10th CONFERENCE
on
DYNAMICAL SYSTEMS THEORY AND APPLICATIONS
December 07-10, 2009. Lodz, Poland

MODIFIED MURAVSKII MODEL FOR ELASTIC FOUNDATIONS

Igor V. Andrianov, Yurii A. Kirichek, J. Awrejcewicz

Abstract: A frequency equation for the vibration of an engine seating and an equation for pressure under the bottom of the engine are obtained.

1. Introduction

In general, the problem of suppressing engine seating vibrations is reduced to that of a 3D contact dynamical problem [1]. Although in general one may apply numerical approaches to solve a 3D problem, say FEM, but even using high tech computers this approach causes various difficulties to get a reliable solution in an economical way.

From the point of view of an analytical treatment the so far applied methods may be divided into mathematically rigorous and approximate analytical techniques. Methods of the first group are associated with the application of integral equations [2], and therefore it is necessary to prove the existence and uniqueness theorems. Next, the discussed equations are usually reduced to coupled integral equations which can be transformed to the second order Fredholm type equations by using various approaches (for instance, applying the method of orthogonal polynomials) to get finally a system of linear algebraic equations. The latter ones are tested according to their regularity and then a truncated system is solved numerically.

Practically oriented computational methods include already earlier introduced various physically motivated simplifications. For instance, the engine seating mass is assumed to be concentrated to a point, or a ground reaction is uniformly distributed, and so on. The latter approach is commonly accepted in engineering society. Models proposed by Winkler [3], Pasternak [4] and Vlasov [5] do not allow us to get sufficiently accurate results, whereas the model of an elastic half-plane (or an elastic half-space) is not adequately complex in comparison with simplifications introduced for the foundation. On the other hand when inertial properties of foundation are neglected while computing vertical vibrations of a plate-type engine, the seating may cause serious errors, as both experimental and numerical investigations show [6,7]. Owing to the critical state-of-the art review carried out, the model proposed by Muravskii [8] seems to be one of the most appropriate ones for engineering as it exhibits high practical accuracy of the ground modeling with its simultaneous simplicity. Here we propose some natural modification of this model.

2. Static loading

Next, we consider the plane problem of elasticity for a strip subjected to a rigid punch action (Fig. 1).

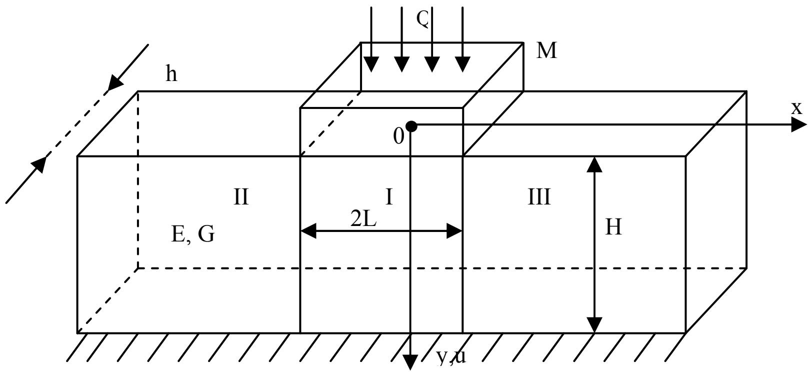

Figure 1. Physical model of a rigid foundation on an elastic layer

The following computational scheme is applied (Fig. 2): on zone I strip is reduced to a rod with stiffness EF, where $F = 2\ell hE$, and E denotes Young modulus; zones II and III are associated with the Muravskii model [8]. The problem defined so far is governed by the following equations

$$E\left[h + F\delta(x)\right]U_{yy} + GhU_{xx} = 0; \tag{1}$$

$$E\left[h + F\delta(x)\right]U_{y} = -p\delta(x) \quad \text{for} \quad y = 0; \tag{2}$$

$$U = 0 \quad \text{for} \quad y = H, \tag{3}$$

$$U \to 0 \quad \text{for} \quad |x| \to \infty,$$

where $\delta(x)$ is the Dirac function; G is the shear modulus; $p = 2\ell hp + Mg$.

A solution to the boundary value problem (1) – (3) is sought in the form satisfying the boundary conditions (3):

$$U(x,y) = \sum_{k=1,3,5,\ldots} U_k(x)\cos\frac{k\pi y}{2H}. \tag{4}$$

Substituting (4) to (1) and after splitting regarding cosines one gets

$$E\left[h + F\delta(x)\right]\left(\frac{n\pi}{2H}\right)^2 U_n(x) - GhU_{nxx}(x) = p\delta(x), \quad n = 1,3,5\ldots. \tag{5}$$

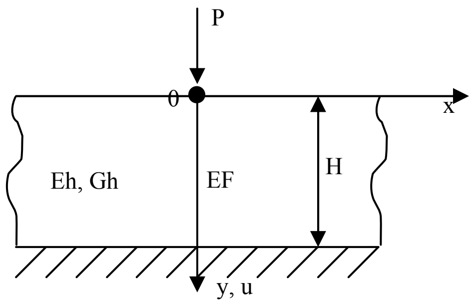

Figure 2. Computational scheme of the static problem

Applying the Fourier transformation

$$\overline{U}_n(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U_n(x) e^{-ixq} dx \tag{6}$$

to (5), one gets

$$Eh\left(\frac{n\pi}{2H}\right)^2 \overline{U}_n(p) + Ghq^2 \overline{U}_n(p) = \frac{1}{\sqrt{2\pi}}\left[p - EF\left(\frac{n\pi}{2H}\right)^2 U_n(0)\right]. \tag{7}$$

Using an inverse Fourier transformation governed by formula (6) one finds $U_n(q)$, and then $U_n(0)$ is defined.

