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External and internal resonances for 3-dof plane pendulum excited by harmonic forces

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Abstract: The dynamical response of a harmonically excited 3 degrees-of-freedom planar physical pendulum is studied in the paper. The investigated system may be considered as a good example for several engineering applications. We adopt the asymptotic method of multiple scale (MS) in order to carry out the analytical calculations. The solution until the third order has been achieved. MS method allows to identify parameters of the system being dangerous due to the resonance and yields time histories for the assumed generalized co-ordinates. The tests for up to 3 simultaneously occurring resonance conditions have been made. The energy transfer from one to another mode of vibrations is illustrated in the graphs. The modulation equations transferred into an autonomous form allows to obtain the frequency response functions and to draw resonance curves. Their stability can be verified.

1. Introduction

Pendulums are relatively simple systems, nevertheless can be used to simulate the dynamics of a wide variety of engineering devices and machine parts. The coupling in the equations of motion describes energy exchange between modes of vibrations and possibility of autoparametric excitation. The energy transfer associated with it is well known in nonlinear dynamics of multi degree-of-freedom systems an is widely discussed by many authors [1, 2].

Nonlinear dynamics of mechanical system with three degrees-of-freedom near resonance is the subject of this paper. The asymptotic method of multiple scales was applied both to solving the equations of motion and to determining the resonance conditions [3].

2. Formulation of the problem

The investigated system consists of a rigid body of mass m suspended at the point A on a massless and linear spring, whose other end is fixed at point O (see Figure 1). The point C is the mass centre of the body. Let $S=AC$ denotes the eccentricity. The dynamical extension of the spring Z and angles φ and γ are used as the general co-ordinates. External excitations i.e. the force $F(t)= F_0(t)\cos(\Omega_1 t)$, the moments $M_\varphi(t)= M_2(t)\cos(\Omega_2 t)$ and $M_\psi(t)= M_3(t)\cos(\Omega_3 t)$ are taken into consideration. The motion of the pendulum is damped by viscous force $C_1\dot{Z}$ and linear moments $C_2\dot{\varphi}$ and $C_3\dot{\gamma}$.

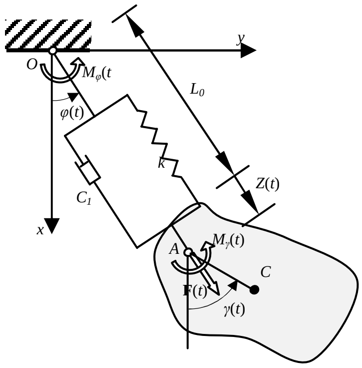

Figure 1. Physical spring pendulum.

Applying the Lagrange equations we obtain the motion equations. Their dimensionless form is as follows

$$\ddot{z} + c_1\dot{z} + w_1^2 z + 1 - \cos(\varphi) - (1+z)(\dot{\varphi})^2 + s\sin(\varphi-\gamma)\ddot{\gamma} - s\cos(\varphi-\gamma)(\dot{\gamma})^2 = f_1\cos(p_1\tau), \tag{1}$$

$$s(1+z)(\sin(\varphi-\gamma)(\dot{\gamma})^2 + s(1+z)\cos(\varphi-\gamma)\ddot{\gamma}) + (1+z)^2\ddot{\varphi} + c_2\dot{\varphi} + (1+z)(\sin(\varphi) + 2\dot{z}\dot{\varphi}) = f_2\cos(p_2\tau), \tag{2}$$

$$\ddot{\gamma} + c_3\dot{\gamma} + w_3^2\sin(\gamma) - w_3^2(1+z)(\sin(\varphi-\gamma)\dot{\varphi}^2 + w_3^2\sin(\varphi-\gamma)\ddot{z} + w_3^2\cos(\varphi-\gamma)((1+z)\ddot{\varphi} + 2\dot{z}\dot{\varphi}) = f_3\cos(p_3\tau) \tag{3}$$

where z, φ, γ are functions of τ, $L = L_0 + \dfrac{mg}{k}$, $z = \dfrac{Z}{L}$, $s = \dfrac{S}{L}$, $\omega_1^2 = \dfrac{k}{m}$, $\omega_2^2 = \dfrac{g}{L}$, $\omega_3^2 = \dfrac{Sg}{i_A^2}$,

$w_1 = \dfrac{\omega_1}{\omega_2}$, $w_3 = \dfrac{\omega_3}{\omega_2}$, $\tau = \omega_2 t$, $c_1 = \dfrac{C_1}{m\omega_2}$, $c_2 = \dfrac{C_2}{mL^2\omega_2}$, $c_3 = \dfrac{C_3}{\omega_2 I_A}$, $f_1 = \dfrac{F_0}{mg}$, $f_2 = \dfrac{M_2}{mgL}$,

