10th CONFERENCE
on
**DYNAMICAL SYSTEMS
THEORY AND APPLICATIONS**
December 7-10, 2009. Łódź, Poland

RESONANCES IN KINEMATICALLY DRIVEN NONLINEAR SPRING PENDULUM

Roman Starosta, Grażyna Sypniewska-Kamińska, Jan Awrejcewicz

Abstract: The weakly non-linear 2-DOF mechanical system parametrically and externally excited is studied. Multiple scales method application allows recognizing resonances occurring in the system. The amplitude response functions for chosen cases are studied and illustrated.

1. Introduction

The parametrical nonlinear systems are in great interest of many researches and we are aimed on a dynamical analysis of such a system. Namely, the pendulum with changing length moving in circular path is investigated (Fig.1). In this paper we are focused on recognizing resonance conditions and analysis of the chosen resonance dynamics.

Dynamical systems including mathematical or physical pendulum play significant role in technology. In such systems one can observe an auto-parametric resonance phenomena, because of the coupling occurring in the equation of motion. The phenomenon of energy transfer from one of the mode of vibration to the other was widely discussed in references [3,4].

The structure investigated in the paper can be recognized as a model of various engineering elements in machines or can simulate the motion of a floating body.

On the other hand, asymptotic methods are intensively developed in last decades and applied for solving nonlinear problems [1,2]. The multiple scale method makes possible to recognize parameters of the system that are dangerous due to the resonances and allows to illustrate frequency-amplitude response functions. In what follows primary, parametric and combined resonances are studied. All calculations were performed with the help of the computer algebra system *Mathematica,* in which several procedures were elaborated in order to automatize most of the operations.

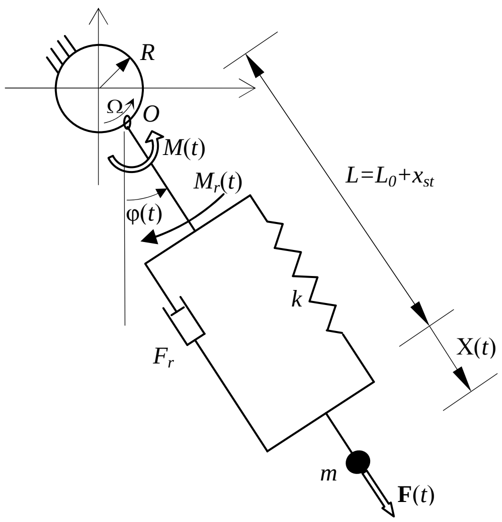

Fig. 1. Spring pendulum moving on circular path

2. Formulation of the problem

We study the spring pendulum moving with angular velocity Ω on a fixed circular path of radius R (Fig. 1) assuming the planar motion of the pendulum. X and ϕ are the generalized co-ordinates. Around the point O act the moment $M(t)=M_0\cos\Omega_2 t$ and the linear viscous damping moment $M_r=B_2\dot{\phi}$. The force $F(t)=F_0\cos\Omega_1 t$ acts on the mass m along the pendulum length. Moreover, linear viscous damping $F_r=B_1\dot{X}$ is assumed (B_1 and B_2 are the viscous coefficients).

The kinetic energy of the following system has the form:

$$T=\frac{m}{2}\left(\left(R\,\Omega\cos(\Omega t)+\dot{X}(t)\sin(\phi(t))+\cos(\phi(t))\left(L+X(t)\right)\dot{\phi}(t)\right)^2+\left(R\,\Omega\sin(\Omega t)-\dot{X}(t)\cos(\phi(t))+\sin(\varphi(t))\left(L+X(t)\right)\dot{\phi}(t)\right)^2\right),\quad(1)$$

whereas the potential energy reads:

$$V=\frac{k}{2}\left(X(t)+\frac{mg}{k}\right)^2-mg\left(R\cos(\Omega t)+\cos(\phi(t))\left(L+X(t)\right)\right),\quad(2)$$

where L denotes length of the statically stretched pendulum at $\phi=0$, m is its mass, k denotes stiffness of the spring and g is the Earth's acceleration.

