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DETERMINATION OF THE LIMITS OF THE UNSTABLE ZONES OF THE UNSTATIONARY NON-LINEAR MECHANICAL SYSTEMS

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Abstract—The paper presents a method of determining the unstable zones of mechanical systems circumscribed with the ordinary non-linear differential equations with periodically variable coefficients, on the assumption that the parametric excitation and non-linearity in the system are small (of the order of small parameters μ and ε). In this method the solutions are searched in the form of power series in relation to two small independent parameters μ and ε.

INTRODUCTION

The small parameter method, consisting in the searching for a solution in the form of power series in relation to a small parameter has been successfully applied in solving the ordinary non-linear differential equations with constant coefficients or linear differential equations with periodically variable coefficients [1]. Other authors [2] have applied this method to the unstationary non-linear mechanical systems, artificially conditioning other parameters on the small one.

This paper presents the method of determining the limits of the loss of stability for the non-linear and unsteady mechanical systems by assuming two small independent parameters and searching for a form of solution which, after equating one parameter to zero, is reduced to the classical method described in [1].

The general procedure for determining the unstable zones of the discrete material systems with a finite number of degrees of freedom has been presented for the system of non-linear differential equations with periodical coefficients of the first order, whereas the computational examples have been presented for the parametric non-linear systems with one or two degrees of freedom.

THE METHOD

Let us consider the system of equations having the form:

$$\frac{dx_s}{dt} = a_{s1}x_1 + \ldots + a_{sn}x_n + \mu[f_{s1}(t,\mu)x_1 + \ldots + f_{sn}(t,\mu)x_n] + \varepsilon F_s(x_1,\ldots,x_n) \qquad (1)$$

where:

$$fsj(t,u) = f^{0}_{sj}(t) + \mu fsj(t) + \mu^2 fsj^{(2)}(t) + \ldots$$
$$fsj^{(k)}(t) = fsj^{(k)}(t+T), \qquad s = 1,\ldots,n, \qquad j = 1,\ldots,n, \qquad k = 1,2,\ldots$$

The quantities μ and ε are small parameters (not necessarily of the same order), while the functions f_{sj} (t,μ) can be represented as a power series of a small parameter and are periodical with their period T.

Because for $\varepsilon = 0$ the characteristic exponents of the system (1) are the function of μ, then for $\varepsilon \neq 0$ let us search for them as a function of two small parameters μ and ε, having the form:

$$\alpha_i = \lambda_i + a_i(\mu,\varepsilon), \qquad (2)$$

assuming at the same time that λ_i are single imaginary characteristic roots of the system obtained from (1) after assuming $\mu = \varepsilon = 0$, of the form:

$$\frac{dx_s}{dt} = a_{s1}x_1 + \ldots + a_{sn}x_n. \tag{3}$$

We shall assume that the forcing of εF_s is so small that there exists a particular solution of (1) having the form:

$$x_s(t) = e^{[\lambda_i + a_i(\mu,\varepsilon)]t}, \tag{4}$$

where $y_s(t)$ is a periodic function (having the period T for $\varepsilon = 0$), whereas

$$a_i(\mu,\varepsilon) = \mu a_i^{(1,0)} + \mu^2 a_i^{(2,0)} + \ldots + \varepsilon(a_i^{(0,1)} + \mu a_i^{(1,1)} + \ldots) + \varepsilon^2(a_i^{(0,2)} + \mu a_i^{(1,2)} + \ldots) + \ldots \tag{5}$$

After substituting (4) and (5) in (1) we get:

$$\frac{dy_s}{dt} = a_{s1}y_1 + \ldots + a_{sn}y_n + \mu[fsi(t,\mu)y_1 + \ldots + fsn(t,\mu)y_n] - [\lambda_i + a_i(\mu,\varepsilon)]y_s$$
$$+ \varepsilon e^{-[\lambda_i + a_i(\mu,\varepsilon)]t} F_s[e^{[\lambda_i + a_i(\mu,\varepsilon)]t}y_1, \ldots, e^{[\lambda_i + a_i(\mu,\varepsilon)]t}y_n], \tag{6}$$

where $y_s(t)$ is a periodic function.

The solutions of (6) will be sought in the form:

$$y_s(t) = y_s^{(0,0)} + \mu y_s^{(1,0)} + \ldots + \varepsilon(y_s^{(0,1)} + \mu y_s^{(1,1)} + \ldots) + \varepsilon^2(y_s^{(0,2)} + \mu y_s^{(1,2)} + \ldots) + \ldots \tag{7}$$

