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Two kinds of evolution of strange attractors for the example of a particular non-linear oscillator

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1. Introduction

Physical systems governed by nonlinear differential equations at least of third order, where chaotic motion was detected, are widely described in the literature (see, for example, references [1–13]). There are so many studies of nonlinear dynamics in simple systems because they provide the advantage of a less complex analysis of the behaviour of strange attractors. Another reason is that similar behaviour sometimes is likely to be met in complicated systems. Chaotic orbits, however, have usually been sought in a random way and it is difficult to present a general method to discover them.

Ruelle and Takens [14] have shown that strange attractors could arise after a finite sequence of bifurcations which cause complicated irregular motion. This idea has been used here, first to detect analytically bifurcation curves based on the approximate Van der Pol method, and then to prove numerically the existence of chaotic motion near the bifurcation curves.

2. Analysed system and Hopf bifurcations

The equation of motion for the analysed is

$$M\ddot{x} + (c_2x^2 - c_1)\dot{x} + k_0x + k_1x^3 + \mu_0 Mg \operatorname{sign} \dot{x} = P_0(t), \tag{1}$$

where $P_0(t) = a\mu m v^2 \cos vt$. This model describes a mechanical oscillator with Van der Pol type damping and Duffing type stiffness, where the exciting force $P_0(t)$ originates from rotating engine rotor with the mass m and the unbalance μ ("a" is the amplification coefficient). From (1) we obtain

$$\begin{aligned} \dot{y} &= z, \\ \dot{z} &= \varepsilon(1 - y^2)\dot{y} - \delta y - \gamma y^3 - \alpha \operatorname{sign} \dot{y} + p_0 v^2 \cos vt, \end{aligned} \tag{2}$$

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where

$$y = (c_2/c_1)^{1/2}x, \quad \varepsilon = c_1/M, \quad \delta = k_0/M, \quad \gamma = (k_1c_1)/(Mc_2),$$
$$\alpha = \mu_0 g(c_2/c_1)^{1/2}, \quad p_0 = (a\mu m)/(M(c_2/c_1)^{-1/2}). \tag{3}$$

The approximate analytical method of Van der Pol is used to solve the system of equations (2) assuming that

$$y = u(t)\cos \nu t - v(t)\sin \nu t,$$
$$z = -\nu(u(t)\sin \nu t + v(t)\cos \nu t), \tag{4}$$

where $u(t)$ and $v(t)$ are slowly changing functions of t. After substituting equations (4) in (2) we obtain

$$\dot{u} = N\sin \nu t,$$
$$\dot{v} = N\cos \nu t, \tag{5}$$

where

$$N = (\delta - \nu^2)\nu^{-1}(u\cos \nu t - v\sin \nu t) + \varepsilon\nu^{-1}(u\cos \nu t - v\sin \nu t)^3$$
$$+ \varepsilon(1 - (u\cos \nu t - v\sin \nu t)^2)(u\sin \nu t + v\cos \nu t)$$
$$+ \frac{\alpha}{\nu}\operatorname{sign}(-\nu(u\sin \nu t + v\cos \nu t)) - p_0\nu\cos \nu t. \tag{6}$$

The right-hand sides of Eq. (5), being periodic functions of t with the period $2\pi/\nu$, are expanded into Fourier series. Taking into consideration that both u and v are slowly changing functions of t and that only the first terms of the expansions are significant the following averaged system of equations

$$\dot{u} = -\frac{\delta - \nu^2}{2\nu}v + \frac{\varepsilon}{2}u - \frac{\varepsilon}{2}u(u^2 + v^2) - \frac{3\gamma}{8\nu}v(u^2 + v^2) - \frac{2\alpha u}{\pi\nu(u^2 + v^2)^{1/2}},$$
$$\dot{v} = \frac{\delta - \nu^2}{2\nu}u + \frac{\varepsilon}{2}v - \frac{\varepsilon}{8}v(v^2 + u^2) + \frac{3\gamma}{8\nu}u(u^2 + v^2) - \frac{2\alpha v}{\pi\nu(u^2 + v^2)^{1/2}} - \frac{\nu p_0}{2} \tag{7}$$

is obtained.

