Numerical modelling and analysis of mechanical systems with impacts and dry friction

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Abstract

The work is devoted to modelling and numerical analysis of technical systems with impacts and dry friction. The model of a mechanical system subjected to unilateral constraints, together with the model of stability, are developed. Examples of analysis are presented for plane triple physical pendulum with rigid limiters motions. The model of the piston-connecting rod-crankshaft system of the combustion engine, as a special case of triple pendulum, is also built and gives results conforming experiments. Then an experimental rig of the triple physical pendulum with the first body periodically forced is built. A mathematical model of the real pendulum is created, where friction in joints is modelled as a composition of dry friction and damping. Good agreement between model and real system is obtained.

1 Introduction

Modern achievements in the development of mathematics, mechanics and associated with them numerical calculation techniques allow for more exact modelling of the real dynamic phenomena that are exhibited by various physical objects. Following historical overview of the natural sciences mentioned, a significant role is played by a physical pendulum, which is the very useful mechanism used in the design of various real processes.

A pendulum as a simple non-linear systems is still a subject of interest of scientists from all the world. It is caused by simplicity of that system on the one hand, and due to many fundamental and spectacular phenomena exhibited by a single pendulum on the other hand. In mechanics and physics investigations of single and coupled pendulums are widely applied. Lately, even the monograph on the pendulum has been published [7]. This is a large study on this simple system also from the historical point of view.

Although a single or a double pendulum (in their different forms) are quite often studied experimentally [8, 9], a triple physical pendulum is rather rarely presented in literature from a point of view of real experimental object. For example, in the work [10] the triple pendulum excited by horizontal harmonic motion of the pendulum frame is presented and a few examples of chaotic attractors are reported. Experimental rigs of any pendulums are still of interest of many researchers dealing with dynamics of continuous multi-degrees-of-freedom mechanical systems. The model having such a properties has been analysed in work [11]. It consists of a chain of N identical pendulums coupled by dumped elastic joints subject to vertical sinusoidal forcing on its base.

On the other hand, it is well known that impact and friction accompany almost all real behaviour, leading to non-smooth dynamics. The non-smooth dynamical systems can be modelled as the so-called piece-wise smooth systems (PWS) and are interesting also from a theoretical point of view, since they can exhibit certain non-classical phenomena of non-linear dynamics. It was a motivation of the first studies of the authors [1, 2, 3] on a triple physical pendulum. In those studies a numerical model of triple physical pendulum with rigid limiters of motion was formulated. Such a system can exhibit impacts as well as sliding solutions, with permanent contact with the obstacle on some time intervals. Special numerical tools for non-linear dynamics analysis of that system exhibiting discontinuities was developed and tested. In addition, application of the numerical model was presented: the piston - connecting rod - crankshaft system of a combustion engine [2].

In February, 2005, in the Department of Automatics and Biomechanics, the experimental rig of triple physical pendulum was finished and activated. This stand has been constructed and built in order to investigate experimentally various phenomena of non-linear dynamics, including regular and chaotic motions, bifurcations, coexisting attractors, etc. It is clear that to have more deep insight into dynamics of the real pendulum, the corresponding mathematical model is required. In references [4, 6] the suitable mathematical modelling and numerical analysis have been performed, where the viscous damping in the pendulum joints (constructed by the use of rolling bearings) has been assumed. In the next step [5], we have also taken into account the dry friction in the joints with many details and variants. Here we present the model of friction taking into account only essential details.

2 Triple pendulum with rigid limiters of motion

In physics and technology, many dynamical systems can be found, which work in the so called different modes, where the system passes from one region to another very quickly. In such cases it is possible to assume that that passing is immediate and discrete. In such a way the system is modelled as a piece-wise-smooth (PWS) dynamical system and can be described by the use of the following equation

$$\dot{x} = f(x), \tag{1}$$

where $x = x(t) \in \mathbb{R}^n$ represents state of the system in the t instance, whilst the map $f : \mathbb{R}^n \to \mathbb{R}^n$ is the piece-wise smooth function, i.e. the phase space $D \in \mathbb{R}^n$ is divided into a finite number of sub-domains V_i on which the function f is smooth, separated by $(n - 1)$ -dimensional hyper-surfaces $\sum_{i,j}$, which are also (at least) piece-wise-smooth.