After equating the expressions representing the same powers of the small parameter μ and ε and the same powers of their products $\mu^m \varepsilon^1$ $(m, 1 = 1, 2, \ldots)$, the recurrent systems of linear differential equations are obtained, which have the form:

$$\frac{dy_s^{(0,0)}}{dt} = a_{s1}y_1^{(0,0)} + \ldots + a_{sn}y_n^{(0,0)} - \lambda_i y_s^{(0,0)},$$

$$\frac{dy_s^{(1,0)}}{dt} = a_{s1}y_1^{(1,0)} + \ldots + a_{sn}y_n^{(1,0)} - \lambda_i y_s^{(1,0)} + [fs1^{(0)}(t)y_1^{(0,0)} + \ldots + fsn^{(0)}(t)y_n^{(0,0)}] - a_i^{(1,0)}y_s^{(0,0)},$$

$$\frac{dy_s^{(2,0)}}{dt} = a_{s1}y_1^{(2,0)} + \ldots + a_{sn}y_n^{(2,0)} - \lambda_i y_s^{(2,0)} - y_s^{(0,0)}a_i^{(2,0)} - y_s^{(1,0)}a_i^{(1,0)}$$
$$+ [fs1^{(1)}(t)y_1^{(0,0)} + \ldots + fsn^{(1)}(t)y_n^{(0,0)}] + [fs1^{(0)}(t)y_1^{(1,0)} + \ldots + fsn^{(0)}(t)y_n^{(1,0)}],$$

$$\vdots \qquad \vdots$$

$$\frac{dy_s^{(0,1)}}{dt} = a_{s1}y_1^{(0,1)} + \ldots + a_{sn}y_n^{(0,1)} - \lambda_i y_s^{(0,1)} + F_s[e^{\lambda it}y_1^{(0,0)}, \ldots, e^{\lambda it}y_n^{(0,0)}], \tag{8}$$

$$\frac{dy_s^{(1,1)}}{dt} = a_{s1}y_1^{(1,1)} + \ldots + a_{sn}y_n^{(1,1)} - \lambda_i y_s^{(1,1)} - a_i^{(1,1)}y_s^{(0,0)} - a_i^{(0,1)}y_s^{(1,0)}$$
$$+ F_s(e^{\lambda it}y_1^{(0,0)}, e^{\lambda it}y^{(1,0)}, \ldots, e^{\lambda it}y_n^{(0,0)}, e^{\lambda it}y_n^{(1,0)}),$$

$$\frac{dy_s^{(2,1)}}{dt} = a_{s1}y_1^{(2,1)} + \ldots + a_{sn}y_n^{(2,1)} - \lambda_i y_s^{(2,1)} - a_i^{(2,1)}y_s^{(0,0)} - a_i^{(1,1)}y_s^{(1,0)}$$
$$- a_i^{(2,0)}y_s^{(0,1)} - a_i^{(1,0)}y_s^{(1,1)} + F_s[e^{\lambda it}y_1^{(0,0)}, e^{\lambda it}y_1^{(1,0)}, e^{\lambda it}y_1^{(0,1)}, e^{\lambda it}y_1^{(1,1)}, \ldots$$
$$\ldots e^{\lambda it}y_n^{(0,0)}, e^{\lambda it}y_n^{(1,0)}, e^{\lambda it}y_n^{(0,1)}, e^{\lambda it}y_n^{(1,1)}],$$

The particular components of the expression (5) are determined from (8) from the condition of avoiding the secular terms, putting

$$e^{\lambda i t} = e^{i\bar{\lambda} i t} = \cos \bar{\lambda}_i t + i \sin \bar{\lambda}_i t.$$

It is known that the unstable zones are found near such frequency of parametric excitation $\omega\left(T = \frac{2\pi}{\omega}\right)$, that the connections $\pm\,\omega = \frac{\lambda_k \pm \lambda_i}{N_i}$ are fulfilled. In that case, the characteristic roots of the first equation of the system (8) are $\pm N\omega i$ $(N = 1, 2, 3, \ldots)$. The initial conditions are assumed only for that equation; they equal zero for the other equations. Thus, for these equations, only the particular solutions are searched.

Example 1

Let us consider the standard example of vibrations of a system with one degree of freedom, of the form:

$$m\frac{d^2x}{dt^2} + (k_0 + k_1 \cos 2t)x + k_2 x^3 = 0, \tag{9}$$

where m is the mass of the body, and k_0, k_1 and k_2 are rigidities. After assuming the parameters $\lambda^2 = \frac{k_0}{m}$, $\mu = \frac{k_1}{k_0}$, and $\varepsilon = \frac{k_2}{m}$, we obtain the equation:

$$\frac{d^2x}{dt^2} + \lambda^2(1 + \mu \cos 2t)x + \varepsilon x^3 = 0. \tag{10}$$

For $\mu = 0$, we obtain the Duffing equation, and for $\varepsilon = 0$, the Mathieu one.

Let us develop the quantity λ^2 and x into a power series of small parameters μ and ε

$$\begin{aligned} \lambda^2 &= n^2 + a^{(1,0)}\mu + \mu^2 a^{(2,0)} + \ldots + \varepsilon(a^{(0,1)} + \mu a^{(1,1)} + \ldots) \\ &\quad + \varepsilon^2(a^{(0,2)} + \mu a^{(1,2)} + \mu^2 a^{(2,2)} + \ldots) + \ldots \end{aligned} \tag{11}$$

$$\begin{aligned} x &= x^{(0,0)} + \mu x^{(1,0)} + \mu^2 x^{(2,0)} + \ldots + \varepsilon(x^{(0,1)} + \mu x^{(1,1)} + \mu^2 x^{(2,1)} + \ldots) \\ &\quad + \varepsilon^2(x^{(0,2)} + \mu x^{(1,2)} + \mu^2 x^{(2,2)} + \ldots) + \ldots \end{aligned} \tag{12}$$