In this paper we follow the method of slowly varying variables, which was used to obtain Hopf bifurcation curves of a particular forced Van der Pol oscillator by Arrowsmith and Taha [15]. Assuming that

$$\omega = \frac{\delta - \nu^2}{\nu}, \quad \varepsilon = 8, \quad \frac{3\gamma}{8\nu} = 1, \quad \frac{2\alpha}{\pi\nu} = 1,$$
$$\nu = 2P, \quad p_0 = 1 \tag{8}$$

we obtain

$$\dot{u} = -\omega v + 4u - u(u^2+v^2) - v(u^2+v^2) - u(u^2+v^2)^{-1/2},$$
$$\dot{v} = \omega u + 4v - v(u^2+v^2) + u(u^2+v^2) - v(u^2+v^2)^{-1/2} - P. \tag{9}$$

In order to determine the Hopf bifurcations, the terms of Eq. (9), containing the square root, are expanded into Taylor series around the point of equilibrium (u_0, v_0). The expansion is limited to linear terms in u and v. As a result, we have

$$\begin{aligned}\dot{u} &= -\omega v + (4-(u^2+v^2))u - (u^2+v^2)v - \frac{u_0}{(u_0^2+v_0^2)^{1/2}} \\ &\quad - \frac{v_0^2}{(u_0^2+v_0^2)^{3/2}}(u-u_0) + \frac{u_0 v_0}{(u_0^2+v_0^2)^{3/2}}(v-v_0), \\ \dot{v} &= \omega u + (4-(u^2+v^2))v + (u^2+v^2)u - \frac{v_0}{(u_0^2+v_0^2)^{1/2}} \\ &\quad - \frac{u_0^2}{(u_0^2+v_0^2)^{3/2}}(v-v_0) + \frac{u_0 v_0}{(u_0^2+v_0^2)^{3/2}}(u-u_0) - P.\end{aligned} \tag{10}$$

Let us now proceed to the new coordinate system (u', v') whose origin is the singular point (u_0, v_0). From Eq. (10), after linearization, we obtain

$$\begin{aligned}\dot{u}' &= \left(4 - 3u_0^2 - v_0^2 - 2u_0v_0 - \frac{v_0^2}{(u_0^2+v_0^2)^{3/2}}\right)u' \\ &\quad + \left(-\omega - u_0 - 3v_0^2 - 2u_0v_0 + \frac{u_0v_0}{(u_0^2+v_0^2)^{3/2}}\right)v', \\ \dot{v}' &= \left(\omega + 3u_0^2 + v_0^2 - 2u_0v_0 + \frac{u_0v_0}{(u_0^2+v_0^2)^{3/2}}\right)u' \\ &\quad + (4 - u_0^2 - 3v_0^2 + 2u_0v_0 - \frac{u_0^2}{(u_0^2+v_0^2)^{3/2}}\Bigg)v'.\end{aligned} \tag{11}$$

The characteristic equation of (11) has the form

$$\delta^2 - (A+D)\delta + AD - BC = 0, \tag{12}$$

where

$$\begin{aligned}A &= 4 - 3u_0^2 - v_0^2 - 2u_0v_0 - \frac{v_0^2}{(u_0^2+v_0^2)^{3/2}}, \\ B &= -\omega - u_0^2 - 3v_0^2 - 2u_0v_0 + \frac{u_0v_0}{(u_0^2+v_0^2)^{3/2}}, \\ C &= \omega + 3u_0^2 + v_0^2 - 2u_0v_0 + \frac{u_0v_0}{(u_0^2+v_0^2)^{3/2}}, \\ D &= 4 - u_0^2 - 3v_0^2 + 2u_0v_0 - \frac{u_0^2}{(u_0^2+v_0^2)^{3/2}}.\end{aligned} \tag{13}$$

In order for the Hopf bifurcation to exist, it is necessary for the roots of the characteristic Eq. (12) to be strictly imaginary. This leads to the set of bifurcation equations

$$\frac{1}{4(u_0^2+v_0^2)^{1/2}} = 2-(u_0^2+v_0^2),$$

$$-\omega v_0 + (4-(u_0^2+v_0^2))u_0 - (u_0^2+v_0^2)v_0 - \frac{u_0}{(u_0^2+v_0^2)^{1/2}} = 0, \qquad (14)$$

$$\omega u_0 + (4-(u_0^2+v_0^2))v_0 + (u_0^2+v_0^2)u_0 - \frac{v_0}{(u_0^2+v_0^2)^{1/2}} - P = 0.$$

This set of equations, after replacing the cartesian coordinates (u_0, v_0) by polar ones (r_0, Θ_0) acquires the following form

$$4r_0(2-r_0^2) = 1,$$

$$-\omega r_0 \sin\Theta_0 + (4-r_0^2)r_0\cos\Theta_0 - r_0^3\sin\Theta_0 - \cos\Theta_0 = 0, \qquad (15)$$

$$\omega r_0\cos\Theta_0 + (4-r_0^2)r_0\sin\Theta_0 + r_0^3\cos\Theta_0 - \sin\Theta_0 - P = 0.$$

