Analytical and numerical studies of effective reduction of a weakly nonlinear two degrees-of-freedom mechanical system

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Summary. Dynamics of a weakly nonlinear two degree-of-freedom system is analyzed. The problem is reduced to the so-called effective equation of the system internal motion. Novel non-linear stationary and non-stationary phenomena exhibited by the system are detected, analyzed and discussed.

Introduction

Energy exchange and non-stationary processes appear in many engineering dynamical systems and they are of great interest of many researchers. This problem has been widely discussed in references [3, 5]. Owing to strongly nonlinear differential equations governing majority of the so far mentioned problems, mainly numerical approaches have been applied [3]. However, one may observe in recent years a great interest in a successful application of modern asymptotic methods to engineering oriented problems [1, 4]. In particular, a novel idea for an effective study of nonlinear dynamical systems is linked with a concept of the so-called limiting phase trajectories LPT (see [2]). In this report we are aimed mainly on analysis of unsteady-state dynamics of our 2-DOF nonlinear system using the LPT approach.

Formulation of the problem

The investigated system consists of a physical pendulum coupled with a small mass supported by a nonlinear spring as shown in Figure 1.

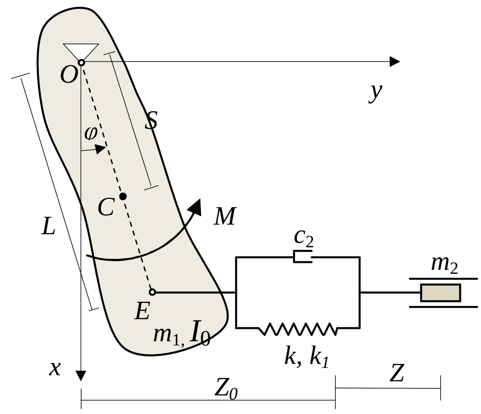

Fig. 1. Nonlinear 2-DOF system.

The pendulum moves in the neighborhood of its static equilibrium position. The systems exhibits flat motion, and the pendulum is excited by the external moment *M*. Viscous model of damping is taken and we assume in our further asymptotic analysis that $m_2 << m_1$.
Kinetic and potential system energies have the form

$$T = \frac{1}{2} m_2 \dot{Z}^2(t) + \frac{1}{2} I_0 \dot{\varphi}^2(t), \tag{1}$$

$$V = -gLm_2 - gm_1 S\cos\varphi(t) + \frac{1}{2}k\big(Z(t) - L\sin\varphi(t)\big)^2 + \frac{1}{4}k_1\big(Z(t) - L\sin\varphi(t)\big)^4 . \tag{2}$$

The external loading and damping are taken into consideration as generalized forces. Equations of motion are derived from the second type Lagrange equations, and they have the following form

$$\begin{aligned} & gm_1 S\sin\varphi(t) + kL\cos\varphi(t)\big(L\sin\varphi(t) - Z(t)\big) + k_1 L\cos\varphi(t)\big(L\sin\varphi(t) - Z(t)\big)^3 + \\ & + \frac{1}{2}L\big(2c_1 + c_2 L + c_2 L\cos 2\varphi(t)\big)\dot{\varphi}(t) - c_2 L\cos\varphi(t)\,\dot{\varphi} + I_0\ddot{\varphi}(t) = M\cos(\Omega_0 t) \end{aligned} \tag{3}$$

$$k\big(Z(t) - L\sin\varphi(t)\big) + k_1\big(Z(t) - L\sin\varphi(t)\big)^3 + c_2\dot{Z}(t) + m_2\ddot{Z}(t) - c_2 L\cos\varphi(t)\,\dot{\varphi}(t) = 0 , \tag{4}$$

where I_0, g, m_1, m_2, k, k_1, c_1, c_2, L and S are the known parameters.

The equations (3)-(4) should be supplemented by the initial conditions for generalized co-ordinates and their first derivatives, i.e. one gets

$$Z(0) = Z_0,\ \dot{Z}(0) = V_0,\ \varphi(0) = \varphi_0,\ \dot{\varphi}(0) = \omega_0. \tag{5}$$

The problem is then transformed to the dimensionless form.

The effective equation governing the system internal motion

The carried out analysis of equations of motion consists of two general steps. At first equations (3) and (4) are transformed into one Duffing type effective equation. In the second step, new complex variables are introduced, and phase trajectories of the system are studied in the case of a main resonance.

Dimensionless time $\tau = \omega t$, where $\omega^2 = \frac{k\left(L^2 m_2 + I_0\right)}{I_0 m_2}$ and a new dimensionless variables $z(\tau) = \frac{Z(\tau)}{Z_0}$ and $\phi(\tau) = \frac{L\varphi(\tau)}{Z_0}$ are introduced into equations (3) and (4). Next we introduce the variable describing internal motion $y(\tau) = z(\tau) - \phi(\tau)$.

Using the idea presented in [6], the original problem can be reduced to the effective Duffing type equation,

$$\ddot{y} + \gamma_e \dot{y} + y + \eta_e y^3 = P\cos(\Omega\tau + \Phi), \tag{6}$$

with the initial conditions

$$y(0) = y_0,\ \dot{y}(0) = v_0, \tag{7}$$

where γ_e, η_e, P, Ω, Φ are nonlinear parameters being functions of I_0, g, m_1, m_2, k, k_1, c_1, c_2, L, Ω_0 and S.

Complex representation and limiting phase trajectories

Non-steady forced vibrations of weakly non-linear oscillator are studied applying the method presented in [2]. Equation (6) is transformed to the new form by introducing complex variables $\psi = v + iy$ and $\overline{\psi} = v - iy$, where $v = \frac{dy}{d\tau}$.

Multiple time scales methods is further applied. After introduction of $\psi = \chi e^{i\tau}$, $\overline{\psi} = \overline{\chi} e^{-i\tau}$, a solution to the studied complex problem is further being sought in the form

$$\chi(\tau;\varepsilon) = \sum_{k=1}^{k=2} \varepsilon^k \chi_k(\tau_0, \tau_1) + O(\varepsilon^3), \tag{8}$$

where $\tau_0 = \tau$ and $\tau_1 = \varepsilon\tau$ are time scales.

Then, a first integral of the solvability condition yielded by the asymptotic approach is analyzed. Various regimes of internal motion are detected using the phase plane of amplitude and phase shift of oscillations.

Conclusions

Analytical study of the 2-DOF nonlinear dynamical system is presented. It is shown how the investigated two degree-of-freedom system is reduced to the 1-DOF system representing the system internal dynamics. In particular, non-steady forced system vibrations are investigated analytically and the obtained results are verified numerically. Both qualitative and quantitative complex analyses have been performed. It has been shown that the most intensive energy transfer in the system is governed by behavior of the so-called limiting phase trajectories. Important non-linear dynamical transition type phenomena are detected, monitored and discussed, among other.

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