Let us consider the first unstable zone ($n = 1$). After substituting (11) and (12) into (10), and equating the expressions representing the same powers μ and ε and their combinations, we obtain the following recurrent system of linear differential equations:

$$\begin{aligned}
&\ddot{x}^{(0,0)} + x^{(0,0)} = 0, \\
&\ddot{x}^{(1,0)} + x^{(1,0)} = -x^{(0,0)}\cos 2t - a^{(1,0)}x^{(0,0)}, \\
&\ddot{x}^{(2,0)} + x^{(2,0)} = -a^{(2,0)}x^{(0,0)} - a^{(1,0)}x^{(1,0)} - a^{(1,0)}\cos 2t\, x^{(0,0)} - x^{(1,0)}\cos 2t, \\
&\vdots \\
&\ddot{x}^{(0,1)} + x^{(0,1)} = (x^{(0,0)})^3 - a^{(0,1)}x^{(0,0)}, \\
&\ddot{x}^{(1,1)} + x^{(1,1)} = -3(x^{(0,0)})^2 x^{(1,0)} - a^{(1,0)}x^{(0,1)} - a^{(0,1)}x^{(0,0)}\cos 2t - a^{(1,1)}x^{(0,0)} \\
&\qquad - x^{(0,1)}\cos 2t, \\
&\ddot{x}^{(2,1)} + x^{(2,1)} = -3x^{(0,0)}(x^{(1,0)})^2 - 3(x^{(0,0)})^2 x^{(2,0)} - a^{(2,1)}x^{(0,0)} - a_{1,1}x^{(1,0)} \\
&\qquad - a^{(0,1)}x^{(2,0)} - a^{(0,1)}x^{(1,0)}\cos 2t - a^{(2,0)}x^{(0,1)} - x^{(0,1)}a^{(1,0)}\cos 2t \\
&\qquad - x^{(1,1)}a^{(1,0)} - x^{(1,1)}\cos 2t, \\
&\vdots \\
&\ddot{x}^{(0,2)} + x^{(0,2)} = -3(x^{(0,0)})^2 x^{(0,1)} - a^{(0,2)}x^{(0,0)} - a^{(0,1)}x^{(0,1)}. \\
&\vdots
\end{aligned} \tag{13}$$

Let us assume the initial conditions $x(t = 0) = A_0$, $\dot{x}(t = 0) = B_0$. From the first equation of the system (13) we obtain:

$$x^{(0,0)} = A_0 \cos t + B_0 \sin t. \tag{14}$$

Substituting (14) into the second equation of the system (13), from the condition of avoiding the secular terms, we get: 1^0 $a^{(1,0)} = -\frac{1}{2}$ and $B_0 = 0$ or 2^0 $a^{(1,0)} = \frac{1}{2}$ and $A_0 = 0$. Let us consider the first case: then $x^{(1,0)} = \frac{1}{16} A_0 \cos 3t$ and, passing on to the third equation, we have:

$$a^{(2,0)} = \frac{7}{32}, \qquad x^{(2,0)} = -\frac{9}{256} A_0 \cos 3t + \frac{A_0}{768} \cos 5t.$$

Acting analogously we obtain:

$$a^{(0,1)} = -\frac{3}{4} A_0^2, \qquad x^{(0,1)} = \frac{1}{32} A_0^3 \cos 3t,$$

$$a^{(1,1)} = \frac{5}{16} A_0^2, \qquad x^{(1,1)} = -\frac{11 A_0^3}{256} \cos 3t + \frac{A_0^3}{384} \cos 5t,$$

$$a^{(2,1)} = \frac{75}{1024} A_0^2, \qquad x^{(2,1)} = \frac{73}{24\,576} A_0^3 \cos 3t - \frac{235}{73\,728} A_0^3 \cos 5t + \frac{16 A_0^3}{147\,456} \cos 7t,$$

$$a^{(0,2)} = -\frac{3}{128} A_0^4, \qquad x^{(0,2)} = \frac{3}{1024} A_0^5 \cos 3t + \frac{3}{3072} A_0^5 \cos 5t.$$

Taking the above calculations into account in (11), we obtain:

$$\lambda^2 = 1 - \frac{1}{2}\mu + \frac{7}{32}\mu^2 + \ldots + \varepsilon A_0^2\left(-\frac{3}{4} + \frac{5}{16}\mu + \frac{75}{1024}\mu^2 + \ldots\right) + \varepsilon^2\left(-\frac{3}{128} A_0^4 + \ldots\right) + \ldots$$

For $\varepsilon = 0$ we obtain one branch of the first limit of stability loss of the Mathieu equation, and for $\mu = 0$—the dependence of the frequency on the amplitude for a conservative system with the Duffing characteristic.