The governing equations of the system are as follows:

$$\ddot{x}(t)+c_1\dot{x}(t)-\left(1+x(t)\right)\left(\dot{\varphi}(t)\right)^2+\omega_1^2x(t)+\omega_2^2\left(1-\cos\left(\varphi(t)\right)\right)-r\,\Omega^2\cos(t\Omega-\varphi(t))=f_1\cos(\Omega_1 t)\ ,\quad(3)$$

$$\left(1+x(t)\right)^2\ddot{\varphi}(t)+\left(c_2+2\left(1+x(t)\right)\dot{x}(t)\right)\dot{\varphi}(t)+\omega_2^2\sin\left(\varphi(t)\right)\left(1+x(t)\right)-r\,\Omega^2\left(1+x(t)\right)\sin(t\Omega-\varphi(t))=f_2\cos(\Omega_2 t)\ ,\quad(4)$$

where $L=L_0+\dfrac{mg}{k}$, $x=\dfrac{X}{L}$, $r=\dfrac{R}{L}$, $\omega_1^2=\dfrac{k}{m}$, $\omega_2^2=\dfrac{g}{L}$, $c_1=\dfrac{B_1}{m}$, $c_2=\dfrac{B_2}{mL^2}$, $f_1=\dfrac{F_0}{mL}$, $f_2=\dfrac{M_0}{mL^2}$.

The equations (3,4) should be supplemented by the adequate initial conditions.

3. Solution method

To solve the governing equations and to obtain the resonance conditions the multiple scale method is applied.

Since the motion in a small neighborhood of the static equilibrium position is considered, the amplitudes of vibrations are assumed to be of order of a small parameter ε, where $0<\varepsilon<<1$, and hence $x=\varepsilon\tilde{x}, \phi=\varepsilon\tilde{\phi}$.

Owing to the above assumption trigonometric functions in (3) and (4) can be written as follows

$$\sin\phi \cong \phi-\frac{1}{3!}\phi^3,\ \cos\phi \cong 1-\frac{1}{2}\phi^2,$$

$$\sin(t\Omega-\phi)\cong \sin t\Omega-\cos t\Omega-\frac{1}{2}\phi^2\sin t\Omega,\ \cos(t\Omega-\phi)\cong \cos t\Omega+\sin t\Omega-\frac{1}{2}\phi^2\cos t\Omega .$$

The generalized forces are assumed as follows: $c_i=\varepsilon^2\tilde{c}_i,\ f_i=\varepsilon^3\tilde{f}_i, r=\varepsilon^2\tilde{r}$, i=1,2.

The functions x and ϕ, which satisfy the equations of motion are sought in the form:

$$\begin{aligned}\tilde{x}(t;\varepsilon)&=\sum_{k=1}^{k=3}\varepsilon^k\tilde{x}_k(T_0,T_1,T_2)+O(\varepsilon^4),\\ \tilde{\phi}(t;\varepsilon)&=\sum_{k=1}^{k=3}\varepsilon^k\tilde{\phi}_k(T_0,T_1,T_2)+O(\varepsilon^4),\end{aligned}\tag{5}$$

where $T_0=t$, $T_1=\varepsilon t$ and $T_2=\varepsilon^2 t$ are various time scales.

The derivatives with respect to time t are calculated in terms of the new time scales as follows

$$\begin{aligned}\frac{d}{dt}&=\frac{\partial}{\partial T_0}+\varepsilon\frac{\partial}{\partial T_1}+\varepsilon^2\frac{\partial}{\partial T_2},\\ \frac{d^2}{dt^2}&=\frac{\partial^2}{\partial T_0^2}+2\varepsilon\frac{\partial^2}{\partial T_0\,\partial T_1}+\varepsilon^2\left(\frac{\partial^2}{\partial T_1^2}+2\frac{\partial^2}{\partial T_0\,\partial T_2}\right)+O(\varepsilon^3).\end{aligned}\tag{6}$$

