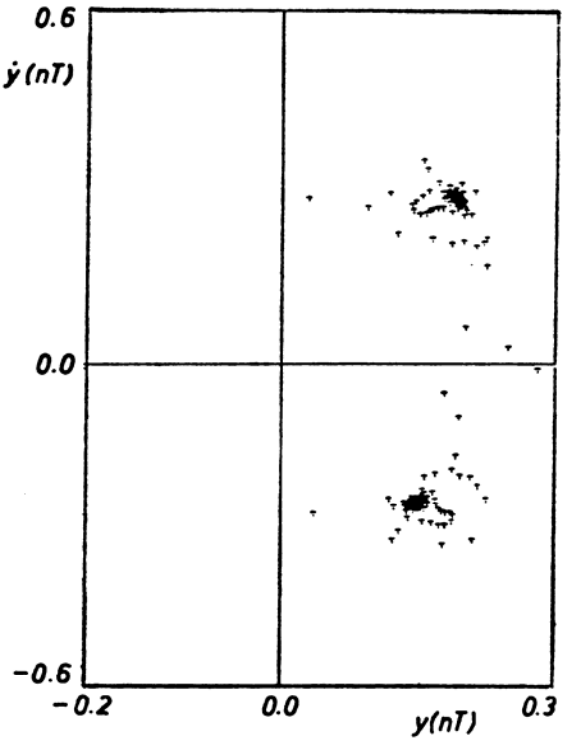

Fig. 4. e)

efficient, we can observe many bifurcations which lead to subharmonic and quasiperiodic motions. In this case we have not detected chaotic motions.

The second example is presented in Fig. 4. For $\delta = 0.6$ one can observe quasiperiodic motion and, for $0.6 < \delta < 0.7$ Hopf bifurcation of the stationary state occurs. We have presented in Fig. 4b the chaotic transitional phenomena which persist to 1400 periods and then converge to one point (the map has 2000 points). For $\delta = 0.76$ the situation is similar to that previously described but suddenly, for $\delta = 0.762$ (see Fig. 4d), the stable fixed point vanishes and two new, stable symmetric points appear. It is strongly evident that the new, stable fixed point appears in the middle of the domain previously free of points and, further, that new domains free of points inside the attractor are located at the same place where, earlier, the single fixed point was lying. Generally the shape of attractor remains unchanged during changes of δ.

Therefore, using the example of this simple anharmonic oscillator, we have detected a new phenomenon indicative of the simultaneous coexistence of one stable and two unstable fixed points with long transitional chaos. For $0.76 < \delta < 0.762$ bifurcation appears so that the stability for these three points is changed with the previous one stable point becoming unstable and the two unstable becoming stable. However, this change of stability of the fixed points appears to have no influence on the transitional chaotic phenomena because the shape of the Poincaré map remains unchanged. With further increase of δ the chaotic transitional phenomena become shorter and shorter. During these changes, however, the two stable fixed points remain at their previous positions.

Eventually, the transitional chaotic behaviorour disappears (Fig. 4e) and we find two stable fixed points. The changes in the behaviour of chaotic transitional phenomena have no apparent influence on the other stable fixed points and vice versa.

4. Concluding Remarks

The aim of this work is to provide a method to discover strange chaotic attractors and subsequently, to observe new nonlinear dynamical phenomena. Using an approximate analytical method, we have obtained a set of bifurcation equations, assuming that bifurcation of the stationary regular motion with one and two frequencies takes place. In the former case we have determined bifurcation curves, while in the latter we have found only two isolated solutions.

In the first case we have presented some examples of strange chaotic attractors. All of them possess a horizontal axis of symmetry even though the static load in the considered equation was expected to cause nonsymmetry.

In Fig. 2 we have shown how with the changes of stability of the previously stable fixed point the long chaotic transitional phenomena shift into a chaotic attractor. This new complex attractor can be considered as a mixed one in which a strange chaotic attractor and an unstable fixed point coexist.

We have described a route from a periodic motion to an irregular one with the change of the nonlinear rigidity μ. With the increase of μ chaotically transitional phenomena last longer and then a strange chaotic attractor appears.

The analog scheme of transition from regular to irregular motion is also presented when we have discovered chaos by changing the parameter δ. Similar as in two earlier cases inside the domain of the strange attractor there is a region without points. This space was previously covered by the points of the stable attractor, which now is unstable and coexists with the strange chaotic attractor.

With reference to bifurcation of the stationary state with two frequencies, two isolated critical values of the parameters were found. For the first set many bifurcations appear with a variation of the coefficient δ. However, this does not lead to chaos. Near the second set of bifurcation parameters analysed, the evolution of simultaneously coexisting stable and unstable fixed points with chaotically transitional phenomena (Fig. 4) is detected. This evolution shows that fixed points and these transitional phenomena are "uncoupled" in the sense that changes in one behaviour do not have any influence on the other. In our investigations we have used the discrete power spectrum. It has allowed us to observe the evolution of the Fourier components with time.

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