From the first equation of (15) three values of r_0 are obtained:

$$r_{01} = 1.347, \quad r_{02} = 0.126 \quad \text{and} \quad r_{03} = -1.473.$$

After substituting these values into the other two equations (15) three necessary conditions for the Hopf bifurcation are obtained

$$P^2 = 1.814\omega^2 + 6.50\omega + 2.274,$$

$$P^2 = 0.016\omega^2 + 0.001\omega + 2.01, \qquad (16)$$

$$P^2 = 2.17\omega^2 + 9.415\omega - 34.409.$$

Fig. 1 presents the Hopf bifurcation curves. Because of the symmetry of the equations with respect to P, the figure only presents the half plane $P \geq 0$. The bifurcation curves marked in Fig. 1 as 1, 2 and 3 correspond to the values r_{01}, r_{02} and r_{03}.

3. The evolution of strange attractors

Equation (1) was solved numerically using a variable-order, variable-step Gear method. In this method the accuracy of the integration and interpolation is controlled. If the appropriate condition is not satisfied, the stepsize is reduced. Numerical results are presented as phase portraits and Poincaré maps. The first digital integrations run for a long time (about 300 s) until all transients have decayed. Then $y(t)$ and $\dot{y}(t)$ are recorded either in the short time, to present phase trajectories, or are recorded to the

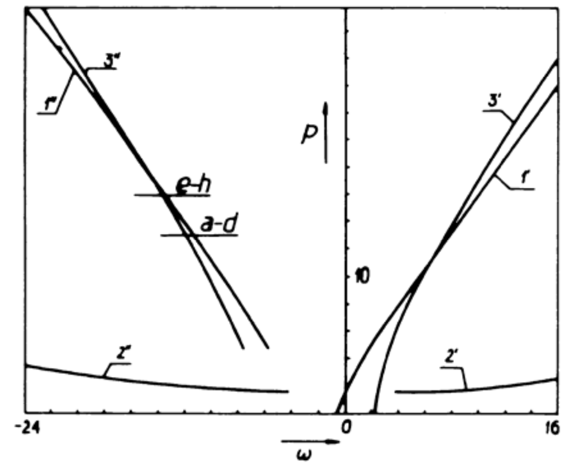


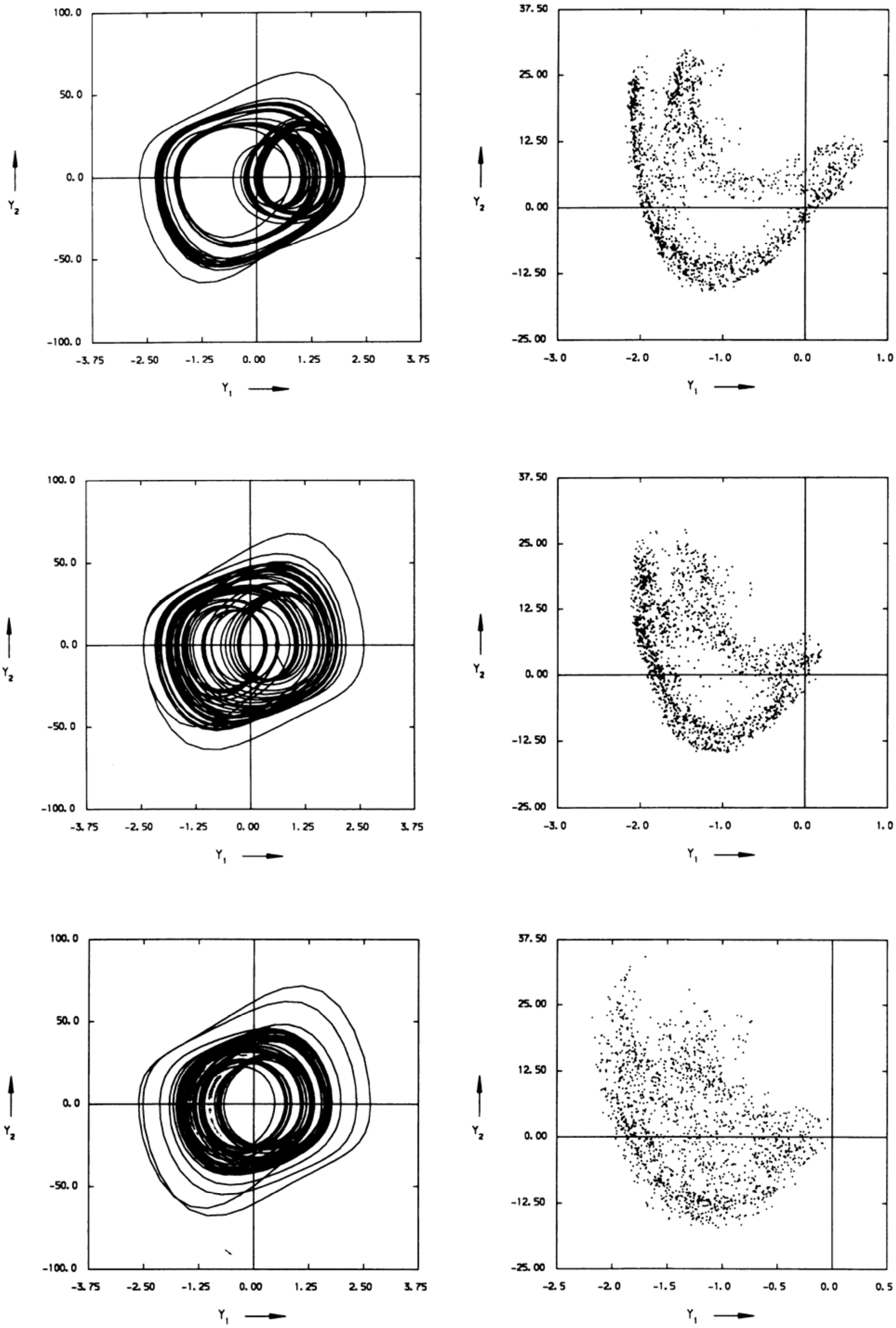
Figure 1
Hopf bifurcation curves obtained using an analytical method.