Example 2

Let us consider the unsteady non-linear system with two degrees of freedom, the movement of which is circumscribed with the following differential equations:

$$\begin{aligned} M\ddot{z}_1 + c(\dot{z}_1 - \dot{z}_2) + k(z_1 - z_2) + k_1(z_1 - z_2)^3 &= 0, \\ m\ddot{z}_2 - c(\dot{z}_1 - \dot{z}_2) - k(z_1 - z_2) - k_1(z_1 - z_2)^3 &= -(k_3 - k_0 \cos 2\omega t) z_2, \end{aligned} \tag{15}$$

where: m and M are masses, k, k_1, k_3 and k_0 are rigidities, and c is a damping coefficient. The frequencies of free vibrations of the conservative system will be calculated after assuming $c = k_1 = k_0 = 0$ in (15). They amount to:

$$\alpha_{1,2}^2 = \frac{1}{2}\left[\frac{k}{M} + \frac{k + k_3}{m} \mp \sqrt{\left(\frac{k}{M} + \frac{k + k_3}{m}\right)^2 - \frac{4kk_3}{Mm}}\right]. \tag{16}$$

We shall limit ourselves to calculating the first simple parametric resonance around the frequency α_1.

Equation (15) will be rearranged to the form:

$$\ddot{y}_1 + \lambda_1^2 y_1 = -\varepsilon \bar{M} \lambda_1^2 (\varepsilon_1 y_2 - \varepsilon_2 y_1)^3 - \mu \delta_1 \lambda_1 (\varepsilon_1 \dot{y}_2 - \varepsilon_2 \dot{y}_1) + \mu \rho_1 \lambda_1^2 (y_1 - y_2) \cos 2\tau,$$

$$\ddot{y}_2 + \lambda_1^2 \nu^2 y_2 = -\varepsilon \bar{M} \lambda_1^2 \nu^2 (\varepsilon_1 y_2 - \varepsilon_2 y_1)^3 - \mu \delta_1 \lambda_1 \nu (\varepsilon_1 \dot{y}_2 - \varepsilon_2 \dot{y}_1) + \mu \rho_2 \lambda_1^2 \nu^2 (y_1 - y_2) \cos 2\tau, \tag{17}$$

where:

$$\mu = \frac{k_0}{k_1}, \quad \varepsilon = \frac{k_1}{k}, \quad \tau = \omega t, \quad \lambda_1^2 = \frac{\alpha_1^2}{\omega^2}, \quad \frac{M}{m} = \bar{M}, \quad \ddot{y}_i = \frac{d^2 y_i}{d\tau^2} (i = 1, 2),$$

$$z_1 = \beta_1 y_2 - \beta_2 y_1, \quad z_2 = \psi(y_1 - y_2), \quad \frac{\alpha_2}{\alpha_1} = \nu, \quad \delta_1 = \frac{M \alpha_1 c}{mk},$$

$$\rho_1 = \frac{\gamma_1 k_1 \psi}{m \alpha_1^2}, \quad \rho_2 = \frac{\gamma_2 k_1 \psi}{m \alpha_i^2}, \quad \gamma_1 = \frac{k - M\alpha_1^2}{k}, \quad \gamma_2 = \frac{k - M\alpha_2^2}{k}, \quad \psi = \frac{1}{\gamma_1 - \gamma_2},$$

$$\varepsilon_1 = \beta_1 + \psi, \quad \beta_1 = \frac{m\gamma_1}{M(\gamma_1 - \gamma_2)}, \quad \varepsilon_2 = \beta_2 + \psi, \quad \beta_2 = \frac{m\gamma_2}{M(\gamma_1 - \gamma_2)}. \tag{18}$$

We seek the solutions of (17), of the form:

$$y_1 = y_1^{(0,0)} + \mu y_1^{(1,0)} + \mu^2 y_1^{(2,0)} + \dots + \varepsilon y_1^{(0,1)} + \varepsilon^2 y_1^{(0,2)} + \dots + \mu\varepsilon y_1^{(1,1)} + \dots$$

$$y_2 = y_2^{(0,0)} + \mu y_2^{(1,0)} + \mu^2 y_2^{(2,0)} + \dots + \varepsilon y_2^{(0,1)} + \varepsilon^2 y_2^{(0,2)} + \dots + \mu\varepsilon y_2^{(1,1)} + \dots$$

$$\lambda_1^2 = 1 + \mu a_1^{(1,0)} + u^2 a_1^{(2,0)} + \dots + \varepsilon a_1^{(0,1)} + \varepsilon^2 a_1^{(0,2)} + \dots + \mu\varepsilon a_1^{(1,1)} + \dots$$

$$\lambda_1 = 1 + \frac{a_1^{(1,0)}}{2}\mu + \frac{a_1^{(0,1)}}{2}\varepsilon + \frac{a_1^{(1,1)}}{2}\mu\varepsilon + \mu^2 \frac{a_1^{(2,0)}}{4} + \varepsilon^2 \frac{a_1^{(0,2)}}{4} + \dots \tag{19}$$