If the motion of the system goes on inside one of the sub-domains V_i, then the system behaves like a smooth one. When the orbit crosses one of the surfaces $\sum_{i,j}$, the system undergoes a discontinuity. Piece-wise-smooth systems, dependently on the kind of discontinuity, can be classified as follows:

1. Systems with discontinuous Jacobian Df, with continuous but non-smooth vector field $\mathbf{f}$ and smooth state $\mathbf{x}$ ($\mathbf{f}$ is a C^0 class function).

2. Systems with discontinuous vector field $\mathbf{f}$ and continuous but non-smooth state $\mathbf{x}$.

3. Systems with discontinuous state $\mathbf{x}$. In this case, every time when the system undergoes a contact with one of the discontinuity surfaces $\sum_{i,j}$, its state undergoes jump $\mathbf{x}^+ = \mathbf{g}(\mathbf{x}^-)$, where $\mathbf{x}^-$ means state of the system just before the contact with the surface $\sum_{i,j}$, while $\mathbf{x}^+$ is the system state just after that contact, and $\mathbf{g}(x)$ is a piece-wise-smooth (at least) on surfaces $\sum_{i,j}$ function.

PWS systems of the second group (also first group as a subclass) are sometimes called as Filippov systems, for which a special theory is developed. Some electronic systems with elements possessing characteristics, which can be described as discontinuous (diodes, transistor, etc.) are often modelled as PWS systems. Among the mechanical systems we can often meet dynamical systems with dry friction, which can be modeled as systems with discontinuous damping characteristic (second group). Moreover, mechanical system can also posses discontinuous stiffness characteristic. Particularly, mechanical systems with impacts can be modelled as systems with discontinuous stiffness (second group), if we assume flexible bodies model. However, usually we assume rigid body model for impacting systems. In this case the system is a PWS system of the third group, because during the contact of the system with a surface $\sum_{i,j}$ a jumping change of velocity takes place. Here the Newton's restitution rule is applied in calculation of the post-impact velocity. Such a system is actually a system with unilateral constraints described by the use of algebraic inequalities. It should be noted that it is equivalent to impulse of the function $\mathbf{f}$ (Dirac's delta type) on one side of the surface $\sum_{i,j}$. However, as it has been shown many times, model of flexible impacting bodies is convergent to impacting model based on the restitution coefficient, if the stiffness of bodies tends to infinity. Moreover, in some systems, different types of discontinuity can be present simultaneously. As an example we can indicate a mechanical system with impacts and dry friction. Both that phenomena can be independent, or the model of impact can take into account friction phenomena on surfaces of impacting bodies.

When the mathematical model is built, the next problem to be solved is to obtain numerical solution of such defined PWS equations, i.e. simulation of the system. Oberve that it can be done by gluing smooth solutions (between the successive points of non-smoothness) obtained by the use of classical methods and the points of non-smoothness should be detected. In each point of non-smoothness it is possible the necessity of mapping of the system state to the another state according to the function $\mathbf{g}$. It is sufficient in the case of finite number of non-smoothness points in a finite time interval. It is however possible that the solution on some time interval has permanent contact with some surface $\sum_{i,j}$ and then it is necessary to formally reduce number of degrees of freedom or to describe the problem by the use of differential-algebraic equations (DAEs). Before the permanent contact between the solution and the non-smoothness surface $\sum_{i,j}$, the trajectory can cross this surface infinite number of times. In order to solve this problem numerically, we can artificially determine certain shortest time interval between two successive crossings. If the time between two successive crossings becomes shorter, the permanent contact starts.

