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DYNAMICS OF A 3D PHYSICAL PENDULUMS WITH NONAUTONOMOUS SYSTEM UNIVERSAL JOINTS

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Abstract: Physical and mathematical models of 3D nonautonomous single and double pendulums under gravity with axial excitation are presented. First body is suspended on a universal joint (also called Cardan-Hooke's joint) and linked with a second body by second universal joint, so that the entire mechanical system has 5 degrees-of-freedom. Dimensions, masses, moments of inertia and positions of the centres of mass of both rigid bodies are known. Drive shaft of suspension joint is excited by external constant moment of force, which induces its axial rotation with constant angular velocity and rotation of both bodies. In addition, stiffness coefficients of joints are included into the mathematical model. Orientation of each body is described by three Euler angles. Position transformation from fixed coordinate system to appropriate local systems is governed by the corresponding rotation matrix. System of ordinary differential equations obtained from Lagrange's equations is solved numerically with a help of specially developed simulation software. A few nonlinear system regimes are illustrated and discussed.

1. Introduction

There are a wide variety of a single and multi-body pendulum applications in history of our science. The first known mechanical system which includes a simple pendulum was a clock. Original accurate one was invented in 1657 by Christiaan Huygens (Dutch physicist, astronomer and mathematician), which was a breakthrough in time measuring. This invention was widely described by Huygens [1] himself, as well as Taylor and Van Kersen [2]. He also made several observations of pendulum mechanics, e.g. an resonance effect between two pendulums. Second famous pendulum applicant was Léon Foucault (physicist from France), whose novel mechanical construction showed consequence of Earth's rotation on its axis. He caused a great sensation in the Panthéon in Paris in 1851 hanging his long and heavy pendulum (67 m, 28 kg) beneath the central dome. Phillips [3] explains this interesting mechanical effects in his publication and Aczel [4] aims at its historical background. Similar experimental measurement function of a pendulum was presented by British physicist Captain Henry Kater in 1817 constructing his reversible pendulum to precisely measure the acceleration of gravity.

Nowadays, pendulums and systems of pendulums are used not only as a form of measuring equipment but more frequently as a model of complex mechanical systems, e.g. recovery and optimalization of human walking apparatus presented by Donelan *et al.* [5]. Pendulums are also used to find methods of stabilization and optimization, e.g. in seismology to find way to stabilize buildings during the earthquakes Matta and De Stefano [6], or even in quantum mechanics as it is shown by Richter *et al.* [7].

In this paper we present a novel three-dimensional physical and mathematical models of a single and double pendulums driven by external force in the gravitational field. Three degrees-of-freedom universal joints are very rarely used in applications, probably due to several problems with modeling and verification of real models. Besides the analytical calculation and numerical simulations, we show some drafts of experimental setup, which will be controlled by a specially written computer simulation software to confirm the validity of the computations.

2. The model

As introduced in previous section, we consider a pendulum in 3D space, driven by external gravitational field. First body is suspended on a universal joint (also called Cardan-Hook's joint) and linked with second body by second universal joint so that the entire mechanical system has five degrees-of-freedom. We assume that both rigid bodies are identified, i.e. we know their geometrical properties, masses, moments of inertia and positions of the body centers. Also stiffness coefficients of joints are known and included in the mathematical model. For the purpose of faster calculation we assume that both joints are massless. Input shaft of first suspension joint is excited by external constant moment of force, which makes the system axially rotating with a constant angular velocity.