$f_3 = \dfrac{M_3}{\omega_2^2 I_A}$, $p_i = \dfrac{\Omega_i}{\omega_2}$, $i = 1,2,3$, k denotes stiffness of the spring, L_0 is its length, when it is unstretched, g is the Earth's acceleration, I_A, i_A are moment and radius of inertia of the body with respect to the axis which passes through A and is perpendicular to the plane of motion, respectively, The Eqs. 1–3 are supplemented by adequate initial conditions (quantities $u_{01},...,u_{06}$ are known)

$$z(0) = u_{01},\ \dot{z}(0) = u_{02},\ \varphi(0) = u_{03},\ \dot{\varphi}(0) = u_{04},\ \gamma(0) = u_{05},\ \dot{\gamma}(0) = u_{06}. \tag{4}$$

3. Solution method

The asymptotic method of multiple scales in time is used to obtain the solution and to determine resonance conditions. Trigonometric functions in Eqs. 1–3 are approximated by the power series of 3rd order. The amplitudes of vibrations are assumed to be of the order of a small parameter ε , where $0<\varepsilon<<1$, hence $z=\varepsilon\zeta$, $\varphi=\varepsilon\phi,\ \gamma=\varepsilon\chi$. The functions $\zeta,\ \phi$ and χ are sought in the form

$$\zeta=\sum_{k=1}^{k=3}\varepsilon^k\zeta_k(\tau_0,\tau_1,\tau_2)+O(\varepsilon^4),\quad \phi=\sum_{k=1}^{k=3}\varepsilon^k\phi_k(\tau_0,\tau_1,\tau_2)+O(\varepsilon^4),\chi=\sum_{k=1}^{k=3}\varepsilon^k\chi_k(\tau_0,\tau_1,\tau_2)+O(\varepsilon^4)\quad(5)$$

where $\tau_0=\tau,\ \tau_1=\varepsilon\tau,\ \tau_2=\varepsilon^2\tau,$ are various time scales.

The derivatives with respect to time τ are equivalent to the following differential operators

$$\frac{d}{d\tau}=\frac{\partial}{\partial\tau_0}+\varepsilon\frac{\partial}{\partial\tau_1}+\varepsilon^2\frac{\partial}{\partial\tau_2},\quad \frac{d^2}{d\tau^2}=\frac{\partial^2}{\partial\tau_0^2}+2\varepsilon\frac{\partial^2}{\partial\tau_0\partial\tau_1}+\varepsilon^2\left(\frac{\partial^2}{\partial\tau_1^2}+2\frac{\partial^2}{\partial\tau_0\partial\tau_2}\right)+o(\varepsilon^3)\qquad(6)$$

The amplitudes of generalized forces, damping coefficients and eccentricity are assumed in the form

$$c_i=\varepsilon^2\tilde{c}_i,\quad f_i=\varepsilon^3\tilde{f}_i,\quad i=1,2,3,\quad s=\varepsilon^2\tilde{s},\qquad(7)$$

The quantities $\tilde{c}_i,\tilde{f}_i,\tilde{s}$ are of the order of unity.

Substituting the assumptions given by Eq. 5 and then arranging them according to the powers of the small parameter, we obtain the set of nine partial linear differential equations

– first order equations (i.e. the order of ε)

$$\frac{\partial^2\zeta_1}{\partial\tau_0^2}+w_1^2\zeta_1=0,\quad \frac{\partial^2\phi_1}{\partial\tau_0^2}+\phi_1=0,\quad \frac{\partial^2\chi_1}{\partial\tau_0^2}+w_3^2\left(\chi_1+\frac{\partial^2\phi_1}{\partial\tau_0^2}\right)=0,\qquad(8)$$

– second order equations (i.e. the order of ε^2)