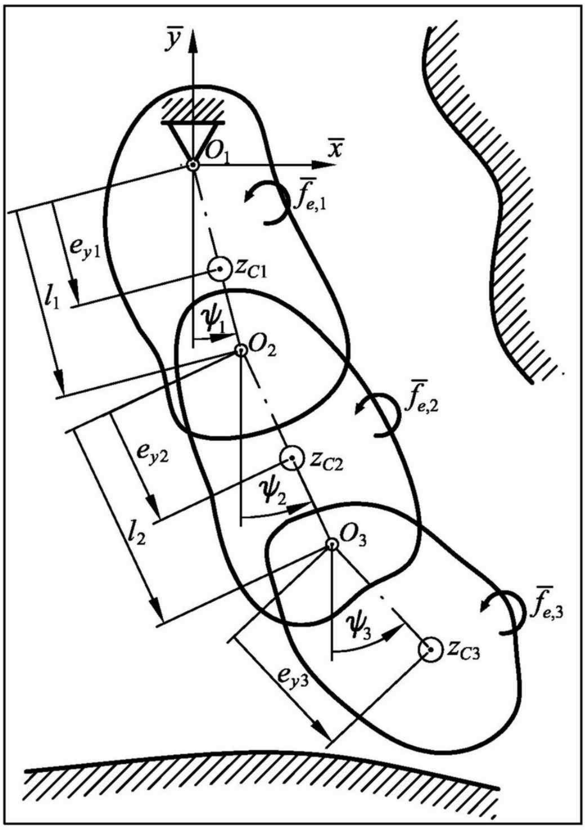

Figure 1: Triple physical pendulum model.

The so far presented approach has been used in modelling and numerical analysis of plane triple physical pendulum with rigid limiters of motion [1, 2, 3], shown in Figure 1. The system has been modelled as the system of the third group. The obstacles are the surfaces of non-smoothness in the state space, and crossing of anyone of them by the orbit is detected and the system state undergoes jump according to $\mathbf{g}$ function. Function $\mathbf{g}$ defines an impact law. Here we have assumed that the obstacle surface is smooth, i.e. there is no dry friction accompanying impact. In other words it is assumed that the impact impulse is perpendicular to the impact surface. This condition and additional equation determining the change of the velocity vector in direction perpendicular to the impact surface according to the so-called restitution coefficient, allow for determining velocities after impact with singular obstacle. In our model also some parts of trajectory can be in permanent contact with obstacles. This is very often preceded by the infinite number of impacts in finite time interval. It has been solved in a way described previously. Permanent contact demands determination of a reaction force generated by an obstacle.

In the next step the authors build the model of the piston - connecting-rod - crankshaft system of the combustion engine as a special case of triple physical pendulum, where the piston in the cylinder barrel moves with backlash and impact an appear [3]. Figure 2 presents simulation results of this system with restitution coefficient equal to 0.5. This is periodic solution and typical behaviour is seen. Namely, the piston six times moves from one side of the cylinder barrel to the other one during one cycle of the engine work.

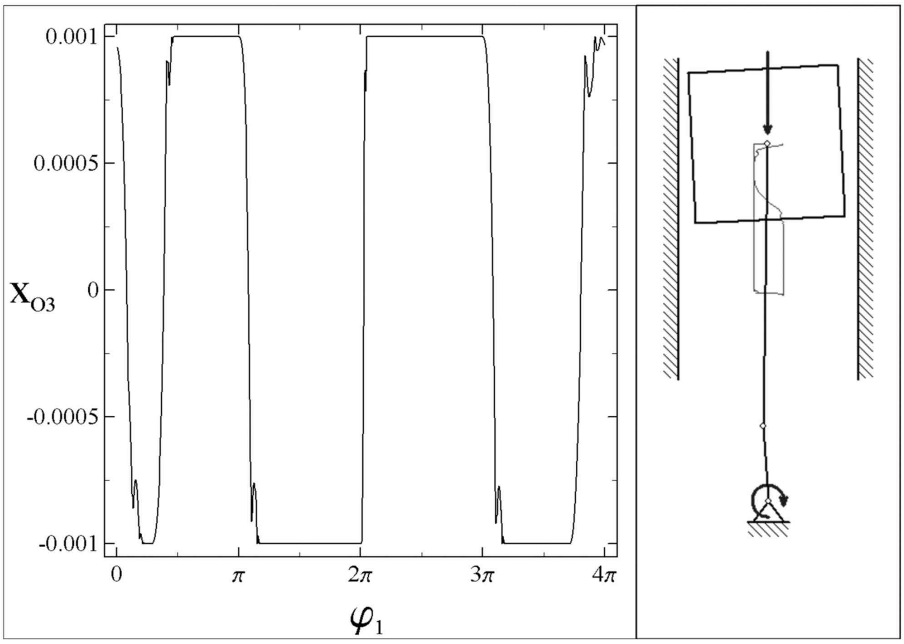

Figure 2: Response of the piston - connecting-rod - crankshaft system for the restitution coefficient $e = 0.5$ (x_{O3} is the pin position coordinate along direction perpendicular to cylinder axis, ψ_1 is angular position of the shaft).