Orientation of each body is described by three Euler angles φ_i, θ_i i ψ_i, where i is the index of each two bodies (see Fig. 1a). Position of each body in fixed coordinate system is governed by matrix of rotation **R** (see Eq. (1)) matching rotation by each Euler angles. Any rotation in 3D space can be combined from three basic rotations about corresponding orthogonal axes and represented by three rotation matrices (and three Euler angles) in several conventions (orders of transformations). In this paper we have chosen Y-X-Z order.

$$\mathbf{R}(\varphi_i,\theta_i,\psi_i)=\begin{bmatrix}\cos\theta_i\cos\psi_i+\sin\varphi_i\sin\theta_i\sin\psi_i & -\cos\theta_i\sin\psi_i+\sin\varphi_i\sin\theta_i\cos\psi_i & \cos\varphi_i\sin\theta_i\\ \cos\varphi_i\sin\psi_i & \cos\varphi_i\cos\psi_i & -\sin\varphi_i\\ \sin\varphi_i\cos\theta_i\sin\psi_i-\sin\theta_i\cos\psi_i & \sin\varphi_i\cos\theta_i\cos\psi_i+\sin\theta_i\sin\psi_i & \cos\varphi_i\cos\theta_i\end{bmatrix}. \quad (1)$$

Construction details of the pendulum are shown in Figure 1a. L_i and e_i is the length and position of mass center C_i of the link (angular velocities $\dot{\psi}_1, \dot{\theta}_1, \dot{\varphi}_1$ are shown in Figure 1b.)

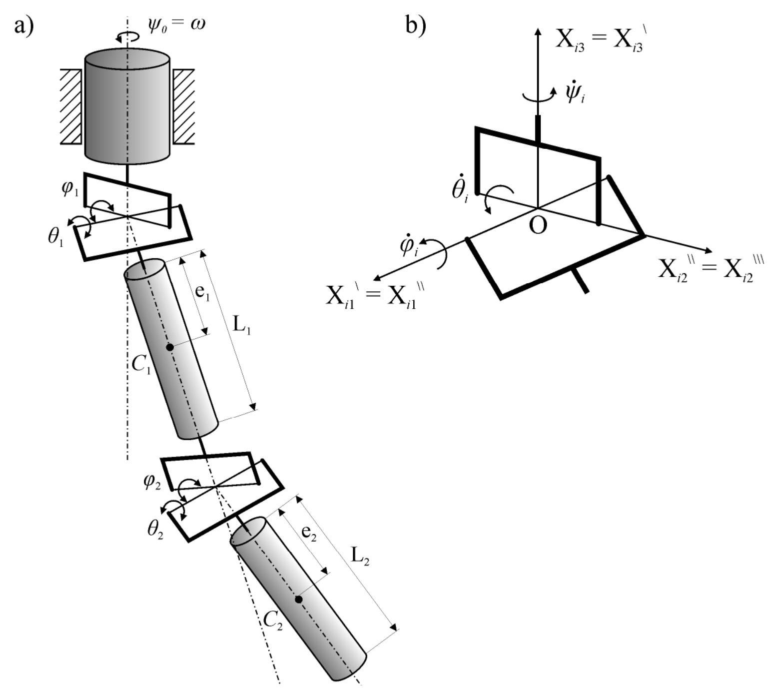

Figure 1. Coupled pendulums (a), and three Euler angles rotations
with the corresponding angular velocities $\dot{\psi}_1, \dot{\theta}_1, \dot{\varphi}_1$(b)

To derive Lagrange's equations of mechanical system, we define its kinetic energy of translation and rotation of each i-th links by

$$T_i = \frac{1}{2} m_i \sum_{i=1}^{N} \left(\dot{x}_{1Ci}^2 + \dot{x}_{2Ci}^2 + \dot{x}_{3Ci}^2 \right) + \frac{1}{2} \left(I_{1j} \sum_{i=1}^{N} \omega_{1Ci}^2 + I_{2j} \sum_{i=1}^{N} \omega_{2Ci}^2 + I_{3j} \sum_{i=1}^{N} \omega_{3Ci}^2 \right), \tag{2}$$

where m_i denote the mass of i-th body, $\dot{x}_{aCi}$ is the linear velocity of i-th body mass center C_i around axis OX_a, I_{aj} is the moment of inertia of i-th body around axis OX_a, ω_{aCi} is the angular velocity of i-th body around axis OX_a in reference to its mass center position..