After substituting (19) into (17) we obtain the following system of linear differential equations:

$$\ddot{y}_1^{(0,0)} + y_1^{(0,0)} = 0,$$

$$\ddot{y}_1^{(1,0)} + y_1^{(1,0)} = -a_1^{(1,0)} y_1^{(0,0)} - \delta_1(\varepsilon_1 \dot{y}_2^{(0,0)} - \varepsilon_2 \dot{y}_1^{(0,0)}) + \rho_1 (y_1^{(0,0)} - y_2^{(0,0)}) \cos 2\tau,$$

$$\begin{aligned}\ddot{y}_1^{(2,0)} + y_1^{(2,0)} = {} & -a_1^{(1,0)} y_1^{(1,0)} - a_1^{(2,0)} y_1^{(0,0)} - \delta_1 \frac{a_1^{(1,0)}}{2} (\varepsilon_1 \dot{y}_2^{(0,0)} - \varepsilon_2 \dot{y}_1^{(0,0)}) \\ & - \delta_1 (\varepsilon_1 y_2^{(1,0)} - \varepsilon_2 y_1^{(1,0)}) + \rho_1 a_1^{(1,0)} (y_1^{(0,0)} - y_2^{(0,0)}) \cos 2\tau \\ & + \rho_1 (y_1^{(1,0)} - y_2^{(1,0)}) \cos 2\tau,\end{aligned}$$

$$\ddot{y}_1^{(0,1)} + y_1^{(0,1)} = -a_1^{(0,1)} y_1^{(0,0)} - \bar{M}(\varepsilon_1 y_2^{(0,0)} - \varepsilon_2 y_1^{(0,0)})^3,$$

$$\begin{aligned}\ddot{y}_1^{(0,2)} + y_1^{(0,2)} = {} & -a_1^{(0,1)} y_1^{(0,1)} - a_1^{(0,2)} y_1^{(0,0)} - \bar{M} a_1^{(0,1)} (\varepsilon_1 y_2^{(0,0)} - \varepsilon_2 y_1^{(0,0)})^3 \\ & - \bar{M}\{3\varepsilon_1^3 (y_2^{(0,0)})^2 y_2^{(0,1)} - 6\varepsilon_1^2 \varepsilon_2 y_2^{(0,1)} y_1^{(0,0)} + 6\varepsilon_1 \varepsilon_2^2 y_2^{(0,0)} y_1^{(0,0)} y_1^{(0,1)} \\ & + 3\varepsilon_1 \varepsilon_2^2 y_2^{(0,1)} (y_1^{(0,0)})^2 - 3\varepsilon_2^3 (y_1^{(0,0)})^2 y_1^{(0,1)} - 3\varepsilon_1^2 \varepsilon_2 (y_2^{(0,0)})^2 y_1^{(0,1)}\},\end{aligned}$$

$$\begin{aligned}\ddot{y}^{(1,1)} + y_1^{(1,1)} = {} & -a_1^{(1,1)} y_1^{(0,0)} - a_1^{(0,1)} y_1^{(1,0)} - a_1^{(1,0)} y_1^{(0,1)} - \bar{M} a_1^{(1,0)} (\varepsilon_1 y_2^{(0,0)} - \varepsilon_2 y_1^{(0,0)})^3 \\ & - \bar{M}[3\varepsilon_1^3 (y_2^{(0,0)})^2 y_2^{(1,0)} - 6\varepsilon_1^2 y_2^{(0,0)} y_2^{(1,0)} \varepsilon_2 y_1^{(0,0)} + 6\varepsilon_1 \varepsilon_2^2 y_2^{(0,0)} y_1^{(1,0)} y_1^{(0,0)} \\ & - \delta_1 \frac{a_1^{(0,1)}}{2} (\varepsilon_1 \dot{y}_2^{(0,0)} - \varepsilon_2 \dot{y}_1^{(0,0)}) - \delta_1 (\varepsilon_1 \dot{y}_2^{(0,1)} - \varepsilon_2 \dot{y}_1^{(0,1)}) \\ & + \rho_1 (y_1^{(0,1)} - y_2^{(0,1)}) \cos 2\tau + \rho_1 a_1^{(0,1)} (y_1^{(0,0)} - y_2^{(0,0)}) \cos 2\tau\end{aligned} \tag{20'}$$

$$\ddot{y}_2^{(0,0)} + \nu^2 y_2^{(0,0)} = 0,$$

$$\ddot{y}_2^{(1,0)} + \nu^2 y_2^{(1,0)} = -\nu^2 a_1^{(1,0)} y_2^{(0,0)} - \delta_2 \nu(\varepsilon_1 \dot{y}_2^{(0,0)} - \varepsilon_2 \dot{y}_1^{(0,0)}) + \rho_2 \nu^2 (y_1^{(0,0)} - y_2^{(0,0)}) \cos 2\tau,$$

$$\begin{aligned}\ddot{y}_2^{(2,0)} + \nu^2 y_2^{(2,0)} = {} & -\nu^2 a_1^{(1,0)} y_2^{(1,0)} - \nu^2 a_1^{(2,0)} y_2^{(0,0)} - \delta_2 \nu \frac{a_1^{(1,0)}}{2} (\varepsilon_1 \dot{y}_2^{(0,0)} - \varepsilon_2 \dot{y}_1^{(0,0)}) \\ & - \delta_2 \nu (\varepsilon_1 \dot{y}_2^{(1,0)} - \varepsilon_2 \dot{y}_1^{(1,0)}) + \rho_2 \nu^2 a_1^{(1,0)} (y_1^{(0,0)} - y_2^{(0,0)}) \cos 2\tau \\ & + \rho_2 \nu^2 (y_1^{(1,0)} - y_2^{(1,0)}) \cos 2\tau\end{aligned}$$