The next step is the orbit stability analysis in such kind of systems. In PWS systems, on each part of solution, where the orbit is smooth, the perturbations also behave smoothly according to typical linearized equation. However, in the non-smoothness points we should transform the perturbation state in a special way. Then we can apply classical methods for Lyapunov exponents calculations, periodic orbit stability analysis and classical bifurcation of periodic orbits analysis. In Figure 3 exemplary periodic, quasi-periodic and chaotic solutions exhibited by triple physical pendulum with horizontal barrier and with the first pendulum harmonically forced, are presented. Let us note that because of some parts of the orbit with the permanent contact with the barrier, two Lyapunov exponents (for each orbit) tend to minus infinity.

In PWS systems all kinds of bifurcations typical for smooth systems can be observed and they can be analysed by the use of classical methods with some special modifications described above. Moreover, in PWS systems we can also find non-classical bifurcation, which can be detected only in non-smooth dynamical systems. An example of such a bifurcation is grazing bifurcation. Note also that analysis of

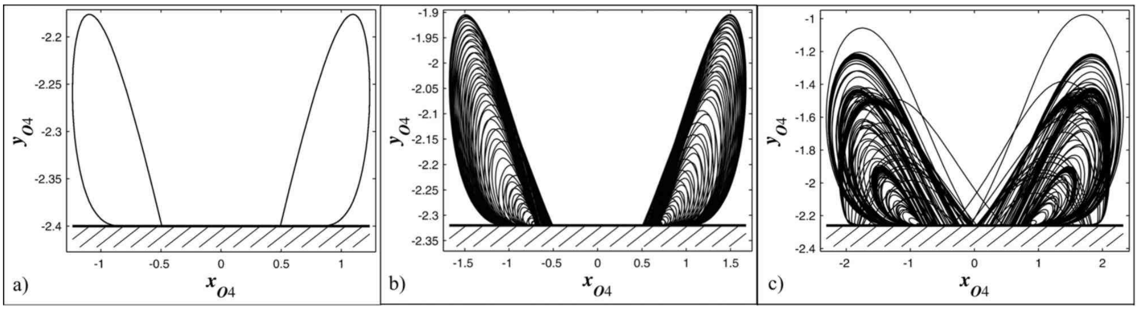

Figure 3: Periodic (a), quasi-periodic (b) and chaotic solution (c) in the triple pendulum system with harmonic excitation and horizontal obstacle (x_{O4} i y_{O4} are position coordinates of the third pendulum end in the Cartesian coordinates); Lyapunov exponents: 0.00, 0, -0.10, -1.05, -2.0, $-\infty$, $-\infty$ (b) 0.01, 0, -0.12, -0.74, -1.92, $-\infty$, $-\infty$ (c); restitution coefficient $e = 0$.

non-classical bifurcations requires special numerical tools.

3 Experimental rig

The experimental rig (see Fig. 4) of the triple physical pendulum consists of the following subsystems: pendulum, driving subsystem and the measurement subsystem. It is assumed that the pendulum is moving in a plane.

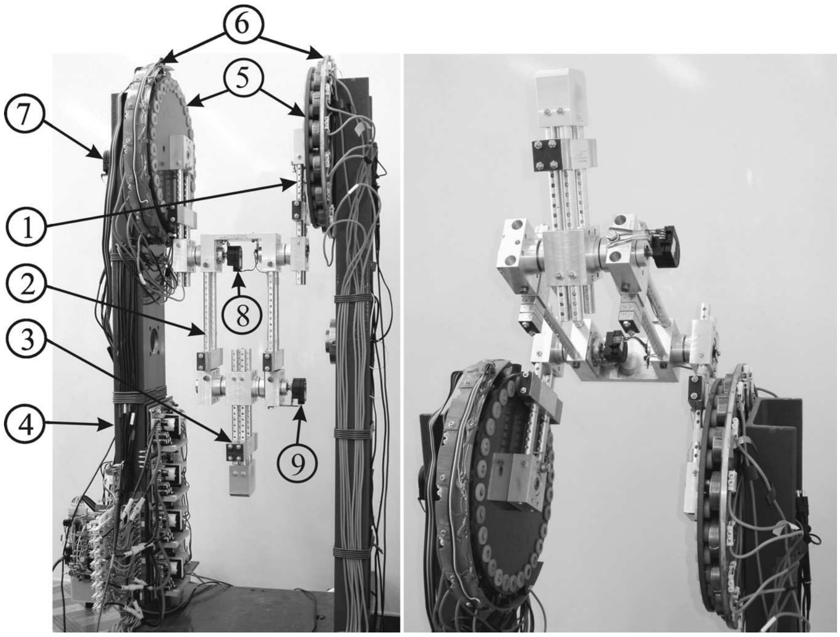

Figure 4: Experimental rig: 1, 2, 3 - links; 4 - stand; 5 - rotors; 6 - stators; 7, 8, 9 - rotational potentiometers.