We also define potential energy of links and stiffness energy of joints as follows

$$V_i = g \sum_{i=1}^{N} x_{3Ci}, \quad U_i = \frac{1}{2} \sum_{i=1}^{N} k_i \left(\theta_i^2 + \varphi_i^2 \right), \tag{3}$$

where g is the gravitational acceleration and k_i is i-th universal link stiffness coefficient.

The external excitation momentum of force is constant and makes the first universal joint input shaft rotating about fixed axis OX_3 with constant angular velocity ω, i.e. $\psi_1 = \omega t$.

The Lagrange's equations follows

$$\frac{d}{dt}\frac{\partial T_i}{\partial \dot{q}_i} - \frac{\partial T_i}{\partial q_i} + \frac{\partial U_i}{\partial q_i} + \frac{\partial V_i}{\partial q_i} = 0, \tag{4}$$

where q_i denote Euler's angles φ_i and θ_i governing the angular position of i-th body.

In that follows we study a single pendulum variant governed by the following ODEs

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_1\cos^2\varphi_1 + I_3\sin^2\varphi_1 \end{bmatrix}\begin{bmatrix} \ddot{\varphi}_1 \\ \ddot{\theta}_1 \end{bmatrix} = -k\begin{bmatrix} \varphi_1 \\ \theta_1 \end{bmatrix} - e_1 gm\begin{bmatrix} \sin\varphi_1\cos\theta_1 \\ \cos\varphi_1\sin\theta_1 \end{bmatrix} - \dot{\theta}_1\begin{bmatrix} \left[I_3\omega + \dot{\theta}_1\left(J_1 - I_3\right)\sin\varphi_1\right]\cos\varphi_1 \\ \left[I_3\omega\cos\varphi_1 + \left(I_3 + J_1\right)\right]\dot{\varphi}_1\sin 2\varphi_1 \end{bmatrix}, \tag{5}$$

where $J_1 = I_1 + {e_1}^2 m$.

System of motion equations of a 3D double pendulum model have the following form

$$[\mathbf{W_1}\ \mathbf{W_2}\ \mathbf{W_3}\ \mathbf{W_4}][\ \ddot{\varphi}_1\ \ \ddot{\theta}_1\ \ \ddot{\varphi}_2\ \ \ddot{\theta}_2]^{\mathrm{T}} = [\ P_1\ \ P_2\ \ P_3\ \ P_4]^{\mathrm{T}}, \tag{6}$$

where

$$\mathbf{W_1} = \begin{bmatrix} \sin^2 t\omega(\sin^2\varphi_2(I_1\sin^2\theta_2 + I_3\cos^2\theta_2) + I_1\cos^2\varphi_2) + \cos^2 t\omega(I_1\cos^2\theta_2 + I_3\sin^2\theta_2) + I_1 - \frac{1}{2}J_4\sin2\theta_2\sin\varphi_2\sin2t\omega + J_1 + J_2 \\ \frac{1}{8}J_4(\cos\varphi_1(\sin2t\omega(3\cos2\theta_2 - 2\cos^2\theta_2\cos2\varphi_2 - 1) + 4\sin2\theta_2\sin\varphi_2\cos2t\omega) + 4\sin\varphi_1(\cos^2\theta_2\sin2\varphi_2\sin t\omega + \sin2\theta_2\cos\varphi_2\cos t\omega)) \\ \cos t\omega(I_1\cos^2\theta_2 + I_3\sin^2\theta_2) + \sin\varphi_2(J_3\cos(\theta_1 - \theta_2)\sin\varphi_1 - J_4\sin\theta_2\cos\theta_2\sin t\omega) + J_3\cos\varphi_1\cos\varphi_2 \\ -J_3\sin(\theta_1 - \theta_2)\sin\varphi_1\cos\varphi_2 \end{bmatrix},$$