The definitions (5) and (6) transform the original equations to the set of the following ordinary linear differential equations

(order ε^1)

$$\begin{aligned}&\frac{\partial^2\tilde{x}_1}{\partial T_0^2}+\omega_1^2\tilde{x}_1=0,\\ &\frac{\partial^2\tilde{\phi}_1}{\partial T_0^2}+\omega_2^2\tilde{\phi}_1=0;\end{aligned}\tag{7}$$

(order ε^2)

$$\begin{aligned}&\frac{\partial^2\tilde{x}_2}{\partial T_0^2}+\omega_1^2\tilde{x}_2=\tilde{r}\Omega^2\cos\Omega T_0-\frac{1}{2}\omega_2^2\tilde{\phi}_1^2-2\frac{\partial^2\tilde{x}_1}{\partial T_0\,\partial T_1}+\left(\frac{\partial\tilde{\phi}_1}{\partial T_0}\right)^2,\\ &\frac{\partial^2\tilde{\phi}_1}{\partial T_0^2}+\omega_2^2\tilde{\phi}_1=\tilde{r}\Omega^2\sin T_0\Omega-\omega_2^2\tilde{x}_1\tilde{\phi}_1-2\frac{\partial\tilde{x}_1}{\partial T_0}\frac{\partial\tilde{\phi}_1}{\partial T_0}-2\tilde{x}_1\frac{\partial^2\tilde{\phi}_1}{\partial T_0^2}-2\frac{\partial^2\tilde{\phi}_1}{\partial T_0\partial T_1}.\end{aligned}\tag{8}$$

(order ε^3)

$$
\begin{gathered}
\frac{\partial^2 \tilde{x}_3}{\partial T_0^2}+\omega_1^2 \tilde{x}_3=\tilde{f}_1 \cos T_0 \Omega_1+\tilde{r} \Omega^2 \tilde{\phi}_1 \sin \Omega T_0-\omega_2^2 \tilde{\phi}_1 \tilde{\phi}_2-\frac{\partial \tilde{x}_1^2}{\partial T_1^2}-\tilde{c}_1 \frac{\partial \tilde{x}_1}{\partial T_0}+2 \frac{\partial \tilde{\phi}_1}{\partial T_1} \frac{\partial \tilde{\phi}_1}{\partial T_0}+\tilde{x}_1\left(\frac{\partial \tilde{\phi}_1}{\partial T_0}\right)^2+ \\
+2 \frac{\partial \tilde{\phi}_1}{\partial T_0} \frac{\partial \tilde{\phi}_2}{\partial T_0}-2 \frac{\partial^2 \tilde{x}_1}{\partial T_0 \partial T_2}-2 \frac{\partial^2 \tilde{x}_2}{\partial T_0 \partial T_1}, \\
\frac{\partial^2 \tilde{\phi}_3}{\partial T_0^2}+\omega_2^2 \tilde{\phi}_3=\tilde{f}_2 \cos T_0 \Omega_2+\tilde{r} \Omega^2 \tilde{x}_1 \sin T_0 \Omega-\tilde{r} \Omega^2 \tilde{\phi}_1 \cos T_0 \Omega-\omega_2^2 \tilde{x}_2 \tilde{\phi}_1-\omega_2^2 \tilde{x}_1 \tilde{\phi}_2-\frac{\partial^2 \tilde{\phi}_1}{\partial T_1^2}-2 \frac{\partial \tilde{x}_1}{\partial T_0} \frac{\partial \tilde{\phi}_1}{\partial T_1} \\
-\tilde{c}_2 \frac{\partial \tilde{\phi}_1}{\partial T_0}-2 \frac{\partial \tilde{x}_1}{\partial T_1} \frac{\partial \tilde{\phi}_1}{\partial T_0}-2 \tilde{x}_1 \frac{\partial \tilde{x}_1}{\partial T_0} \frac{\partial \tilde{\phi}_1}{\partial T_0}-2 \frac{\partial \tilde{x}_2}{\partial T_0} \frac{\partial \tilde{\phi}_1}{\partial T_0}-2 \frac{\partial \tilde{x}_1}{\partial T_0} \frac{\partial \tilde{\phi}_2}{\partial T_0}-2 \frac{\partial^2 \tilde{\phi}_1}{\partial T_0 \partial T_2}-4 \tilde{x}_1 \frac{\partial^2 \tilde{\phi}_1}{\partial T_0 \partial T_1} \\
-2 \frac{\partial^2 \tilde{\phi}_2}{\partial T_0 \partial T_1}-\tilde{x}_1^2 \frac{\partial^2 \tilde{\phi}_1}{\partial T_0^2}-2 \tilde{x}_2 \frac{\partial^2 \tilde{\phi}_1}{\partial T_0^2}-2 \tilde{x}_1 \frac{\partial^2 \tilde{\phi}_2}{\partial T_0^2} .
\end{gathered} \tag{9}
$$