Table 1

		$\varepsilon = 8, p_0 = 1$			
Figure 2	(ω, p)	δ	γ	α	ν
a	$(-13.25, 13)$	$-13.$			
b	$(-12.25, 13)$	39.			
c	$(-11.75, 13)$	65.	69.33	40.84	26.
d	$(-10.25, 13)$	143.			
e	$(-15, 16)$	64.			
f	$(-14, 16)$	128.			
g	$(-13.75, 16)$	144.	85.33	50.27	32.
h	$(-12.5, 16)$	224.			

time $1600T$, where $T = 2\pi/\nu$, in order to obtain Poincaré maps. Because of relations (8) it is possible to observe the behaviour of the system using only two parameters (see Table 1). It must be taken into account that the behaviour of the system changes if we cross through the bifurcation curves when changing one of the two parameters. Generally, chaotic orbits near the bifurcation curves 1 and 3 were found, while near the bifurcation curve 2 regular orbits were found.

Consider the two sets of parameters presented in Table 1 and marked also in Fig. 1. Based on the numerical calculations, two possibilities of evolution of the strange attractors will be analysed. In the first case (Fig. 2a–d) for the fixed values γ, α and ν the coefficient δ is increased. It corresponds in the two parameter plane (P, ω) to increases of the parameter ω for the constant value of P. In Fig. 2a the strange attractor for $\delta = -13$ is presented. Increase of the stiffness coefficient causes the decrease of the area covered by points of the Poincaré map. For $\delta = 143$ (Fig. 2d) the chaotic dynamics of the phase flow are not as strong as in cases presented in Figures 2a–c. The Fourier spectra presented in Figs. 3a–d are broadest for the parameters which correspond to the Poincaré map presented in Fig.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0.0
-50.0
-100.0
-3.75
-2.50
-1.25
0.00
1.25
2.50
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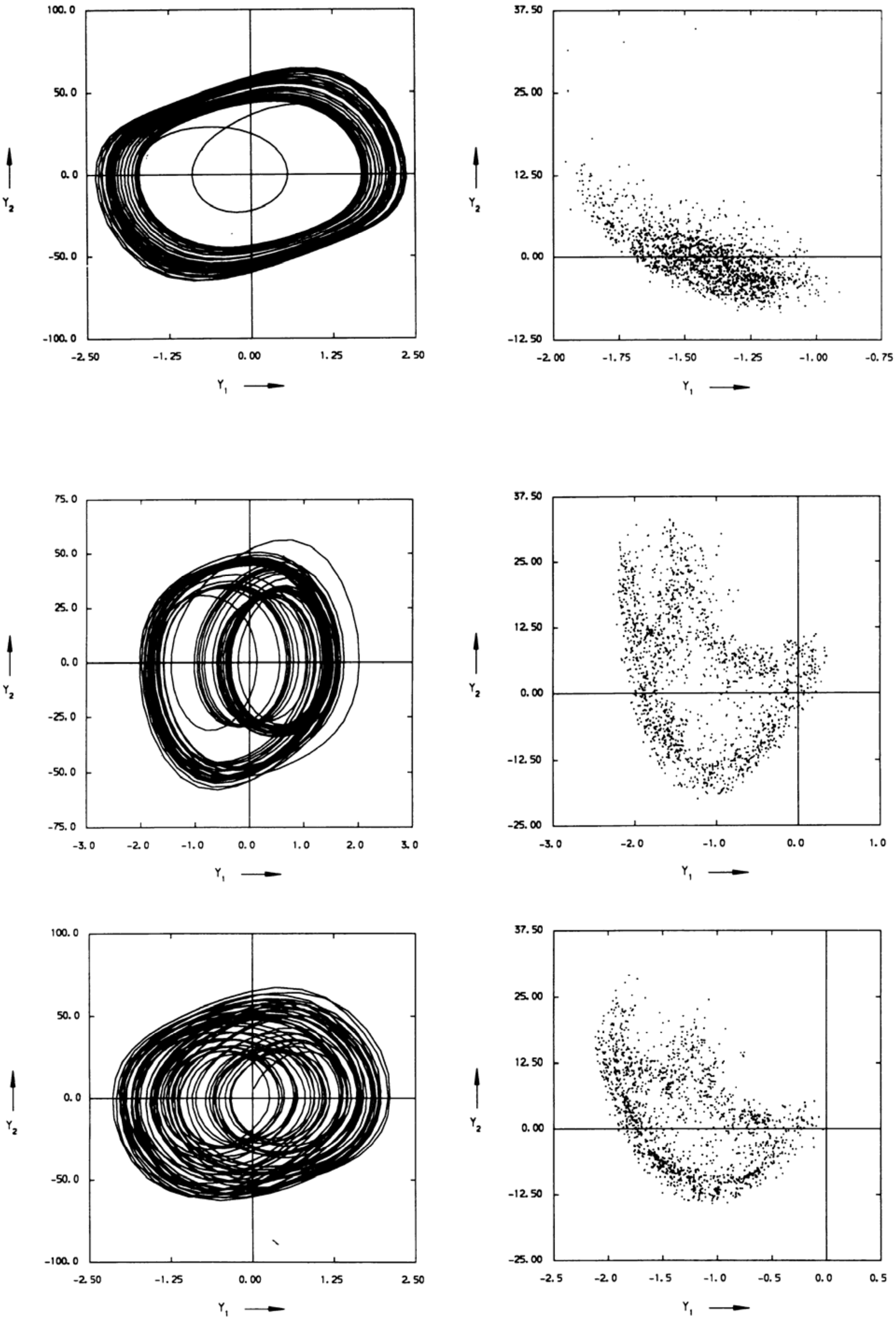
Y_1
Y_2
37.50
25.00
12.50
0.00
-12.50
-25.00
-3.0
-2.0
-1.0
0.0
1.0
-2.5
-1.5
-0.5
0.5

Figure 2 (a)–(c).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0.00
1.25
2.50
Y_2
Y_1
37.50
25.00
12.50
0.00
-12.50
-2.00
-1.75
-1.50
-1.25
-1.00
-0.75
Y_2
Y_1
75.0
50.0
25.0
0.0
-25.0
-50.0
-75.0
-3.0
-2.0
-1.0
0.0
1.0
2.0
3.0
Y_2
Y_1
37.50
25.00
12.50
0.00
-12.50
-25.00
-3.0
-2.0
-1.0
0.0
1.0
Y_2
Y_1
100.0
50.0
0.0
-50.0
-100.0
-2.50
-1.25
0.00
1.25
2.50
Y_2
Y_1
37.50
25.00
12.50
0.00
-12.50
-25.00
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-1.0
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0.5
Y_2
Y_1

Figure 2 (d)–(f).