The links (1, 2, 3) are suspended on the frame (4) and joined by the use of radial and axial needle bearings. The first link is forced by a special direct-current motor of our own construction with optical commutation consisting of two stators (6) and two rotors (5).The construction ensures avoiding the skewing of the structure and forming the forces and moments in planes different that the plane of the assumed pendulum motion. On the other hand the construction allows the full rotations of all the links of the pendulum.

The voltage conveyed to the engine inductors is controlled by the use of special digital system of our own construction together with precise signal generator HAMEG. As a result the square-shape in time forcing (but with some asymmetry - see next sections) with adjustable frequency and desired amplitude is obtained.

The measurement of the angular position of the three links is realized by the use of precise rotational potentiometers (7, 8, 9). Then the LabView measure-programming system is applied for experimental data acquisition and presentation on a computer.

4 Mathematical model

Details on physical modeling, i.e. idealized physical concept of real pendulum presented in Fig. 4, can be found in works [5, 6]. The system is idealized since it is assumed that it is an ideally plane system of coupled links, moving in the vacuum with the assumed model of friction in joints. The system is governed by the following set of differential equations:

$$
\begin{aligned}
&\begin{bmatrix} B_1 & N_{12}\cos(\psi_1-\psi_2) & N_{13}\cos(\psi_1-\psi_3) \\ N_{12}\cos(\psi_1-\psi_2) & B_2 & N_{23}\cos(\psi_2-\psi_3) \\ N_{13}\cos(\psi_1-\psi_3) & N_{23}\cos(\psi_2-\psi_3) & B_3 \end{bmatrix} \begin{Bmatrix} \ddot{\psi}_1 \\ \ddot{\psi}_2 \\ \ddot{\psi}_3 \end{Bmatrix} + \\
&+ \begin{bmatrix} 0 & N_{12}\sin(\psi_1-\psi_2) & N_{13}\sin(\psi_1-\psi_3) \\ -N_{12}\sin(\psi_1-\psi_2) & 0 & N_{23}\sin(\psi_2-\psi_3) \\ -N_{13}\sin(\psi_1-\psi_3) & -N_{23}\sin(\psi_2-\psi_3) & 0 \end{bmatrix} \begin{Bmatrix} \dot{\psi}_1^2 \\ \dot{\psi}_2^2 \\ \dot{\psi}_3^2 \end{Bmatrix} + \\
&+ \begin{Bmatrix} M_{R1}(\dot{\psi}_1) - M_{R2}(\dot{\psi}_1,\dot{\psi}_2) \\ M_{R2}(\dot{\psi}_1,\dot{\psi}_2) - M_{R3}(\dot{\psi}_2,\dot{\psi}_3) \\ M_{R3}(\dot{\psi}_2,\dot{\psi}_3) \end{Bmatrix} + \begin{Bmatrix} M_1\sin\psi_1 \\ M_2\sin\psi_2 \\ M_2\frac{N_{13}}{N_{12}}\sin\psi_3 \end{Bmatrix} = \begin{Bmatrix} M_e(t) \\ 0 \\ 0 \end{Bmatrix},
\end{aligned} \tag{2}
$$

where the pendulum position is governed by three angles ψ_i $(i = 1, 2, 3)$, and where

$$
\begin{aligned}
M_{R1} &= T_1\frac{2}{\pi}\arctan\left(\varepsilon\dot{\psi}_1\right) + 2c\dot{\psi}_1, \\
M_{R2} &= T_2\frac{2}{\pi}\arctan\left(\varepsilon(\dot{\psi}_2-\dot{\psi}_1)\right) + c(\dot{\psi}_2-\dot{\psi}_1), \\
M_{R3} &= T_3\frac{2}{\pi}\arctan\left(\varepsilon(\dot{\psi}_3-\dot{\psi}_2)\right) + c(\dot{\psi}_3-\dot{\psi}_2),
\end{aligned} \tag{3}
$$

are the moments of resistance in the corresponding joints and consisting of two parts: dry friction and viscous damping. The dry friction moment does not depend on the loading of the corresponding bearing and the sign function is approximated by the arctan function. The parameter c is the damping coefficient common for the second and third joint, while in the first joint two times larger damping is taken (since the first joint is built by the use of four bearings, while each other joint contain two bearings).