In order to simplify the notation, the sign ~ (tilde) will be hereinafter omitted.
After eliminating secular terms we obtain the following second and third order solutions:

$$
\begin{aligned}
x_2= & -\frac{4 r \Omega^2 \omega_1^2\left(\omega_1^2-4 \omega_2^2\right) \cos \left(T_0 \Omega\right)+\left(\Omega^2-\omega_1^2\right) \omega_2^2\left(-\omega_1^2+4 \omega_2^2+3 \omega_1^2 \cos 2 T_0 \omega_2\right) D_1^2}{4 \omega_1^2\left(\Omega^2-\omega_1^2\right)\left(\omega_1^2-4 \omega_2^2\right)}- \\
& \frac{\left(\Omega^2-\omega_1^2\right) \omega_2^2\left(\omega_1^2-4 \omega_2^2+3 \omega_1^2 \cos 2 T_0 \omega_2\right) D_2^2+6 \omega_1^2\left(\Omega^2-\omega_1^2\right) \omega_2^2 D_1 D_2 \sin 2 T_0 \omega_2}{4 \omega_1^2\left(\Omega^2-\omega_1^2\right)\left(\omega_1^2-4 \omega_2^2\right)}
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
\phi_2= & -\frac{\omega_2\left(\omega_2^2-\Omega^2\right) C_2\left(D_1\left(-3 \omega_1 \omega_2 \cos T_0 \omega_2 \sin T_0 \omega_1+2\left(\omega_1^2-\omega_2^2\right) \cos T_0 \omega_1 \sin T_0 \omega_2\right)\right)}{\left(-\Omega^2+\omega_2^2\right)\left(\omega_1^3-4 \omega_1 \omega_2^2\right)}+ \\
& \frac{\omega_2\left(\omega_2^2-\Omega^2\right) C_1\left(2 D_2\left(+3 \omega_1 \omega_2 \cos T_0 \omega_1 \sin T_0 \omega_2+2\left(\omega_2^2-\omega_1^2\right) \cos T_0 \omega_2 \sin T_0 \omega_1\right)\right)}{2\left(-\Omega^2+\omega_2^2\right)\left(\omega_1^3-4 \omega_1 \omega_2^2\right)}- \\
& \frac{D_1\left(6 \omega_1 \omega_2 \cos T_0 \omega_1 \cos T_0 \omega_2+4\left(\omega_1^2-\omega_2^2\right) \sin T_0 \omega_2 \sin T_0 \omega_1\right)-r \Omega^2 \omega_1\left(\omega_1^2-4 \omega_2^2\right) \sin \left(T_0 \Omega\right)}{\left(-\Omega^2+\omega_2^2\right)\left(\omega_1^3-4 \omega_1 \omega_2^2\right)}
\end{aligned} \tag{11}
$$