$$\ddot{y}_2^{(0,1)} + \nu^2 y_2^{(0,1)} = -\nu^2 a_1^{(0,1)} y_1^{(0,0)} - \bar{M} \nu^2 (\varepsilon_1 y_2^{(0,0)} - \varepsilon_2 y_1^{(0,0)})^3,$$

$$\begin{aligned}\ddot{y}_2^{(0,2)} + \nu^2 y_2^{(0,2)} = {} & -\nu^2 y_2^{(0,1)} a_1^{(0,1)} - a_1^{(0,2)} \nu^2 y_2^{(0,0)} - \nu^2 \bar{M} a_1^{(0,1)} (\varepsilon_1 y_2^{(0,0)} - \varepsilon_2 y_1^{(0,0)})^3 \\ & - \bar{M} \nu^2 (3\varepsilon_1^3 (y_2^{(0,0)})^2 y_2^{(0,1)} - 6\varepsilon_1^2 \varepsilon_2 y_2^{(0,0)} y_2^{(0,1)} y_1^{(0,0)} \\ & + 6\varepsilon_1 \varepsilon_2^2 y_2^{(0,0)} y_1^{(0,0)} y_1^{(0,1)} + 3\varepsilon_1 \varepsilon_2^2 y_2^{(0,1)} (y_1^{(0,0)})^2 - 3\varepsilon_2^3 (y_1^{(0,0)})^2 y_1^{(0,1)} \\ & - 3\varepsilon_1^2 (y_2^{(0,0)})^2 \varepsilon_2 y_1^{(0,1)}),\end{aligned}$$

$$\begin{aligned}\ddot{y}_2^{(1,1)} + y_2^{(1,1)} = {} & -\nu^2 a_1^{(1,1)} y_2^{(0,0)} - \nu^2 a_1^{(0,1)} y_2^{(1,0)} - \nu_1^2 a_1^{(1,0)} y_2^{(0,1)} \\ & - \nu^2 \bar{M} a_1^{(1,0)} (\varepsilon_1 y_2^{(0,0)} - \varepsilon_2 y_1^{(0,0)})^3 - \nu^2 \bar{M} (3\varepsilon_1^3 (y_2^{(0,0)})^2 y_2^{(1,0)} \\ & - 6\varepsilon_1^2 \varepsilon_2 y_2^{(0,0)} y_2^{(1,0)} y_1^{(0,0)} + 6\varepsilon_1 \varepsilon_2^2 y_2^{(0,0)} y_1^{(1,0)} y_1^{(0,0)} - 3\varepsilon_1^2 \varepsilon_2 (y_2^{(0,0)})^2 y_1^{(1,0)} \\ & + 3\varepsilon_1 \varepsilon_2^2 y_2^{(1,0)} y_1^{2(0,0)} - 3\varepsilon_2^3 (y_1^{(0,0)})^2 y_1^{(1,0)}) - \delta_2 \nu \frac{a_1^{(0,1)}}{2} \cdot \\ & (\varepsilon_1 \dot{y}_2^{(0,0)} - \varepsilon_2 \dot{y}_1^{(0,0)}) - \delta_2 \nu (\varepsilon_1 \dot{y}_2^{(0,1)} - \varepsilon_2 \dot{y}_1^{(0,1)}) \\ & + \rho_2 \nu^2 (y_1^{(0,1)} - y_2^{(0,1)}) \cos 2\tau + \rho_2 \nu^2 a(_1^{(0,1)} (y_1^{(0,0)} - y_2^{(0,0)}) \cos 2\tau. \qquad (20'')\end{aligned}$$

Let us assume the initial conditions $y_1^{(0,0)}(0) = A_0$, $\dot{y}_1^{(0,0)}(0) = B_0$ whereas, because we assume $y_2^{(0,0)} = 0$, then the two remaining initial conditions can be unrestricted. Let us limit ourselves to considering the equations occurring near μ, ε, and $\mu\varepsilon$. After calculations we shall obtain:

$$y_1^{(0,0)} = A_0 \cos \tau + B_0 \sin \tau,$$

$$a_1^{(1,0)} = \pm \sqrt{\frac{\rho_1^2}{4} - \varepsilon_2^2 \delta_1^2},$$

$$y_1^{(1,0)} = -\frac{\rho_1 A_0}{16} \cos 3\tau - \frac{\rho_1 B_0}{16} \sin 3\tau,$$

$$\begin{aligned}y_2^{(1,0)} = {} & -\frac{\left(A_0 \delta_2 \nu \varepsilon_2 + \dfrac{\rho_2 \nu^2 B_0}{2}\right)}{\nu^2 - 1} \sin \tau + \frac{\left(\delta_2 \nu \varepsilon_2 B_0 + \dfrac{\rho_2 \nu^2 A_0}{2}\right)}{\nu^2 - 1} \cos \tau \\ & + \frac{\rho_2 \nu^2 A_0}{2(\nu^2 - 9)} \cos 3\tau + \frac{\rho_2 \nu^2 B_0}{2(\nu^2 - 9)} \sin 3\tau,\end{aligned}$$