$$\mathbf{W_2} = \begin{bmatrix} \frac{1}{8}J_4\left(\cos\varphi_1(\sin2t\omega(3\cos2\theta_2 - 2\cos^2\theta_2\cos2\varphi_2 - 1) + 4\sin2\theta_2\sin\varphi_2\cos2t\omega) + 4\sin\varphi_1(\cos^2\theta_2\sin2\varphi_2\sin t\omega + \sin2\theta_2\cos\varphi_2\cos t\omega)\right) \\ \frac{1}{2}\cos^2\varphi_1(2(\cos^2 t\omega(\sin^2\varphi_2(I_1\sin^2\theta_2 + I_3\cos^2\theta_2) + I_1\cos^2\varphi_2) + \sin^2 t\omega(I_1\cos^2\theta_2 + I_3\sin^2\theta_2) + I_1 + J_1 + J_2) + J_4\sin2\theta_2\sin\varphi_2\sin2t\omega) + \\ +\sin^2\varphi_1(\cos^2\varphi_2(I_1\sin^2\theta_2 + I_3\cos^2\theta_2) + I_1\sin^2\varphi_2 + I_3) + J_4\cos\theta_2\sin2\varphi_1\cos\varphi_2(\sin\theta_2\sin t\omega - \cos\theta_2\sin\varphi_2\cos t\omega) \\ \cos\varphi_1\left(I_1\cos^2\theta_2\sin t\omega + \sin\theta_2(I_3\sin\theta_2\sin t\omega - J_3\cos\theta_1\sin\varphi_2) + \cos\theta_2\sin\varphi_2(J_4\sin\theta_2\cos t\omega + J_3\sin\theta_1)\right) + J_4\sin\theta_2\cos\theta_2\sin\varphi_1\cos\varphi_2 \\ J_3\cos(\theta_1 - \theta_2)\cos\varphi_1\cos\varphi_2 \end{bmatrix},$$

$$\mathbf{W_3} = \begin{bmatrix} \cos t\omega(I_1\cos^2\theta_2 + I_3\sin^2\theta_2) + \sin\varphi_2(J_3\cos(\theta_1 - \theta_2)\sin\varphi_1 - J_4\sin\theta_2\cos\theta_2\sin t\omega) + J_3\cos\varphi_1\cos\varphi_2 \\ \cos\varphi_1\left(I_1\cos^2\theta_2\sin t\omega + \sin\theta_2(I_3\sin\theta_2\sin t\omega - J_3\cos\theta_1\sin\varphi_2) + \cos\theta_2\sin\varphi_2(J_4\sin\theta_2\cos t\omega + J_3\sin\theta_1)\right) + J_4\sin\theta_2\cos\theta_2\sin\varphi_1\cos\varphi_2 \\ I_1\cos^2\theta_2 + I_3\sin^2\theta_2 + J_1 \\ 0 \end{bmatrix},$$

$$\mathbf{W_4} = \begin{bmatrix} -J_3\sin(\theta_1 - \theta_2)\sin\varphi_1\cos\varphi_2 \\ J_3\cos(\theta_1 - \theta_2)\cos\varphi_1\cos\varphi_2 \\ 0 \\ J_1\cos^2\varphi_2 \end{bmatrix}$$

are the columns of symmetric matrix of left-side coefficients, $J_1 = {e_1}^2 m$, $J_2 = {L_1}^2 m$, $J_3 = e_1 L_1 m$, $J_4 = I_1 - I_3$, and P_1, P_2, P_3, P_4 are the right sides of the equation system depending on angular position of first and second links $\varphi_1, \theta_1, \varphi_2, \theta_2$, and their first time derivatives $\dot{\varphi}_1, \dot{\theta}_1, \dot{\varphi}_2, \dot{\theta}_2$.

3. Method of solution

To solve Lagrange's equations (4) we use a symbolic computational software *Wolfram Mathematica®* (other numerical calculations were performed by our software specially written in *CodeGear™ Delphi® 2009).*

The computing procedure consists of: (i) defining variables and basic rotation matrix **R** of both rigid bodies; (ii) defining kinetic, potential and stiffness energies; (iii) building and solving Lagrange's equations; (iv) numerically solving obtained ordinary differential equations; (v) numerical analysis of the nonlinear dynamic of the pendulum; (vi) rendering proper plots and other graphic representation of the results.