The third order approximation is given by

$$
\begin{aligned}
x_3= & \frac{f_1}{\omega_1^2-\Omega_1^2} \cos \left(T_0 \Omega_1\right)-\frac{r \Omega^4\left(2 \Omega \omega_2 \cos \left(T_0 \Omega\right) D_2+\left(\Omega^2-\omega_1^2+\omega_2^2\right) D_1 \sin \left(T_0 \Omega\right)\right) \cos \left(T_0 \omega_2\right)}{\left(\Omega^2-\omega_2^2\right)\left(\Omega^2-\left(\omega_1-\omega_2\right)^2\right)\left(\Omega^2-\left(\omega_1+\omega_2\right)^2\right)}- \\
& \frac{r \Omega^4\left(-2 \Omega \omega_2 \cos \left(T_0 \Omega\right) D_1+\left(\Omega^2-\omega_1^2+\omega_2^2\right) D_2 \sin \left(T_0 \Omega\right)\right) \cos \left(T_0 \omega_2\right)}{\left(\Omega^2-\omega_2^2\right)\left(\Omega^2-\left(\omega_1-\omega_2\right)^2\right)\left(\Omega^2-\left(\omega_1+\omega_2\right)^2\right)}+ \\
& \frac{\left(\omega_1-\omega_2\right) \omega_2 C_1\left(\cos \left(T_0\left(\omega_1+2 \omega_2\right)\right)\left(D_1^2-D_2^2\right)+2 D_1 D_2 \sin \left(T_0\left(\omega_1+2 \omega_2\right)\right)\right)}{16 \omega_1\left(\omega_1+2 \omega_2\right)}+ \\
& \frac{\left(\omega_1+\omega_2\right) \omega_2 C_1\left(\cos \left(T_0\left(\omega_1-2 \omega_2\right)\right)\left(-D_1^2+D_2^2\right)+2 D_1 D_2 \sin \left(T_0\left(\omega_1-2 \omega_2\right)\right)\right)}{16 \omega_1\left(\omega_1+2 \omega_2\right)}+ \\
& \frac{\omega_2 C_2\left(\left(\omega_1^2+2 \omega_2^2\right) \cos \left(T_0 \omega_1\right)\left(D_1^2-D_2^2\right)-6 \omega_1 \omega_2 D_1 D_2 \sin \left(T_0 \omega_1\right)\right) \sin \left(2 T_0 \omega_2\right)}{8 \omega_1\left(\omega_1^2-4 \omega_2^2\right)}- \\
& \frac{\omega_2 C_2\left(2\left(\omega_1^2+2 \omega_2^2\right) \cos \left(T_0 \omega_1\right) D_1 D_2+3 \omega_1 \omega_2\left(D_1^2-D_2^2\right) \sin \left(T_0 \omega_1\right)\right) \cos \left(2 T_0 \omega_2\right)}{8 \omega_1\left(\omega_1^2-4 \omega_2^2\right)}
\end{aligned} \tag{12}
$$

$$\phi_3 = \frac{Function\ of\left(T_0, T_1, T_2, \omega_1, \omega_2, \omega, \Omega_1, \Omega_2 r, f_1, f_2, c_1, c_2\right)}{\left(\Omega^2-\omega_1^2\right)\left(\Omega^2-(\omega_1-\omega_2)^2\right)\left(\Omega^2-(\omega_1+\omega_2)^2\right)\left(\omega_1^2-4\omega_2^2\right)\left(-\Omega^2+\omega_2^2\right)\left(\omega_2^2-\Omega_2^2\right)} \tag{13}$$

C_1, C_2, D_1, D_2 can be calculated from the initial conditions.

4. Resonances

From the above solutions, many resonance cases can be detected. They are classified into primary resonances (external and parametric): $\Omega_1 = \omega_1, \Omega_2 = \omega_2, \Omega = \omega_1, \Omega = \omega_2$, internal resonance: $\omega_1 = 2\omega_2$ and combined resonances: $\omega = \pm(\omega_1 - \omega_2),\ \omega = \pm(\omega_1 + \omega_2)$.