$$a_1^{(0,1)} = \frac{3}{4} \bar{M} \varepsilon_2^3 (A_0^2 + B_0^2),$$

$$y_1^{(0,1)} = -\frac{1}{8} \bar{M} \varepsilon_2^3 A_0 \left(\frac{A_0^2}{4} - \frac{3}{4} B_0^2\right) \cos 3\tau - \frac{1}{8} \bar{M} \varepsilon_2^3 B_0 \left(\frac{3}{4} A_0^2 - \frac{1}{4} B_0^2\right) \sin 3\tau,$$

$$
\begin{aligned}
y_2^{(0,1)} = {} & \frac{\left(-\nu^2 a_1^{(0,1)} A_0 + \frac{3}{4}\bar{M}\nu^2\varepsilon_2^3 A_0^3 + \frac{3}{4}\bar{M}\nu^2\varepsilon_2^3 A_0 B_0^2\right)}{\nu^2-1}\cos\tau \\
& + \frac{\left(-\nu^2 a_1^{(0,1)} B_0 + \frac{3}{4}\bar{M}\nu^2\varepsilon_2^3 A_0^2 B_0 + \frac{3}{4}\bar{M}\nu^2\varepsilon_2^3 B_0^3\right)}{\nu^2-1}\sin\tau \qquad (21) \\
& + \frac{\left(\frac{1}{4}\bar{M}\nu^2\varepsilon_2^3 A_0^3 - \frac{3}{4}\bar{M}\nu^2\varepsilon_2^3 A_0 B_0^2\right)}{\nu^2-9}\cos 3\tau + \frac{\left(\frac{3}{4}\bar{M}\nu^2\varepsilon_2^3 A_0^2 B_0 - \frac{\bar{M}\nu^2\varepsilon_2^3 B_0^3}{4}\right)}{\nu^2-9}\sin 3\tau.
\end{aligned}
$$

From the sixth equation of (20′) we obtain the expressions for $\cos\tau$ and $\sin\tau$, which we equate to zero in order to avoid the terms unrestrictedly growing in time. We obtain:

$$
\begin{aligned}
& -a_1^{(1,1)} A_0 + \frac{3}{4}\bar{M} a_1^{(1,0)}\varepsilon_2^3 A_0^3 + \bar{M} a_1^{(1,0)}\varepsilon_2^3 A_0 B_0^2 \\
& - 3\bar{M}\varepsilon_1\varepsilon_2^2\Bigg[-\frac{\left(A_0\delta_2\nu\varepsilon_2 + \frac{\rho_2\nu^2 B_0}{2}\right)}{\nu^2-1}\cdot\frac{A_0 B_0}{2} + \frac{\left(\sigma_2\nu\varepsilon_2 B_0 + \frac{\rho_2\nu^2 A_0}{2}\right)}{\nu^2-1}\left(\frac{3}{4}A_0^2 + \frac{1}{4}B_0^2\right) \\
& + \frac{\rho_2\nu^2 A_0}{2(\nu^2-9)}\left(\frac{1}{4}A_0^2 - \frac{1}{4}B_0^2\right) + \frac{\rho_2\nu^2 B_0}{2(\nu^2-9)}\cdot\frac{A_0 B_0}{2}\Bigg] + 3\bar{M}\varepsilon_2^3\left[-\frac{A_0^3}{64}\rho_1 - \frac{A_0\rho_1 B_0^2}{32} + \frac{1}{64}B_0^2\rho_1 A_0\right] \\
& + \delta_1\frac{a_1^{(0,1)}}{2}\varepsilon_2 B_0 - \delta_1\varepsilon_1\cdot\left(\frac{-\nu^2 a_1^{(0,1)} B_0 + \frac{3}{4}\bar{M}\nu^2\varepsilon_2^3 A_0^2 B_0 + \frac{3}{4}\bar{M}\nu^2\varepsilon_2^3 B_0^3}{\nu^2-1}\right) - \frac{\rho_1}{16}A_0\bar{M}\varepsilon_3^2\left(\frac{1}{4}A_0^2 - \frac{3}{4}B_0^2\right. \\
& - \frac{\rho_1}{2}\frac{\left(-\nu^2 a_1^{(0,1)} A_0 + \frac{3}{4}\bar{M}\nu^2\varepsilon_2^3 A_0^3 + \frac{3}{4}\bar{M}\varepsilon_2^3 A_0 B_0^2\right)}{\nu^2-1} + \frac{1}{2}\frac{\left(\frac{1}{4}\bar{M}\nu^2\varepsilon_2^3 A_0^3 - \frac{3}{4}\bar{M}\nu^2\varepsilon_2^3 A_0 B_0^2\right)}{\nu^2-9} \\
& \left. + \frac{1}{2}\rho_1 a_1^{(0,1)} A_0\right] = 0, \qquad (22)
\end{aligned}
$$