Finally the function $U(x,y)$ may be written as follows

$$U(x,y) = \frac{8pH}{\pi^2 EF} \sum_{k=1,3,5,...} \frac{\exp\left(-\frac{\pi\alpha k|x|}{2H}\right)}{k(k+d)} \cos\frac{k\pi y}{2H}, \tag{8}$$

where: $\alpha = \sqrt{\frac{E}{G}}$, $d = 2hH(\pi\alpha F)$.

Displacement of the engine seating is given by formula (8) for $x = y = 0$ and after a summation procedure owing to formula 5.1.6.10 from reference [12] one obtains

$$U(0,0) = \frac{P}{2\pi h\sqrt{EG}}\left[\psi(d+1) - \frac{1}{2}\psi\left(\frac{d+1}{2}\right) + \frac{1}{2}\psi(d) + C + \ln 2\right], \tag{9}$$

where $\psi(z)$ is the so called psi-function; $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$, $\Gamma(z)$ is the gamma-function; C is the Euler constant, $C = 0.5772156649\ldots$.

Let us define now pressure under the engine seating bottom

$$T = EFU_y\big|_{x=0}. \tag{10}$$

Since differentiation of the series (8) term by term is not allowed, first A.N.Krylov's method is applied [13]. We will use the identity

$$\frac{8pH}{\pi^2 EF}\sum_{k=1,3,5,...}\frac{\cos\left(\frac{k\pi y}{2H}\right)}{k^2} = \frac{pH}{EF}\left(1-\frac{y}{H}\right). \tag{11}$$

Finally, combine formula (8) for $x = 0$ and (11) to get

$$U(0,y) = \frac{pH}{EF}\left(1-\frac{y}{H}\right) - \frac{8pHd}{\pi^2 EF}\sum_{k=1,3,5...}\frac{\cos\left(\frac{k\pi y}{2H}\right)}{k^2(k+d)}. \tag{12}$$

The series in expression (12) can be differentiated term by term and after a substitution of formula (12) into (10) one gets

$$T = -p + \frac{4pd}{\pi}\sum_{k=1,3,5,...}\frac{\sin\left(\frac{k\pi y}{2H}\right)}{k(k+d)}. \tag{13}$$

Furthermore, shear stress for $x = 0$ is

$$\tau(y) = \frac{EF}{2}U_{yy}\Big|_{x=0} = \frac{1}{2}T_y. \tag{14}$$

In order to allow the term-by-term differentiation, we derive from the right hand side of (13) the following expression

$$\frac{4pd}{\pi}\sum_{k=1,3,5,...}\frac{\sin\left(\frac{k\pi y}{2H}\right)}{k^2} = \frac{2pd}{\pi}\int_0^{\frac{\pi y}{2H}}\left[\ln\left(2\cos\frac{k\pi t}{4H}\right) - \ln\left(2\sin\frac{k\pi t}{4H}\right)\right]dt. \tag{15}$$

For summation of the series, formula 5.4.2.12 from reference [12] is used.

Now, applying formulas (13) – (15), one gets

$$\tau(y) = \frac{pd}{H}\left[\ln\left(2\cos\left(\frac{k\pi y}{4H}\right)\right) - \ln\left(2\sin\left(\frac{k\pi y}{4H}\right)\right) - 2d\sum_{k=1,3,5,...}\frac{\cos\frac{k\pi y}{2H}}{k(k+d)}\right. \tag{16}$$

3. Dynamic loading

Next, we consider a BVP governed by the following equations (Fig. 3)

$$E\left[h + F\delta(x)\right]U_{yy} + GhU_{xx} - \rho h U_{tt} = 0; \tag{17}$$

$$E\left[h + F\delta(x)\right]U_y = -M\delta(x)U_{tt} \quad \text{for} \quad y = 0; \tag{18}$$

$$U = 0 \quad \text{for} \quad y = H; \tag{19}$$

$$U \to 0 \quad \text{for} \quad |x| \to \infty.$$

We aimed at natural vibrations assuming

$$U = U(x, y)\exp(i\omega t). \tag{20}$$

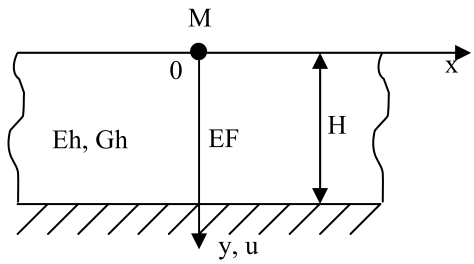

Figure 3. Computational model of dynamical problem

Substituting (20) into the BVP (17) – (19) one gets

$$E\left[h + F\delta(x)\right]U_{yy} + GhU_{xx} + \rho h\omega^2 U = 0; \tag{21}$$