$$\begin{aligned}&\frac{\partial^2\zeta_2}{\partial\tau_0^2}+w_1^2\zeta_2=-\frac{1}{2}\phi_1^2-2\frac{\partial^2\zeta_1}{\partial\tau_0\partial\tau_1}+\left(\frac{\partial\phi_1}{\partial\tau_0}\right)^2,\\ &\frac{\partial^2\phi_2}{\partial\tau_0^2}+\phi_2=-\zeta_1\left(\phi_1+2\frac{\partial^2\phi_1}{\partial\tau_0^2}\right)-2\frac{\partial\zeta_1}{\partial\tau_0}\frac{\partial\phi_1}{\partial\tau_0}-2\frac{\partial^2\phi_1}{\partial\tau_0\partial\tau_1},\\ &\frac{\partial^2\chi_2}{\partial\tau_0^2}+w_3^2\left(\chi_2+\frac{\partial^2\phi_2}{\partial\tau_0^2}\right)=-w_3^2\left((\phi_1-\chi_1)\frac{\partial^2\zeta_1}{\partial\tau_0^2}+\zeta_1\frac{\partial^2\phi_1}{\partial\tau_0^2}\right)-2\frac{\partial^2\chi_1}{\partial\tau_0\partial\tau_1}-2w_3^2\left(\frac{\partial^2\phi_1}{\partial\tau_0\partial\tau_1}+\frac{\partial\zeta_1}{\partial\tau_0}\frac{\partial\phi_1}{\partial\tau_0}\right)\end{aligned}\qquad(9)$$

– third order equations (i.e. the order of ε^3)

$$\begin{aligned}&\frac{\partial^2\zeta_3}{\partial\tau_0^2}+w_1^2\zeta_3=\frac{\partial\phi_1}{\partial\tau_0}\left(2\frac{\partial\phi_2}{\partial\tau_0}+2\frac{\partial\phi_1}{\partial\tau_1}+\zeta_1\frac{\partial\phi_1}{\partial\tau_0}\right)-\tilde{c}_1\frac{\partial\zeta_1}{\partial\tau_0}-\frac{\partial\zeta_1^2}{\partial\tau_1^2}-2\frac{\partial^2\zeta_1}{\partial\tau_0\partial\tau_2}-2\frac{\partial^2\zeta_2}{\partial\tau_0\partial\tau_1}-\\ &-\phi_1\phi_2+\tilde{f}_1\cos(\tau_0 p_1),\end{aligned}$$

$$
\begin{aligned}
&\frac{\partial^2\phi_3}{\partial\tau_0^2}+\phi_3=\tilde{f}_2\cos(\tau_0 p_2)+\frac{\phi_1^3}{6}-\zeta_2\phi_1-\zeta_1\phi_2-\frac{\partial^2\phi_1}{\partial\tau_1^2}-2\left(\frac{\tilde{c}_2}{2}+\frac{\partial\zeta_1}{\partial\tau_1}+\zeta_1\frac{\partial\zeta_1}{\partial\tau_0}+\frac{\partial\zeta_2}{\partial\tau_0}\right)\frac{\partial\phi_1}{\partial\tau_0}-\\
&-2\frac{\partial\zeta_1}{\partial\tau_0}\left(\frac{\partial\phi_1}{\partial\tau_1}+\frac{\partial\phi_2}{\partial\tau_0}\right)-2\left(\frac{\partial^2\phi_1}{\partial\tau_0\partial\tau_2}+\frac{\partial^2\phi_2}{\partial\tau_0\partial\tau_1}+\zeta_2\frac{\partial^2\phi_1}{\partial\tau_0^2}\right)-\zeta_1^2\frac{\partial^2\phi_1}{\partial\tau_0^2}- \qquad (10)\\
&-2\zeta_1\left(2\frac{\partial^2\phi_1}{\partial\tau_0\partial\tau_1}+\frac{\partial^2\phi_2}{\partial\tau_0^2}\right)-\tilde{s}\frac{\partial^2\chi_1}{\partial\tau_0^2},
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial^2\chi_3}{\partial\tau_0^2}+w_3^2\left(\chi_3+\frac{\partial^2\phi_3}{\partial\tau_0^2}\right)=\tilde{f}_3\cos(\tau_0 p_3)+\frac{w_3^2}{6}\chi_1^3-\tilde{c}_3\frac{\partial\chi_1}{\partial\tau_0}-\frac{\partial^2\chi_1}{\partial\tau_1^2}-2\left(\frac{\partial^2\chi_1}{\partial\tau_0\partial\tau_2}+\frac{\partial^2\chi_2}{\partial\tau_0\partial\tau_1}\right)-\\
&+w_3^2(\phi_1-\chi_1)\left(\left(\frac{\partial\phi_1}{\partial\tau_0}\right)^2-2\frac{\partial^2\zeta_1}{\partial\tau_0\partial\tau_1}-\frac{\partial^2\zeta_2}{\partial\tau_0^2}\right)-2w_3^2\left(\left(\frac{\partial\zeta_1}{\partial\tau_1}+\frac{\partial\zeta_2}{\partial\tau_0}\right)\frac{\partial\phi_1}{\partial\tau_0}+\left(\frac{\partial\phi_1}{\partial\tau_1}+\frac{\partial\phi_2}{\partial\tau_0}\right)\frac{\partial\zeta_1}{\partial\tau_0}\right)-\\
&-w_3^2\left(\frac{\partial^2\phi_1}{\partial\tau_1^2}+2\frac{\partial^2\phi_1}{\partial\tau_0\partial\tau_2}+2\frac{\partial^2\phi_2}{\partial\tau_0\partial\tau_1}\right)-w_3^2(\phi_2-\chi_2)\frac{\partial^2\zeta_1}{\partial\tau_0^2}+\frac{w_3^2}{2}(\phi_1-\chi_1)^2\frac{\partial^2\phi_1}{\partial\tau_0^2}-w_3^2\zeta_2\frac{\partial^2\phi_1}{\partial\tau_0^2}-\\
&-w_3^2\zeta_1\left(2\frac{\partial^2\phi_1}{\partial\tau_0\partial\tau_1}+\frac{\partial^2\phi_2}{\partial\tau_0^2}\right).
\end{aligned}
$$