It should be noticed that the system behavior is very complex, especially when the natural frequencies satisfy certain resonance conditions. The parametric $\Omega \approx \omega_1$ and external $\omega_2 \approx \Omega_2$ resonances appearing simultaneously are discussed below. In order to study the resonances, we introduce detuning parameters σ_1 and σ_2 in the following way

$$\Omega = \omega_1 + \varepsilon\sigma_1,\ \Omega_2 = \omega_2 + \varepsilon\sigma_2 . \tag{14}$$

Substitution (14) into (8) and (9) allows to obtain the frequency response functions:

- for parametric resonance

$$\left(-\sigma_1 a_1 + \frac{\omega_2^2\left(7\omega_2^2-\omega_1^2\right)a_1 a_2^2}{4\omega_1\left(\omega_1^2-4\omega_2^2\right)}\right)^2 + \frac{c_1^2}{4}a_1^2 = \frac{R^2\omega^4}{4\omega_1^2}, \tag{15}$$

- for external resonance

$$\left(-\sigma_2 a_2 - \frac{\omega_2\left(\omega_1^2-7\omega_2^2\right)a_2 a_1^2}{4\left(\omega_1^2-4\omega_2^2\right)} + \frac{\omega_2^3\left(\omega_1^2+8\omega_2^2\right)a_2^3}{16\omega_1^2\left(\omega_1^2-4\omega_2^2\right)}\right)^2 + \frac{c_2^2}{4}a_2^2 = \frac{f_2^2}{4\omega_2^2}, \tag{16}$$

where a_1 and a_2 are amplitudes of the longitudinal and swing vibrations, respectively.

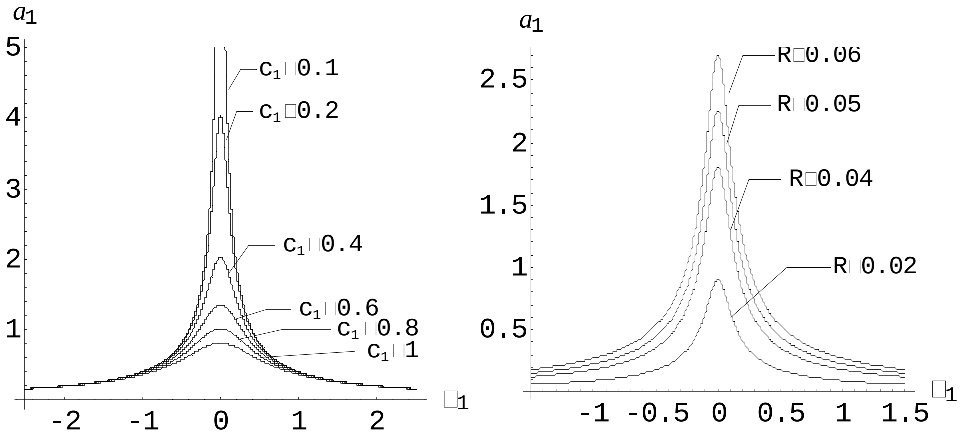

Fig. 2 Amplitudes a_1 vs. detuning parameter for different c_1 (left) and for different radius R (right)

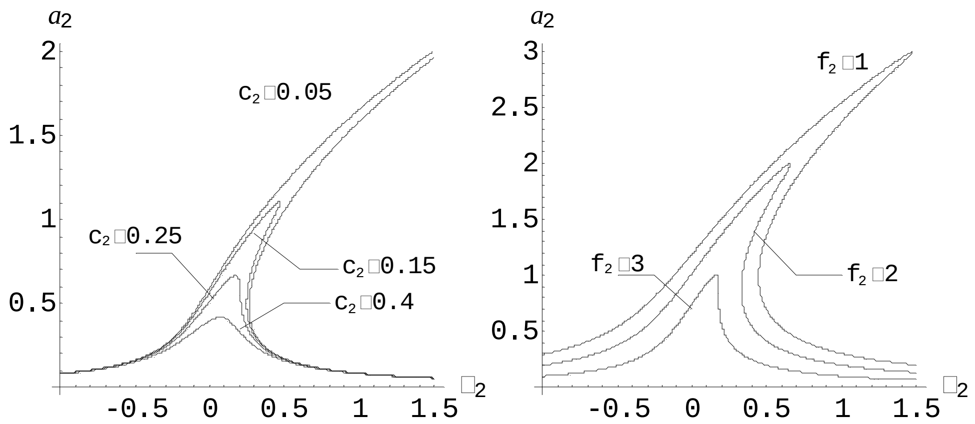

Fig. 3 Amplitudes a_2 vs. detuning parameter for different c_2 (left) and for different f_2 (right)