4. Results

We introduced a simplification by neglecting kinetic energy of rotation of the second link. It made the motion equations easier to investigate without losing much of the interesting nonlinear dynamics effects. All presented computations are carried out for the following fixed parameters: $m_1 = m_2 = 1$, $L_1 = L_2 = 20$, $e_1 = e_2 = 10$, $I_1 = I_2 = 0.005$ and $I_3 = 33$, which corresponds to cylindrical shape (of radius 0.1) of both pendulum links (other parameters and the values of ω and k are shown in the figures).

Besides the time history plot of each angle of each link, $\varphi_i(\theta_i(t))$ plots and Poincaré maps $\varphi_2(\theta_2(t))$ of first link are reported.

Figure 2 shows the single pendulum motion from $t_0 = 0$ (it looks like a simple spherical pendulum).

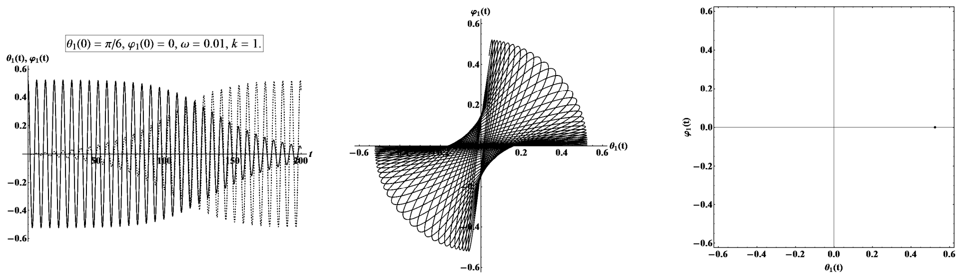

Figure 2. Single pendulum motion – time history of angles $\varphi_1(t)$ and $\theta_1(t)$, $\varphi_1(\theta_1(t))$ and Poincaré map $\varphi_1(\theta_1(t))$ regarding $2\pi/\omega$.

Figure 4 shows the Poincaré maps $\varphi_1(\theta_1(t))$ with a step of $2\pi/\omega$ for constant angular velocity $\omega = 0.5$ and variable k. As in the previous presented computations we neglected first 5000 time steps of 40000. This series of cross sections reveals the potential nonlinear dynamic properties of this mechanical system. As we can see, when stiffness coefficient $k \approx 35.005$ the Poincaré maps shows regular shape of attractors.