We start solving these equations from Eqs. (8), because of the recursive nature of the system (8) – (10). Each of Eqs. (8) is homogenous and their solutions have the form

$$
\zeta_1=A_1\,\mathrm{e}^{\mathrm{i}w_1\tau_0}+\bar{A}_1\,\mathrm{e}^{-\mathrm{i}w_1\tau_0},\quad \phi_1=A_2\,\mathrm{e}^{\mathrm{i}\tau_0}+\bar{A}_2\,\mathrm{e}^{-\mathrm{i}\tau_0},\quad \chi_1=A_3\,\mathrm{e}^{\mathrm{i}\tau_0 w_3}+\bar{A}_3\,\mathrm{e}^{-\mathrm{i}\tau_0 w_3}+\frac{w_3^2}{w_3^2-1}\left(A_2\,\mathrm{e}^{\mathrm{i}\tau_0}+\bar{A}_2\,\mathrm{e}^{-\mathrm{i}\tau_0}\right) \quad (11)
$$

where A_1, A_2, A_3 are unknown complex functions of τ_1 and τ_2, $\bar{A}_i$ is conjugate to A_i.

Introducing all of these solutions into Eqs. (9) implies that there appear secular terms. Conditions of the eliminating secular terms are conducive to the request that the functions A_i depend only on the variable τ_2. After rejection of secular terms we get the second order solutions. The solution of Eqs. (10), we obtain in the similar way. We do not write here the solutions of the second and third order due to their extensive form.

4. The conditions of resonance

The resonance conditions can be obtained based on the analytical form of the solutions. All resonances detected in this way can be recognized as:

– primary external $p_1 = w_1$, $p_2 = 1$, $p_3 = w_3$,

– internal $w_1 = 2$, $w_1 = 2w_3$, $w_1 = w_3 + 1$, $w_1 = w_3 – 1$, $w_3 = 1$, $w_3 = 1 + 2w_1$, $w_3 = 3$, $w_1 = w_3$,

$w_3 = 2w_1 - 1,\ 1 = 3w_3.$

5. External resonances

The case of simultaneously occurring three primary external resonances is discussed below. The resonance effects are reflected in the secular terms, when we introduce the detuning parameters

$$p_1 = w_1 + \sigma_1, \quad p_2 = 1 + \sigma_2, \quad p_3 = w_3 + \sigma_3. \tag{12}$$

We assume the detuning parameters are of the order of small parameter, i.e. $\sigma_i = \varepsilon\tilde{\sigma}_i$. Introducing the parameters σ_i into Eqs. (1–3) causes the appearance of the secular terms. As a result of their elimination we get the conditions of solvability of the problem. They are the system of six differential equations with partial derivatives of unknown functions A_i. This system shows that the functions A_i depend only on the slowest time scale τ_2. The unknown functions A_i are then represented by real functions $\tilde{a}_i(\tau_2)$ and $\psi_i(\tau_2)$

$$A_i = \frac{\tilde{a}_i(\tau_2)}{2} e^{i\psi_i(\tau_2)}, \quad a_i = \varepsilon\tilde{a}_i, \quad i = 1, 2, 3. \tag{13}$$

The functions $a_i(\tau_2)$ are amplitudes and $\psi_i(\tau_2)$ phases.

