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LETTER TO THE EDITOR

Chaotic behaviour of an anharmonic oscillator with almost periodic excitation

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Abstract. The influence of the almost periodic excitation on the chaotic behaviour of the anharmonic oscillator is reported.

Recently aperiodic solutions of the non-linear systems have attracted increasing attention. Several examples of chaotic solutions which form 'strange attractors' are known [1–7]. One of the best known is Duffing's oscillator which plays an important role in many physical problems [8–12]. In the present letter the special Duffing's oscillator excited by almost periodic force:

$$\ddot{x} + a\dot{x} + x^3 = B \cos \omega t \cos \Omega t = \tfrac{1}{2}B[\cos(\Omega - \omega)t + \cos(\Omega + \omega)t] \tag{1}$$

is considered. The unperturbed system has a homoclinic orbit and for $\omega = 0$ and $a = 0.1$, $\Omega = 1.0$, $B \in [9.9, 13.3]$ the chaotic behaviour was found by Ueda [8]. Equation (1) is a special case of the system with two external periodic forces which was investigated in [12] and of the general equation [13]. Now we are interested in the influence of the frequency ω on the chaotic behaviour of the system.

For characterising the chaotic behaviour we have calculated the maximum one-dimensional Lyapunov exponent $\lambda_{\max}$. For regular behaviour (periodic or quasi-periodic) we have $\lambda_{\max} = 0$ and for chaotic behaviour $\lambda_{\max} > 0$. The one-dimensional Lyapunov exponent has been determined by casting (1) into an autonomous system of first-order differential equations ($x_1 = x$, $\dot{x}_2 = x_1$, $\dot{x}_3 = \Omega - \omega$, $x_4 = \Omega + \omega$) and then solving this system together with its variational system:

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= -a y_2 - 3x_1^2 y_1 - \tfrac{1}{2}B[(\sin x_3)y_3 + (\sin x_4)y_4] \\ \dot{y}_3 &= 0 \\ \dot{y}_4 &= 0 \end{aligned} \tag{2}$$

where $x_3(0) = x_4(0) = 0$. Without loss of generality we can put $y_3 = y_4 = 1$. The one-dimensional Lyapunov exponents are defined by:

$$\lambda(x_1(0), x_2(0), y_1(0), y_2(0)) = \lim_{T\to\infty} T^{-1} \ln\|y(T)\|$$

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where we select the biggest rate by varying the initial values $x_1(0)$, $x_2(0)$, $y_1(0)$, $y_2(0)$. The Lyapunov exponent is independent of the norm. We use $\|y\| := \Sigma_{i=1}^{2} |y_i|$. We let digital time integrations run for a long time so that all transients have decayed and then allow a 'single trajectory' to wander over the final attractor.

The plots of maximum one-dimensional Lyapunov exponent against ω for different values of B have been shown in figure 1. In this figure we observed an interesting fact that for $\omega = 0.5$ and 0.75 we obtained $\lambda_{max} = 0$ and regular behaviour of the system (1).

For these values of ω equation (1) has the following forms:

$$\ddot{x} + a\dot{x} + x^3 = \tfrac{1}{2}B[\cos \tfrac{1}{2}t + \cos \tfrac{3}{2}t] \qquad (3)$$

and

$$\ddot{x} + a\dot{x} + x^3 = \tfrac{1}{2}B[\cos \tfrac{1}{4}t + \cos \tfrac{7}{4}t]. \qquad (4)$$

As equations (3) and (4) have the following symmetry under the transformations

$$S_1 : (x, x, t) \to (x, x, t + 4\pi)$$

equation (3) and

$$S_2 : (x, x, t) \to (x, x, t + 8\pi)$$

equation (4) we can compute Poincaré maps $M_{1,2} \subset R^2$ defined as the following sets:

$$M_1 = \{(x(t), \dot{x}(t) \mid t = 4k\pi, k = 1, 2, 3, \ldots\}$$

for equation (3) and

$$M_2 = \{(x(t), \dot{x}(t) \mid t = 8k\pi, k = 1, 2, 3, \ldots\}$$

where $x(t)$ is a solution of equations (3) and (4). Finite approximations of $M_{1,2}$ have been calculated numerically by the Runge–Kutta method [14].

Examples of such maps are shown in figure 2. At first sight they seem to represent 'strange attractors', however after the calculation of about 900 points, the attractors turn to converge to the almost periodic solution of 11 components for $\omega = 0.5$ and 13 components for $\omega = 0.75$. Figure 3 shows the amplitudes of the Fourier components against frequency for $x(t)$.

To summarise the results presented above we find that the existence of the second frequency 'weakens the chaotic behaviour'. The chaotic behaviour of the system (1) was found for $B \in [9.9, 13.3]$ and $\omega \in [0, 0.95)$. In the interval of ω we find isolated points 0.5 and 0.75 for which the system has almost periodic solutions with complicated form.

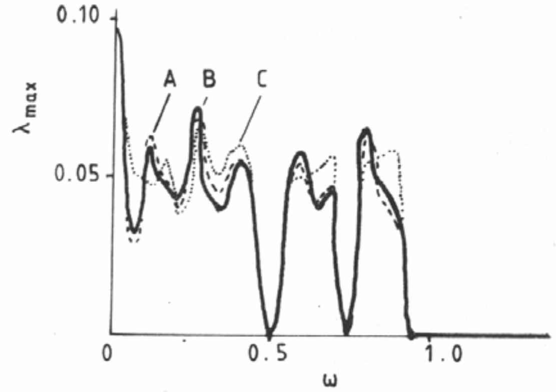

Figure 1. Maximum one-dimensional Lyapunov exponent λ_{max} against ω: $a = 0.1$, $\Omega = 1.0$. A, $B = 10.0$; B, $B = 11.5$; C, $B = 13.0$.