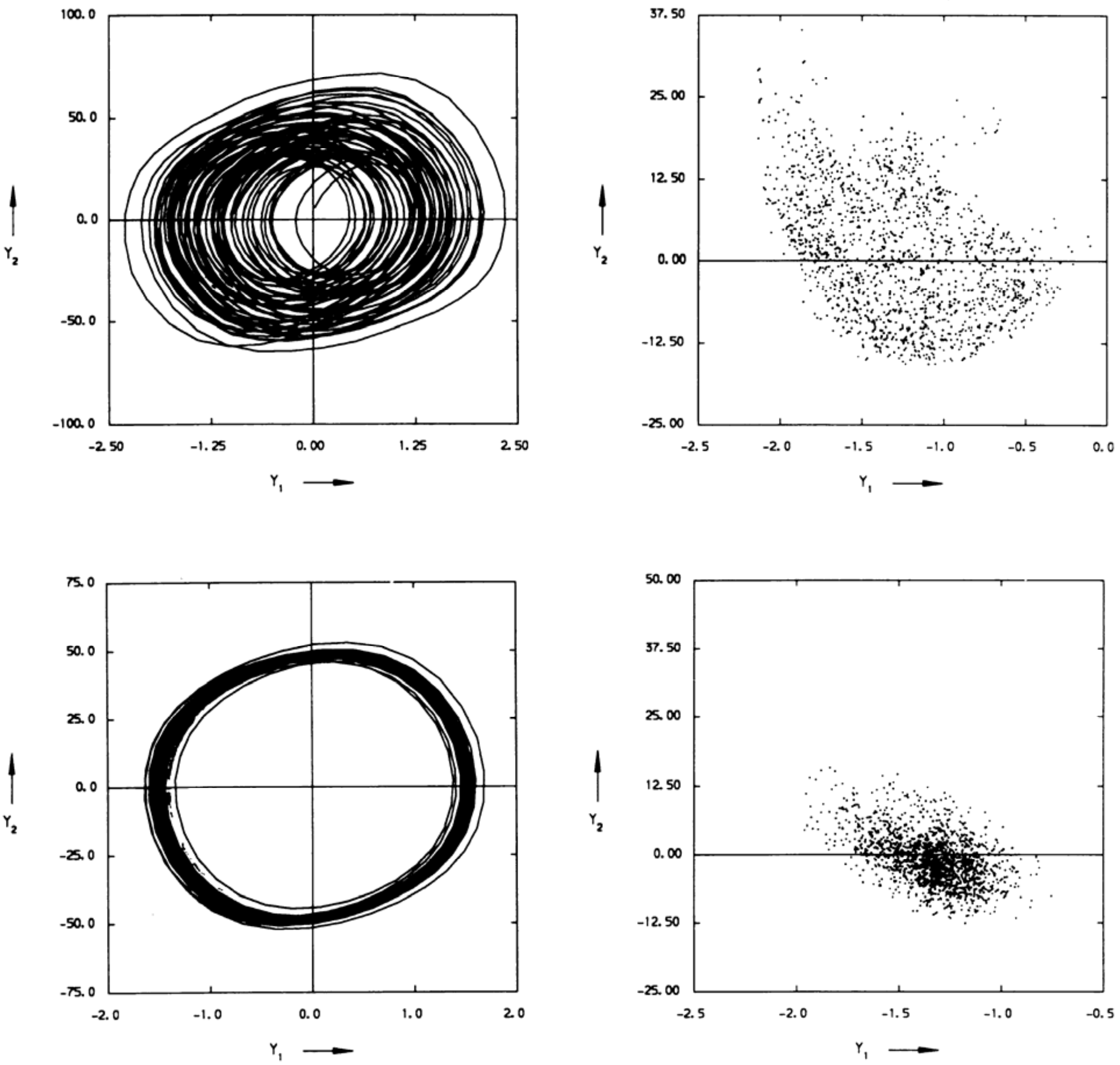


Figure 2
Phase portraits (left) and Poincarè maps (right) for the parameters as in Table 1 ($y_1 = x$, $y_2 = \dot{x}$).

2a. Increase of the δ value causes the decrease in the magnitude of the strange attractor, and the corresponding frequency spectra become narrower and smoother.

One can expect that the situation radically changes when another set of parameters is taken into consideration. In the second example the amplitude of the exciting force is equal to 1024 (note that for the first presented example it was 676). As in the previous case, keeping P constant and increasing ω, δ is increased. As shown in Fig. 2e–h the increase of δ causes a decrease in the shape of strange attractors. The shapes of the corresponding Fourier spectra presented in Fig. 3e–h become narrow and smoother with the increase of δ. The evolution of this strange attractor is very similar to that presented in the first case. The area covered by the chaotic orbits in this case is very close to those presented in Fig. 3a–d. It means that in both