It can be seen in Fig. 2 that the amplitude a_1 is a function monotonically increasing with R and decreasing with c_1. Similarly, it can be seen from Fig. 3 that the amplitude a_2 monotonically increases with the excitation f_2 amplitude and decreases with c_2. The curves are bent to the right, giving rise to the jump phenomenon.

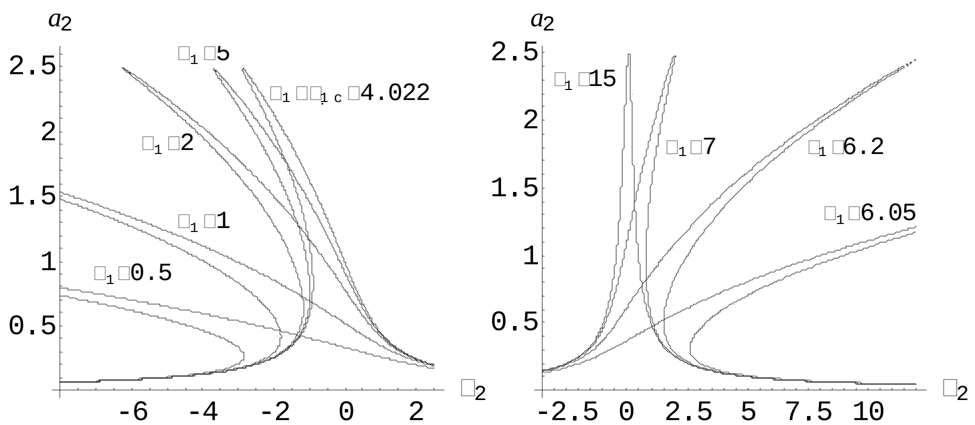

Fig. 4 Amplitudes a_2 against detuning parameter (effects of natural frequency ω_1 variation)

The influence of ω_1 on frequency response for a_2 is presented in Fig. 4. For $\omega_1 > 2\omega_2$ a soft spring effect is observed (curves are bent to the right). For $\omega_1 < 2\omega_2$ a hard spring effect is observed (curves are bent to the left). An interesting phenomenon here occurs – at critical value of $\omega_1 = \omega_{1c}$ the plot slope is minimal.

The graph in Fig.5 presents dependence of the coefficient which is multiplied by a_2^3 in (16) versus ω_1. It explains the slope changes of the resonance curves. In Fig. 6 the variation of coefficient at a_2^3 versus ω_1 is shown.

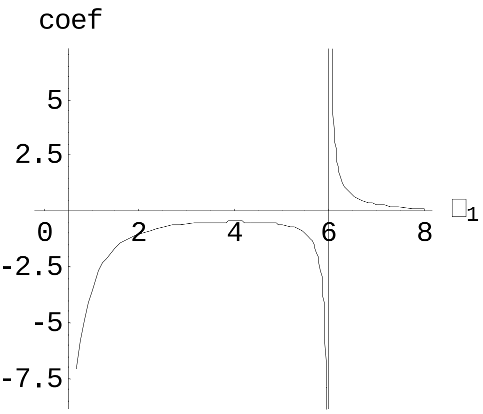

Fig. 5 The plot of the coefficient multiplied by a_2^3 in (16) versus ω_1

Similar behavior as above can be observed in the external resonance for a_2 for various ω_2. Resonance curves and the coefficient accompanying a_2^3 in (16) for this case are presented in Figs. 6 and 7.