In reference [6] more complex model of friction has been investigated, where the dry friction moment consists of two part: the first one being proportional to the normal loading in the bearing, and the second one being constant and present also in the lack of loading. Moreover, the friction is a function of relative velocity due to the Stribeck's curve. As a result of those investigations we have concluded that in our case the model of friction can be simplified to the one presented by the Eq. (3), without any loss of precision.

The external excitation in the pendulum model can be an arbitrary time function, and in particular, it can be the same function as applied (and recorded to a file) in real system (it is useful in the parameter estimation process). On the other hand, it is possible to apply a forcing due to the following formula:

$$M_e(t) = \begin{cases} q & \text{if } \omega t + \phi_0 \mod 2\pi \leq 2\pi a \\ -q & \text{if } \omega t + \phi_0 \mod 2\pi > 2\pi a \end{cases}, \tag{4}$$

which imitates the square-shape in time forcing (applied in the real pendulum), with adjustable angular velocity ω, initial phase ϕ_0, amplitude q and the coefficient a (for $a \neq 0.5$ there is an asymmetry in the forcing).

5 Model parameters

The model parameters are estimated by the global minimum searching of the criterion-function of the model and real system matching. The matching of model and real system is understood as the matching of the corresponding output signals $\psi_i(t)$ (i =1,2,3) from model integrated numerically and from the real pendulum, assuming the same inputs to both model and real systems. The sum of squares of deviations between corresponding samples of signals from the model and the experiment, taken for few different solutions serves as a criterion function. Together with the model parameters also initial conditions of the numerical simulation are estimated. A minimum is searched applying the simplex method. In order to avoid the local minima, the simplex method is stopped from time to time and a random searching is then applied. After random searching the simplex method is restarted again.

If we divide final value of criterion-function by the number of samples used in calculation of criterion-function, we obtain average square of deviation between two signals (obtained from the model and the experiment). In what follows we denote this parameter by F_{cr}. Now this parameter can be used for comparison of matching of different sets of experimental data and corresponding numerical solutions.

In Table 1 the part of the results of the parameter estimations performed in work [6] is presented. Three different sets of parameters, correspondingly to three variants of the model of resistance in the joints are presented. The set A_1 corresponds to the model with dry friction only. The model B_1 contains also viscous damping. The next model (C_1) is a development of the previous one (B_1): the parameter ε is added to the set of the identified parameters.

Table 1: Parameter estimations.

	A_1	B_1	C_1
B_1 [kg·cm^2]	1650.3	1634.7	1641.3
B_2 [kg·cm^2]	1386.3	1378.7	1390.9
B_3 [kg·cm^2]	163.32	166.56	164.50
N_{12} [kg·cm^2]	1111.2	1104.5	1112.6
N_{12} [kg·cm^2]	198.99	201.47	199.92
N_{23} [kg·cm^2]	255.96	259.16	257.15
M_1 [N·cm]	879.76	874.38	875.00
M_2 [N·cm]	632.37	628.53	633.13
T_1 [N·mm]	56.83	72.73	97.53
T_2 [N·mm]	25.06	15.16	13.77
T_3 [N·mm]	11.07	4.58	6.61
c [N·mm·s]	*0*	1.057	0.532
ε [s]	*1000*	*1000*	6.77
$10^3 \cdot F_{cr}$ [rad^2]	0.3255	0.3059	0.2809

In all the identification processes, the same set of experimental solutions is used: five periodic solutions with the forcing frequencies ($f = \omega/2\pi$): $f = 0.2$, 0.35, 0.6, 0.85 and 1.1 Hz (for each the solution the 20 sec of motion was recorded, after ignoring the transient motions) and one decaying solution, which starts from the periodic attractor with forcing frequency $f = 0.5$ Hz (after few seconds of the recorded motion, the forcing was switched off and the total length of the recorded motion was 24 sec). Note that in our work, we do not measure actual value of the forcing, but only the control signal is recorded (determining the sign of the forcing), since we assume the constant forcing amplitude $q = 1.718$ Nm (determined before the identification experiments).