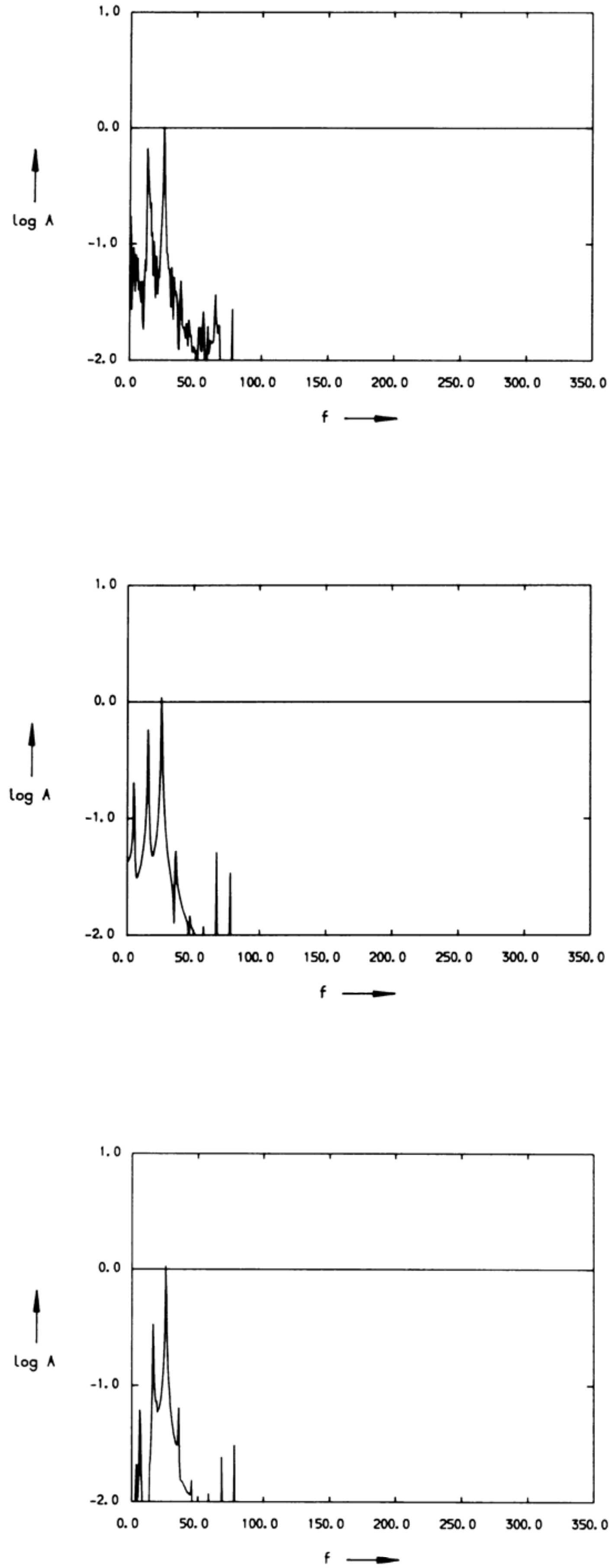
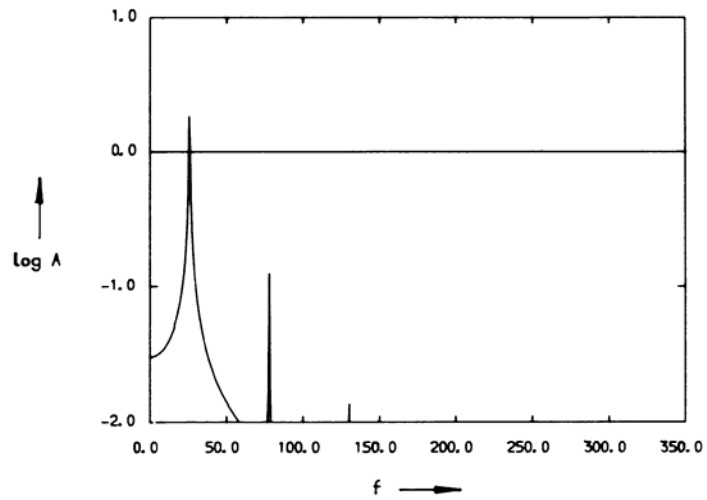


Figure 3 (a)–(c).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log A
-1.0
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100.0
150.0
200.0
250.0