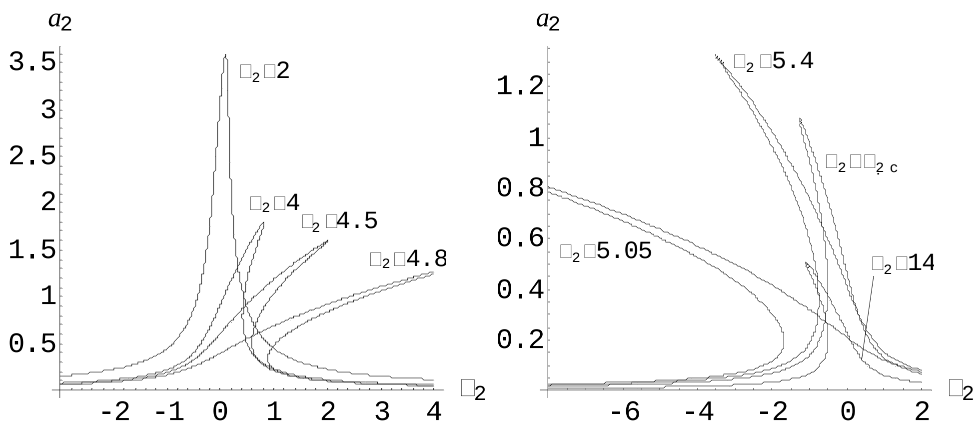

Fig. 6 Amplitudes a_2 vs. detuning parameter (influence of natural frequency ω_1 variation)

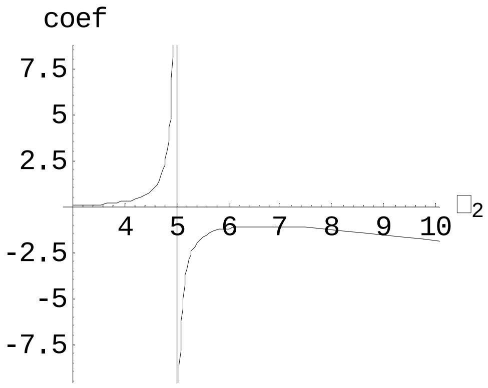

Fig. 7 The coefficient multiplied by a_2^3 in (16) versus ω_2

4. Conclusions

The third order solution for the equations of motion of the studied system is achieved. Since in general the solution of the frequency response function regarding the stability of the system are of special interest, our results are presented in graphical form. In particular two chosen resonance cases have been discussed.

Increase or decrease of ω_1 or ω_2 gives rise to the effect of hard or soft spring, respectively, as the curve is bent to the right or to the left, leading to the jump phenomenon (Figs. 4 and 6). The graphs of the coefficient occurring at the third power of a_2 in frequency response function (Figs. 5 and 7) explains the behavior of resonance curves.

References

[1] Shivamoggi B. K., *Perturbation methods for differential equations*, Birkhauser Boston,
[2] Awrejcewicz J., Krysko V. A., *Introduction to the modern asymptotic methods*, WNT, Warszawa, 2004, in Polish
[3] Sado D. *Energy transfer in nonlinear coupled 2-dof systems*, OWPW Warszawa, 1997, in Polish
[4] Starosta R., Awrejcewicz J., Analysis of parametrical systems in resonances, *proceeding, I Kongres Mechaniki Polskiej*, Warszawa, 2007, in Polish

Roman Starosta. Poznan University of Technology, Institute of Applied Mechanics, ul. Piotrowo 3, 60-965 Poznań, roman.starosta@put.poznan.pl.

Grażyna Sypniewska-Kamińska. Poznan University of Technology, Institute of Applied Mechanics, ul. Piotrowo 3, 60-965 Poznań, grazyna.sypniewska-kaminska@put.poznan.pl.

Jan Awrejcewicz. Technical University of Łódź, Department of Automatics and Biomechanics, ul. Stefanowskiego 1/15, 90-924 Łódź, awrejcew@p.lodz.pl.