6 Simulation results

In the upper part of Fig. 5 the final model C_1 and real system matching for vanishing motion (started from the periodic attractor with the forcing frequency $f = 0.5$ Hz), obtained during the identification process, is presented. In this scale we see almost perfect matching of the corresponding behaviours. The bottom part of Fig. 5 presents enlargement of the final phase of decaying of the same motion, where in addition the simulation of the model A_1 is shown. Here we can observe in details certain aspect of the difference between models A_1 and C_1.

Figure 6 show results of investigation of the forcing frequency region 0.13-0.14 Hz. It is an example that the developed model with its parameters can predict real pendulum dynamics exhibited also for forcing frequencies f outside the region 0.2-1.1 Hz (containing all the periodic solutions studied during the identification process).

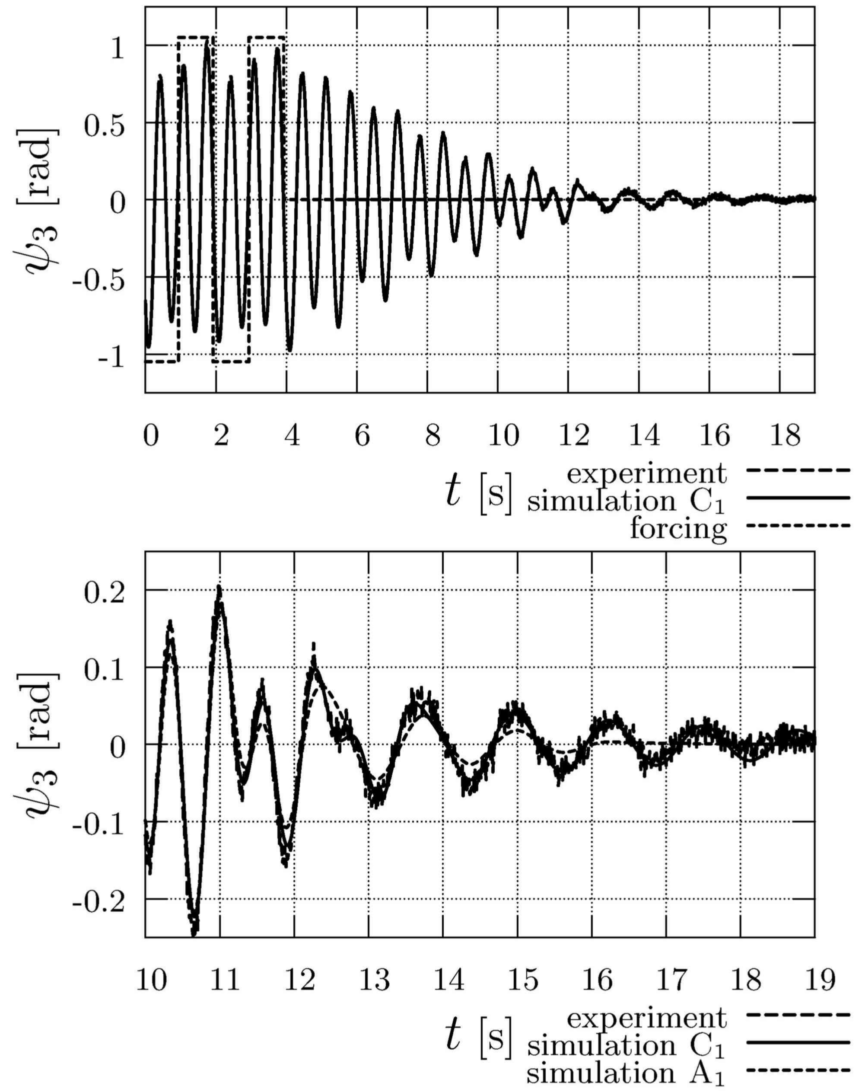

Figure 5: Final model (C_1 and A_1) and real system matching for vanishing motion