$$
\begin{aligned}
& -a_1^{(1,1)} B_0 + \frac{1}{4}\bar{M} a_1^{(1,0)}\varepsilon_2^3\cdot 3A_0^2 B_0 + \frac{3}{4}\bar{M} a_1^{(1,0)}\varepsilon_2^3 B_0^3 - 3\bar{M}\varepsilon_1\varepsilon_2^2\cdot \\
& \Bigg[-\frac{\left(A_0\delta_2\nu\varepsilon_2 + \frac{\rho_2\nu^2 B_0}{2}\right)}{\nu^2-1}\left(\frac{1}{4}A_0^2 + \frac{3}{4}B_0^2\right) + \frac{\left(\delta_2\nu\varepsilon_2 B_0 + \frac{\rho_2\nu^2 A_0}{2}\right)}{\nu^2-1}\cdot\frac{A_0 B_0}{2} \\
& - \frac{1}{4}\frac{\rho_2\nu^2 A_0^2 B_0}{(\nu^2-9)} + \frac{\rho_2\nu^2 B_0}{2(\nu^2-9)}\left(-\frac{1}{4}B_0^2 + \frac{1}{4}A_0^2\right)\Bigg] \\
& + 3\bar{M}\varepsilon_2^3\left[-\frac{\rho_1 A_0^2 B_0}{64} + \frac{A_0^2 B_0\rho_1}{32} + \frac{1}{64}B_0^3\rho_1\right] - \frac{\delta_1 a_1^{(0,1)}}{2}\varepsilon_2 A_0 \\
& - \delta_1\varepsilon_1\left(-\frac{\left(-\nu^2 a_1^{(0,1)} A_0 + \frac{3}{4}\bar{M}\nu^2\varepsilon_2^3 A_0^3 + \frac{3}{4}\bar{M}\nu^2\varepsilon_2^3 A_0 B_0^2\right)}{\nu^2-1}\right)
\end{aligned}
$$

$$-\frac{\rho_1}{16}\bar{M}\varepsilon_2^3 B_0\left(\frac{3}{4}A_0^2-\frac{1}{4}B_0^2\right)-\frac{\rho_1}{2}\left(\frac{-\nu^2 a_1^{(0,1)}B_0+\frac{3}{4}\bar{M}\nu^2\varepsilon_2^3A_0^2B_0+\frac{3}{4}\bar{M}\nu^2\varepsilon_2^3B_0^3}{\nu^2-1}\right.$$

$$\left.+\frac{\left(\frac{3}{4}\bar{M}\nu^2\varepsilon_2^3A_0^2B_0-\frac{1}{4}\bar{M}\nu^2\varepsilon_2^3B_0^3\right)}{2(\nu^2-9)}-\frac{\rho_1 a_1^{(0,1)}B_0}{2}\right]=0.$$

In order to avoid time-consuming calculations, we assume that $B_0 = 0$. Then:

$$a_1^{(1,1)}=\frac{3}{4}\bar{M}a_1^{(1,0)}\varepsilon_2^3A_0^2-\bar{M}\varepsilon_1\varepsilon_2^2\frac{\rho_2\nu^2}{2(\nu^2-1)}\cdot\frac{9}{8}A_0^2-\frac{3}{8}\bar{M}\varepsilon_1\varepsilon_2^2\frac{\rho_2\nu^2}{(\nu^2-9)}A_0^2$$

$$-\frac{3}{64}\bar{M}\varepsilon_2^3\rho_1A_0^2-\frac{1}{64}\bar{M}\varepsilon_2^3A_0^2+\frac{\rho_1\nu^2a_1^{(0,1)}}{2(\nu^2-1)}-\frac{3\bar{M}\nu^2\varepsilon_2^3A_0^2}{8(\nu^2-1)}+\frac{1}{8}\frac{\bar{M}\nu^2\varepsilon_2^3A_0^2}{(\nu^2-9)}+\frac{\rho_1a_1^{(0,1)}}{2}$$

Substituting the calculated values $a_1^{(1,0)}$, $a_1^{(0,1)}$, $a_1^{(1,1)}$ into the third equation (19), we obtain the analytic form of the limit of the loss of stability, dependent on μ, ε and the other parameters.

SUMMARY AND CONCLUDING REMARKS

The method presented in this paper makes it possible to find the analytical solution of the equations circumscribing the vibrations of non-linear unstationary systems. The characteristic multipliers and the solutions are sought in the form of power series of small parameters μ (characterising the quantity of non-linearity). In this method after assuming $\mu = 0$ or $\varepsilon = 0$ we obtain the solutions, which could be determined with the use of the standard method of small parameters. Such analytical form of solution makes it easier to analyse the influence of non-linearity on the limit of stability loss, or vice versa, the influence of parametric excitation on the preservation of the non-linear system. Moreover, it is easy to obtain from it the solution for $\mu = 0$ or $\varepsilon = 0$.

As opposed to the standard method, where the idea used to the systems analysed in the paper consists in a stiff dependence of one parameter on the other (which in the case of reducing one of the parameters to zero leads to the reducing to zero of the other one) and in effect searching for solutions in the form of power series in relation to one small parameter, the form of the sought solution proposed by the author makes it possible to determine e.g. connections between parameters μ and ε such as to realize the predetermined conditions concerning the course of the limits of the loss of stability.

The sought form of solution as well as of the characteristic multipliers contains the terms at powers $\varepsilon^a\mu^1$ (a, $1 = 1, 2, \ldots$), which, in the standard method, occur only for $a \leqslant 2$.

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