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ON THE OCCURRENCE OF CHAOS IN DUFFING'S OSCILLATOR

Recently, an increasing interest is to be noted in deterministic non-linear systems whose response to harmonic excitation can be chaotic in the case of some parameter values. Attention has been concentrated on the analysis of Duffing [1] and Van der Pol–Duffing [2] equations.

Some authors have connected the occurrence of chaotic motion with the occurrence of the positive value of the main characteristic Liapunov exponent [3]. Ueda and co-workers assumed the occurrence of a continuous frequency spectrum [1, 2] to be the criterion of chaos. This feature of chaos is closely connected with the autocorrelation function, which quickly drops to zero in a region where the frequency spectrum is continuous.

Another criterion of the occurrence of chaotic motion is presented in what follows here, for the example of the particular Duffing's equation analyzed in reference [4], of the form

$$d^2x/dt^2 + c\, dx/dt + x + x^3 = P \cos \omega t. \qquad (1)$$

For

$$c = 0{\cdot}2, \qquad P = 50, \qquad \omega = 1{\cdot}9 \qquad (2)$$

the strange attractor is placed between the resonances of the second and third order [4]; the phase plane plot is shown in Figure 1. After transformation of equation (1) the

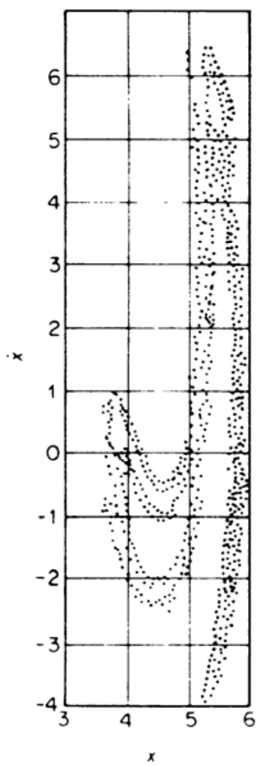

Figure 1. State space plot for the Duffing equation example.

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following is obtained:

$$d^2y/d\tau_1^2+\alpha^2y=\cos\tau_1-h\,dy/d\tau_1-\xi y^3, \tag{3}$$

where $\tau_1=\omega t$, $\alpha^2=1/\omega^2$, $h=c/\omega$, $x=(P/\omega^2)y$, and $\xi=P^2/\omega^6$. Let α slightly differ from $1/n$, and let the following be assumed:

$$\alpha^2=(1/n^2)-(a'/n^2), \qquad \tau_1=n\tau, \qquad h=\varepsilon h', \qquad \xi=\varepsilon\xi', \qquad a'=\varepsilon a. \tag{4}$$

Equation (3) then assumes the form

$$d^2y/d\tau^2+y=n^2\cos n\tau+\varepsilon(ay-nh'\,dy/d\tau-\xi' n^2y^3) \tag{5}$$

and the analysis of the resonance of nth order will be reduced to determining the periodic solutions of equation (5) of period 2π. These solutions are sought in the form

$$y=y_0+\varepsilon y_1+\varepsilon^2y_2+\cdots, \tag{6}$$

where

$$y_0=[n^2/(1-n^2)]\cos n\tau+M_0\cos\tau+N_0\sin\tau. \tag{7}$$

M_0 and N_0 are constants to be determined from the periodic condition of the function y_1, and ε is a small formally introduced parameter.

The condition of periodicity on y_1 for $n=3$ gives

$$\begin{aligned} a'M_0-3hN_0-9\xi(\tfrac{243}{128}M_0-\tfrac{27}{32}(M_0^2-N_0^2)+\tfrac{3}{4}N_0^2M_0+\tfrac{3}{4}M_0^3)&=0,\\ a'N_0+3hM_0-9\xi(\tfrac{243}{128}N_0+\tfrac{27}{16}M_0N_0+\tfrac{3}{4}M_0^2N_0+\tfrac{3}{4}N_0^3)&=0. \end{aligned} \tag{8}$$

Let

$$M_0=A\sin\varphi, \qquad N_0=A\cos\varphi. \tag{9}$$

Upon taking into account equations (9) and (8) the following is obtained:

$$(\tfrac{27}{4}\xi)^2A^4-[\tfrac{27}{2}\xi(a'-\tfrac{2187}{118}\xi)+(\tfrac{243}{32}\xi)^2]A^2+(a'-\tfrac{2187}{118}\xi)^2+9h^2=0. \tag{10}$$

The set of parameters for which no real solution for A exists can be determined from the inequality

$$[\tfrac{27}{2}\xi(a'-\tfrac{2187}{118}\xi)+(\tfrac{243}{32}\xi)^2]^2-4(\tfrac{27}{4}\xi)^2[(a'-\tfrac{2187}{118}\xi)^2+9h^2]<0. \tag{11}$$

Analogically, the following is obtained for $n=2$:

$$(a'+\tfrac{16}{3}\xi-3\xi A^2)^2+4h^2=0. \tag{12}$$

It can be easily checked that the parameters (2) satisfy the inequality (11), and hence that no real solution for A exists that can satisfy equation (12).

Consequently, the chaotic response of Duffing's oscillator to the harmonic excitation (1) has occurred for parameter values such that no real roots of equations (10) and (12) exist.

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