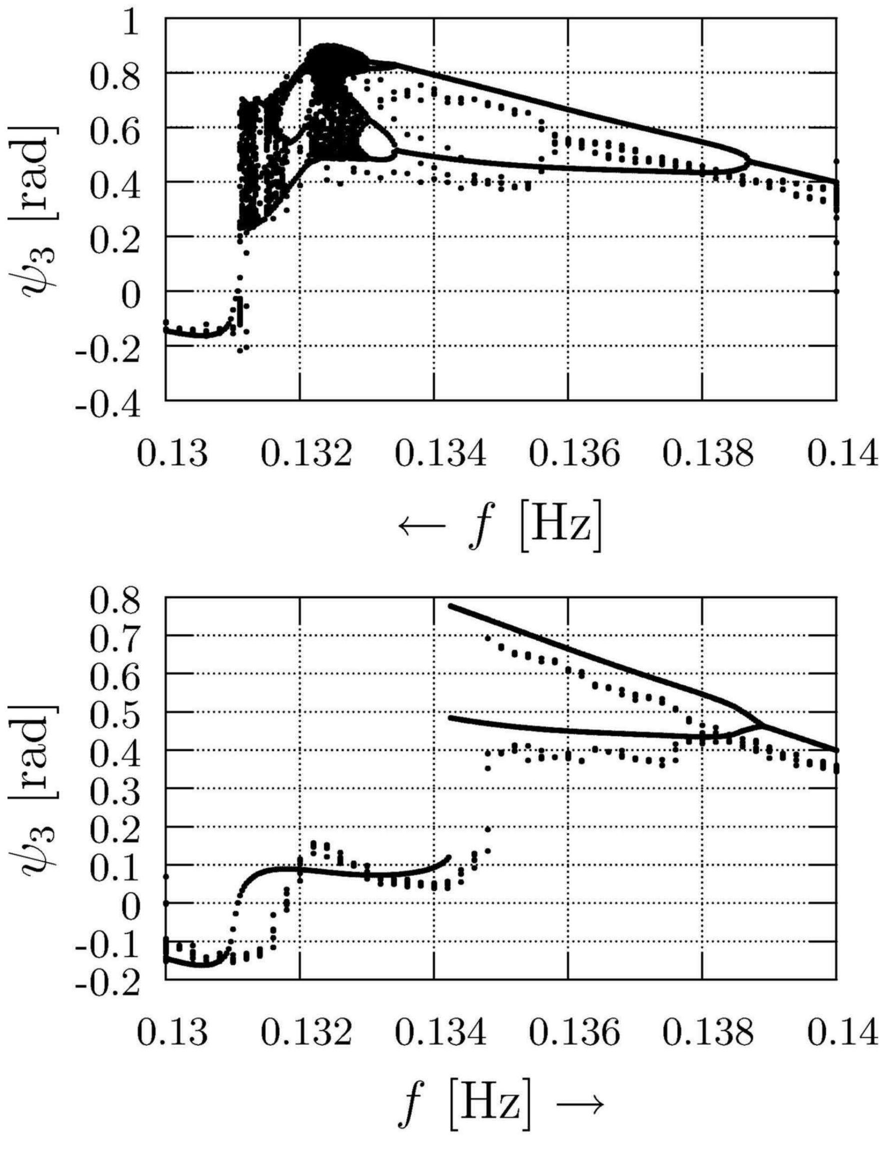

Figure 6: Bifurcation diagrams exhibited by experiment and model (C_1) with the parameter f growing (→) and decreasing (←).

7 Concluding remarks

IThe Aizerman-Gantmakher theory, handling with perturbed solution in points of discontinuity, is used to extend classical method for computing Lyapunov exponents for the multi-degree of freedom mechanical system with rigid barriers imposed on its position. Some examples of identification of attractors in the system of triple pendulum with horizontal barrier are presented, including periodic, quasi-periodic and chaotic attractors with impacts as well as attractors with some segments of trajectory lying on the surface of the obstacle, where the obstacle is permanently active in a certain time interval and the two Lyapunov exponents are degenerated, i.e. their values reach infinity.

Few versions of the model of resistance in the joints have been tested in the identification process. Good agreement between both numerical simulation results and experimental measurements have been obtained and presented for all of the friction model variants. However, one of them (C_1) seems to be optimal, since it gives relatively good results with simultaneous simplicity of the model itself keeping simultaneously high simulation speed.

The model C_1 is better for simulation (higher simulation speed) than others because the ε parameter is much smaller and the characteristic of the friction torque is smooth. Observe that model C_1 gives better results than model B_1, while only the parameter ε is modified one (the result is the smaller value of the parameter ε).

I should be noted, that examples of numerical and experimental simulations presented is section 6 are selective. However, the presented examples show quite good agreement between numerical and experimental results. It leads to conclusion that the used mathematical model of triple pendulum with its parameters estimated can be applied as a tool for quick searching for various phenomena of non-linear dynamics exhibited by a real pendulum as well as for explanation of its rich dynamics.

Acknowledgements

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