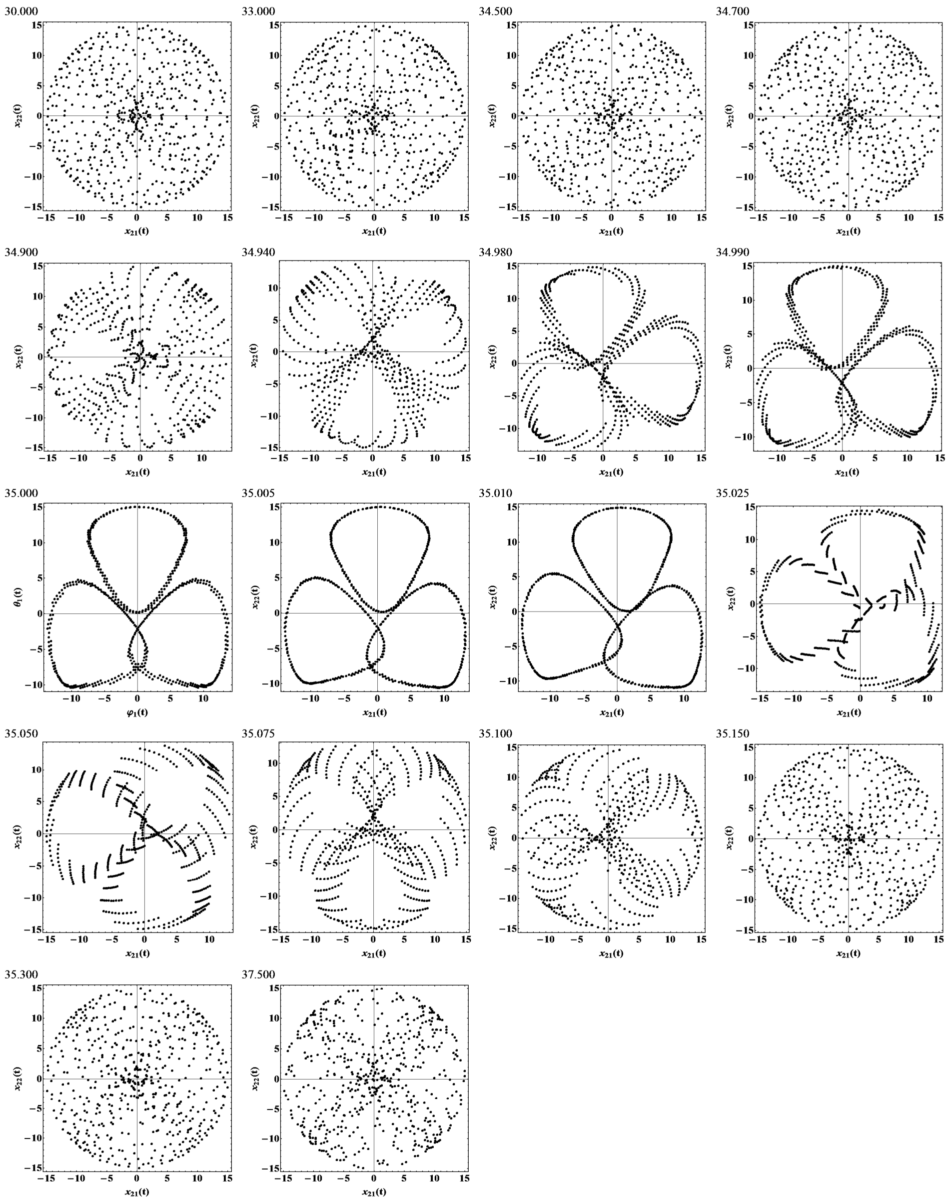

Figure 4. Double pendulum motion – series of Poincaré maps $x_{21}(x_{22}(t))$ about $2\pi/\omega$ for $\omega = 0.5$, $k = [33..35,005..37,5]$ (k value is showed in upper-left corners of the plots).

Figure 4 shows the example plots obtained from the calculation of the double pendulum model. Here we also neglected first 5000 time steps of 40000 to skip transitional period of movement.

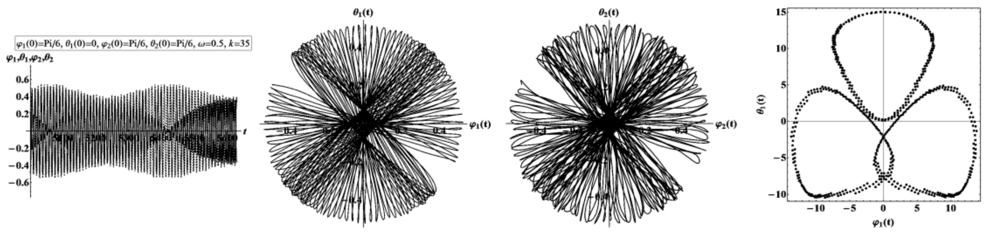

Figure 4. Double pendulum motion – time history of all angles, $\varphi_1(\theta_1(t))$, $\varphi_2(\theta_2(t))$ and Poincaré map $\varphi_2(\theta_2(t))$.

6. Conslusions

Complex physical and mathematical model of novel single and double pendulum mechanical system were introduced. All links are coupled together by two universal joints which corresponds to several rotation matrix transformation to show the position of each link in moving coordinate system and the absolute one. Using the rotation matrix makes the computation very complex but also easy to future extend to describe more than two links of the pendulum. Presentation of the results of single link system shows the expected similarity to Foucault's spherical pendulum. Basic investigation of the double pendulum by analyzing Poincaré maps for $\omega = const$ and variable stiffness coefficient k revealed the existence of some attractors and is now under detailed examination.

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