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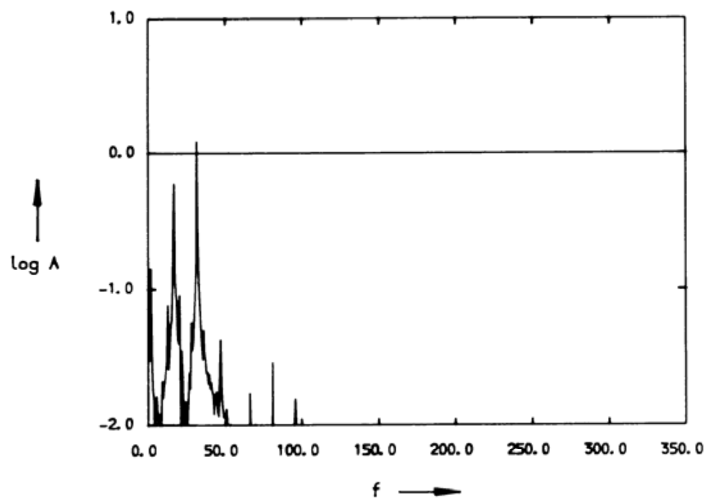
f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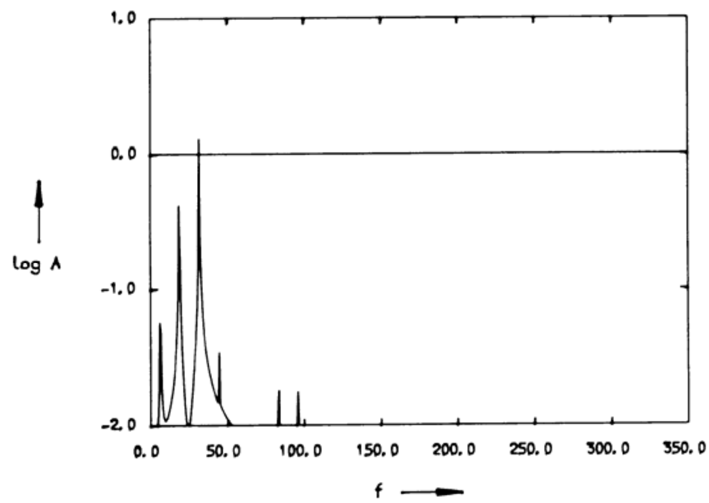
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f