Afterwards we introduce the modified phases

$$\theta_1(\tau_1,\tau_2) = \tau_1\tilde{\sigma}_1 - \psi_1(\tau_2), \quad \theta_2(\tau_1,\tau_2) = \tau_1\tilde{\sigma}_2 - \psi_2(\tau_2), \quad \theta_3(\tau_1,\tau_2) = \tau_1\tilde{\sigma}_3 - \psi_3(\tau_2). \tag{14}$$

Taking advantage of Eq. $(6)_1$ and making some transformations, we obtain autonomous modulation system in the form

$$\frac{da_1}{d\tau} = \frac{f_1}{2w_1}\sin\theta_1 - \frac{1}{2}a_1c_1, \quad \frac{da_2}{d\tau} = \frac{f_2}{2}\sin\theta_2 - \frac{1}{2}a_2c_2, \quad \frac{da_3}{d\tau} = \frac{f_3}{2w_3}\sin\theta_3 - \frac{1}{2}a_3c_3 \tag{15}$$

$$a_1\frac{d\theta_1}{d\tau} = a_1\sigma_1 + \frac{w_1^2 - 7}{4w_1(w_1^2 - 4)}a_1a_2^2 + \frac{f_1}{2w_1}\cos(\theta_1), \tag{16}$$

$$a_2\frac{d\theta_2}{d\tau} = \sigma_2a_2 + \frac{(w_1^4 - 5w_1^2 - 8)a_2^3}{16w_1^2(w_1^2 - 4)} + \frac{(w_1^2 - 7)a_1^2a_2}{4(w_1^2 - 4)} + \frac{a_2sw_3^2}{2(w_3^2 - 1)} + \frac{f_2}{2}\cos\theta_2, \tag{17}$$

$$a_3\frac{d\theta_3}{d\tau} = \sigma_3a_3 - \frac{w_1^4w_3^3a_1^2a_3}{4(w_1^2 - 4w_3^2)} + \frac{(2w_3^3 - w_3^5)a_2^2a_3}{8(w_3^2 - 1)^2} + \frac{w_3a_3^3}{16} + \frac{f_3}{2w_3}\cos\theta_3. \tag{18}$$

Illustration of the amplitude modulations according to the above equations and time history obtained numerically form Eqs. (1–3) are presented in Figure 2. This graphs have been made for the following parameters: σ_1 = 0.03, σ_2 = 0.02, σ_3 = 0.02, w_1 = 5, w_3 = 0.2, s = 0.05, $p_1 = w_1 + \sigma_1$, $p_2 = 1 + \sigma_2$, $p_3 = w_3 + \sigma_3$, f_1 = 0.001, f_2 = 0.001, f_3 = 0.001, c_1 = 0.002, c_2 = 0.002, c_3 = 0,002 and initial conditions: $a_1(0)$ = 0.004, $\theta_1(0)$ = 0, $a_2(0)$ = 0.004, $\theta_2(0)$ = 0, $a_3(0)$ = 0.004, $\theta_3(0)$ = 0.

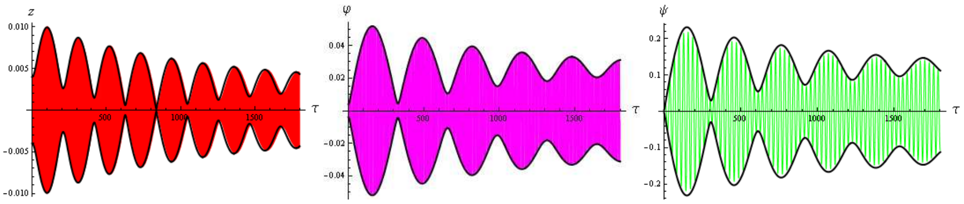

Figure 2. Amplitudes modulation and time history for z, φ and γ.

6. Conclusions

The equations describing the dynamical behavior of the pendulum have been successfully solved by the multiple scales method (MSM). General solution, including the third order of approximation, have been achieved in analytical form. Dimensionless solutions are universal and valid for many similar systems.

The application of MSM allows determining the conditions under which the external and internal resonances appear in the system. The differential equations of amplitude and phases modulation have been obtained for a combination of simultaneously occurring external resonances. The amplitude modulations are consistent with time history of vibrations obtained numerically. The amplitude responses as functions of the detuning parameters have been received for the steady state motion.

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References

1. Sado D., Gajos K.: Analysis of vibrations of three-degree-of-freedom dynamical system with double pendulum. *J. of Theoretical and Applied Mechanics,* 46(1), 2008, 141-156.
2. Starosta R., Sypniewska-Kamińska G., Awrejcewicz J.: Resonances in kinematically driven nonlinear spring pendulum. *DSTA 10th Conference*, 2009, 581-588.
3. Nayfeh A.H., *Perturbation methods.* New York: Wiley, 1973.

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