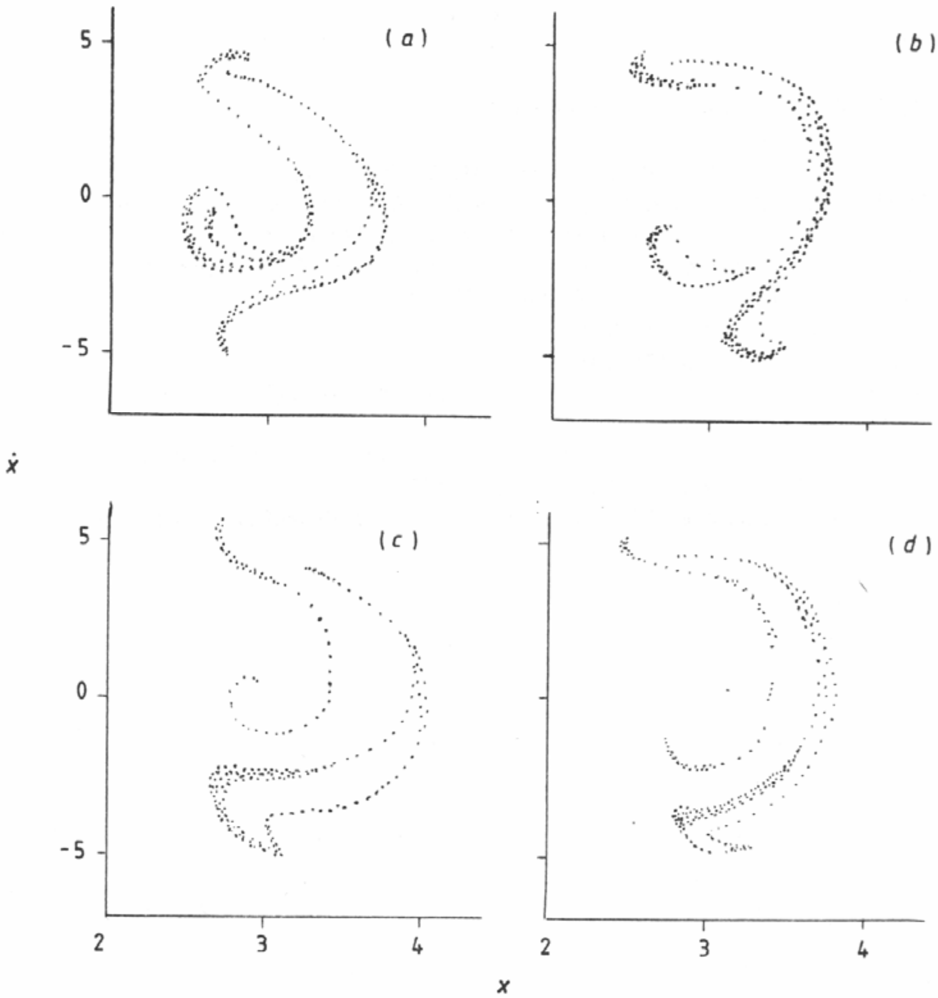

Figure 2. Poincaré maps of the system (1): $a=0.1$, $\Omega=1.0$. (a) $B=10.0$, $\omega=0.5$; (b) $B=10.0$, $\omega=0.75$; (c) $B=11.5$, $\omega=0.5$; (d) $B=11.5$, $\omega=0.75$.

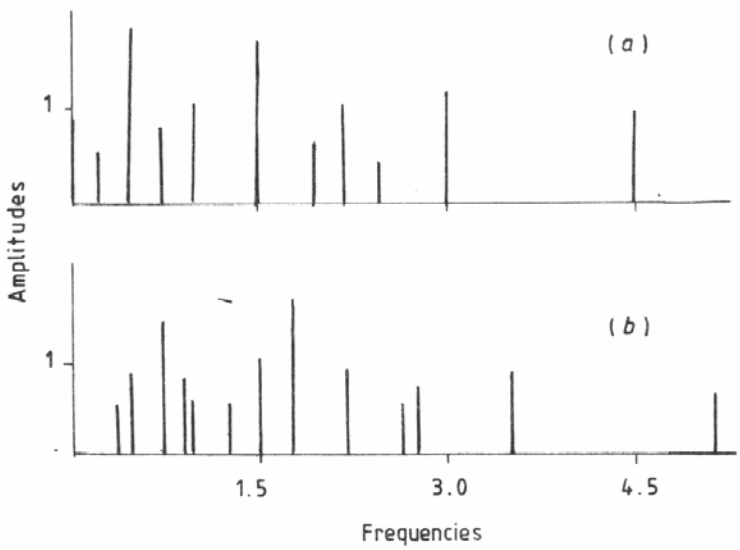

Figure 3. Frequency spectra: $a=0.1$, $\Omega=1.0$, $B=10.0$. (a) $\omega=0.5$; (b) $\omega=0.75$.

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