$$E\left[h + F\delta(x)\right]U_y = -M\delta(x)\omega^2 U \quad \text{for } y = 0; \tag{22}$$

$$U = 0 \quad \text{or} \quad y = H; \tag{23}$$

$$U \to 0 \quad \text{for} \quad |x| \to \infty. \tag{24}$$

Substituting the solution to the BVP (21) – (24) in the form of (4) one obtains

$$E\left[h + F\delta(x)\right]\left(\frac{n\pi}{2H}\right)^2 U_n(x) - GhU_n(x) + \rho h\omega^2 U_n = M\omega^2\delta(x)U_n(x). \tag{25}$$

Now, applying the Fourier transformation (6), one gets

$$E\left(\frac{n\pi}{2H}\right)^2 \overline{U}_n(q) + Ghp^2\overline{U}_n(q) - \rho h\omega^2 U = \frac{1}{\sqrt{2\pi}}\left[M\omega^2 - EF\left(\frac{n\pi}{2H}\right)^2\right]U_n(0). \tag{26}$$

Application of the inversing Fourier transformation defined by formula (6) yields

$$U_n(x) = \frac{\left[M\omega^2 - EF\left(\frac{n\pi}{2H}\right)^2\right]U_n(0)}{4Gh} \times \sqrt{\frac{E}{G}\left(\frac{n\pi}{2}\right)^2 - \frac{\rho H^2}{G}\omega^2} \times \exp\left(-\sqrt{\frac{E}{G}\left(\frac{n\pi}{2H}\right)^2 - \frac{\rho}{G}\omega^2}\,|x|\right).$$

Taking $x = 0$ the following formula for a frequency of natural vibration is finally obtained

$$1 = \frac{M\omega^2 - EF\left(\frac{n\pi}{2H}\right)^2}{Gh}\sqrt{\frac{E}{G}\left(\frac{n\pi}{2}\right)^2 - \frac{\rho H^2\omega^2}{G}}.$$

4. Conclusions

Modification of the Muravskii foundation model allows us to obtain a simple analytical solution of the static and dynamic problem for engine seating on elastic foundations.

References

1. Seimov V.M., *Dynamical Contact Problems*. Naukova Dumka, Kiev, 1976, in Russian.
2. Babeshko V.A., Glushkov E.V., Zinchenko Zh.V., *Dynamics of Heterogeneous Linearly Elastic Media*. Science, Moscow, 1989, in Russian.
3. Selvadurai A.P.S., *Elastic Analysis of Soil-Foundation Interaction*. Elsevier, Amsterdam, 1979.
4. Winkler E., *Die Lehre von der Elastizität und Festigkeit*. Dominikus, Prague, 1867.
5. Vlasov V.Z., Leont'ev U.N., Beams, *Plates and Shells on Elastic Foundations*. Israel Program for Scientific Translations, Jerusalem, 1966.
6. Perelmuter A.V., Slivker V.I., *Numerical Structural Analysis: Methods, Models and Pitfalls*. Springer-Verlag, Berlin, 2003.
7. Kerr A.D., On the determination of foundation model parameters. *J. Geotech. Eng.*, 111(11), 1985, 1134-1140.
8. Kerr A.D., A study of a new foundation model. *Acta Mechanica*, No. 1/2, 1965, 135-147.
9. Vallabhan C.V.G., Das Y.C. Modified Vlasov model for beams on elastic foundations. *J. Geotech. Eng.*, 117(6), 1991, 956-966.
10. Vallabhan C.V.G., Das Y.C., A refined model for beams on elastic foundations. *Int. J. Solids Str.*, 27(5), 1991, 629-637.
11. Muravskii B.G., *Mechanics of Non-Homogeneous and Anisotropic Foundations*. Springer-Verlag, Berlin, 2001.
12. Prudnikov A.P., Brichkov Yu.A., Marichev O.I., *Integrals and Series*. Gordon & Breach, New York, 1986.
13. Kantorovich L.V., Krylov V.I., *Approximate Methods of Higher Analysis*. Noordhoff, Groningen, 1958.

Igor V. Andrianov
Institut für Allgemeine Mechanik
RWTH Aachen (TU Aachen)
Templergraben 64, D-52056 Aachen
Germany

Yurii A. Kirichek
Prydniprovskaya State Academy of Civil Engineering and Architecture
24a Chernyshevskogo St., Dnepropetrovsk UA-49005
Ukraine

Jan Awrejcewicz
Department of Automatics and Biomechanics
Technical University of Lodz
1/15 Stefanowskiego St., 90-924 Lodz
Poland