Figure 3 (d)–(f).

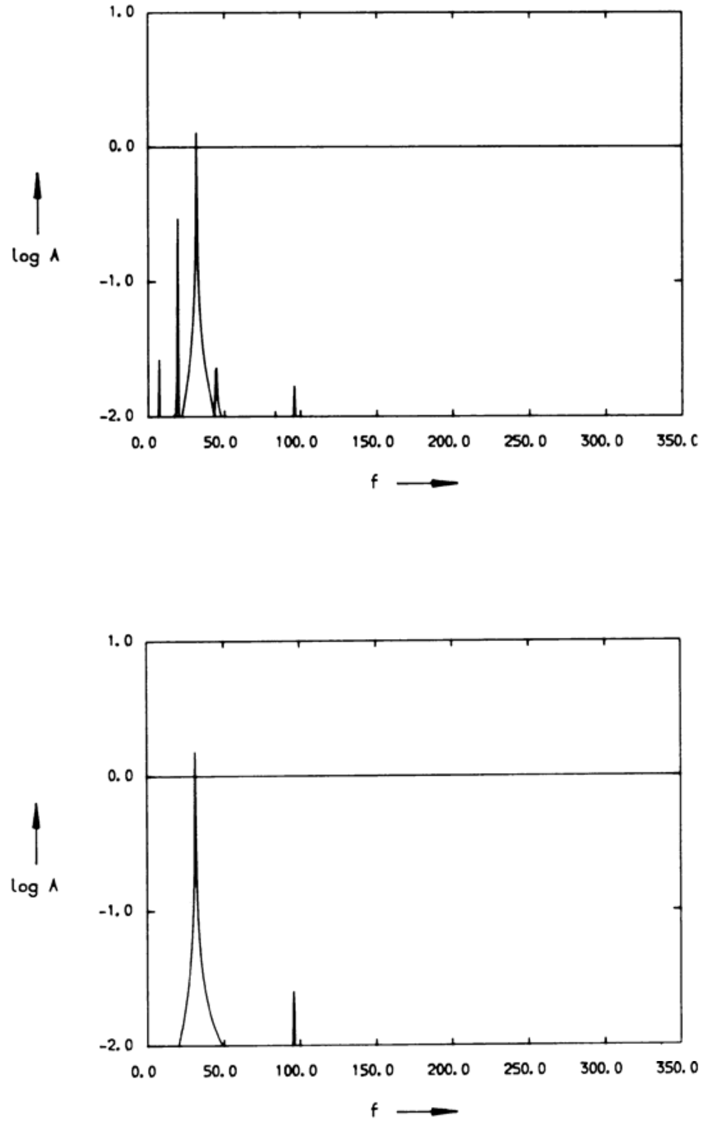

Figure 3
Frequency spectra corresponding to the Poincarè maps presented in Fig. 2.

considered cases the evolution of strange attractors is similar even if the considered parameter sets differ considerably from each other.

4. Concluding remarks

In this paper a new phenomenon of chaotic dynamics is presented. First an averaging method is used to obtain Hopf bifurcation curves for the system of equations (5), and then the behaviour of a system near the bifurcation curves is analysed. It is shown, that the evolution of strange attractors with increasing values of the stiffness coefficient δ is very similar even if the considered parameter sets differ from each other.

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Abstract

The Hopf bifurcation curves for the averaged system of second order differential equations are obtained using an analytical method. Numerical experiments have proved the existence of chaotic motion in the vicinity of these curves. For the different parameter sets, two very similar types of evolution of strange attractors are presented.

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