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INVESTIGATION OF VIBRATIONS OF AUTOPARAMETRICAL SYSTEMS

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Abstract: The paper concerns analysis of nonlinear vibration of smooth autoparametrical autonomous and non-autonomous systems of two-degree-of-freedom. A multiple scales method of investigation of small vibrations is applied to the analysis of resonance. The obtained results have been confirmed numerically. Analytical calculation was made with the use of *Mathematica*.

1. Introduction

The analysis of vibrations of the weak nonlinear smooth auto-parametrical autonomous and non-autonomous two degree of freedom systems was made. The pendulum of changing length is an example of such a system.

Dynamical systems including mathematical or physical pendulum play significant role in technology. In such systems one can observe an auto-parametric resonance phenomena, because of the coupling occurring in the equation of motion. The phenomenon of energy transfer from one of the mode of vibration to the other was widely discussed in [3].

Dynamical analysis of nonlinear vibrations of spring pendulum (Fig. 1) was presented in the paper.

The main goal of the analysis is recognition of primary and parametric resonances using the multiple scales method. The program in the computer algebra system *Mathematica* was elaborated. It enables automatizing many transformations in the perturbation method use for this purpose [6].

The phenomenon of energy transferring between the modes of vibrations occurs in the vicinity of the parametric resonance. The multiple scale method makes possible to recognize parameters of the system that are dangerous due to the resonance and allows to make a time history for the assumed generalized co-ordinates.

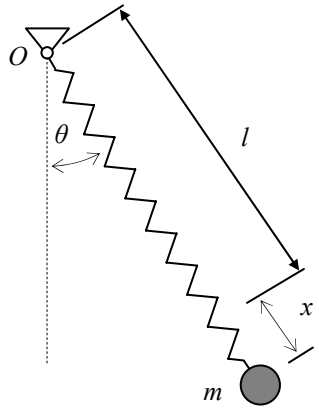

Fig. 1. Spring pendulum.

There are many technical applications of the systems containing various types of pendulums. Discussion on such models with a view to damping vibrations may be found in [4,5].

All the calculations presented in the paper, both analytical and numerical transformations, were made with the help of computer algebra system Mathematica.

2. Spring pendulum

The kinetic and potential energy of the examined system has the form:

$$T = \frac{1}{2} m\left[\dot{x}^2 + (l+x)^2 \dot{\theta}^2\right] \tag{1}$$

$$V = mg(l+x)(1-\cos\theta) + \frac{1}{2}kx^2 \tag{2}$$

where m – mass of the pendulum, l – the length, k – stiffness of the spring, g Earth's acceleration and x and θ – generalized co-ordinates admitted according to Fig.1. Let us introduce dimensionless generalized co-ordinates (with tilde) $\frac{x}{l} = \varepsilon\widetilde{x}$, $\theta = \varepsilon\widetilde{\theta}$ and dimensionless time $t\omega_2 = \tau$.

Expansion of trigonometric functions

$$\sin\theta \cong \theta - \frac{1}{3!}\theta^3 \text{ and } \cos\theta \cong 1 - \frac{1}{2}\theta^2,$$

provides the equations of motion in the dimensionless form

$$\ddot{\tilde{x}} + \Omega^2 \tilde{x} = \varepsilon\left(\dot{\tilde{\theta}}^2 - \frac{1}{2}\tilde{\theta}^2\right)$$
$$\ddot{\tilde{\theta}} + \tilde{\theta} = \varepsilon\left(\tilde{x}\,\tilde{\theta} - 2\dot{\tilde{x}}\,\dot{\tilde{\theta}}\right) \tag{3}$$

where

$$\Omega \equiv \frac{\omega_1}{\omega_2},\ \omega_1^2 \equiv \frac{k}{m},\ \omega_2^2 \equiv \frac{g}{l},$$

$\varepsilon << 1$, is a so-called small parameter.

In order to simplify the notation, the sign ~ (tilde) will be omitted below.

The solution of (3) is sought in the following form

$$x(t;\varepsilon) \approx x_0(T_0, T_1) + \varepsilon\, x_1(T_0, T_1) + O(\varepsilon^2)$$
$$\theta(t;\varepsilon) \approx \theta_0(T_0, T_1) + \varepsilon\, \theta_1(T_0, T_1) + O(\varepsilon^2) \tag{4}$$

Where $T_0 = t$, $T_1 = \varepsilon t$ is a scale of time of slow changing processes.

The original set of equations will be transformed to the set of partial differential equations because of differentiating of the compound functions according to:

$$\frac{d}{dt} = \frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1}$$
$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial T_0^2} + 2\varepsilon \frac{\partial^2}{\partial T_0\, \partial T_1} \tag{5}$$

Substitution of (4) and (5) to (3) leads to the equations:

(order ε^0)

$$\frac{\partial^2 x_0}{\partial T_0^2} + \Omega^2 x_0 = 0$$
$$\frac{\partial^2 \theta_0}{\partial T_0^2} + \theta_0 = 0 \tag{6}$$

(order ε^1)

$$\frac{\partial^2 x_1}{\partial T_0^2} + \Omega^2 x_1 + 2\frac{\partial^2 x_0}{\partial T_0\, \partial T_1} + \frac{1}{2}\theta_0^2 - \left(\frac{\partial \theta_0}{\partial T_0}\right)^2 = 0$$
$$\frac{\partial^2 \theta_1}{\partial T_0^2} + \theta_1 + 2\frac{\partial^2 \theta_0}{\partial T_0\, \partial T_1} - x_0\theta_0 + 2\frac{\partial x_0}{\partial T_0}\frac{\partial \theta_0}{\partial T_0} = 0 \tag{7}$$

The solution of (6) may be found as follows

$$x_0 = A(T_1)\,e^{i\Omega T_0} + \overline{A}(T_1)\,e^{-i\Omega T_0}$$
$$\theta_0 = B(T_1)\,e^{iT_0} + \overline{B}(T_1)\,e^{-iT_0}\,. \tag{8}$$

Substitution of (8) to (7) gives the first order equation in the form

$$\frac{\partial^2 x_1}{\partial T_0^2} + \Omega^2 x_1 + \frac{3}{2}e^{2iT_0}B^2 - B\overline{B} + \frac{3}{2}e^{-2iT_0}\overline{B}^2 + 2ie^{iT_0\Omega}\Omega\frac{dA}{dT_1} - 2ie^{-iT_0\Omega}\Omega\frac{d\overline{A}}{dT_1} = 0$$
$$\frac{\partial^2 \theta_1}{\partial T_0^2} + \theta_1 + e^{iT_0(1+\Omega)}\left(-AB - 2\Omega AB\right) + e^{iT_0(1-\Omega)}\left(-\overline{A}B - 2\Omega\overline{A}B\right) + e^{iT_0(-1+\Omega)}\left(-A\overline{B} + 2\Omega A\overline{B}\right) + \tag{9}$$
$$e^{iT_0(-1-\Omega)}\left(-\overline{A}\,\overline{B} - 2\Omega\overline{A}\,\overline{B}\right) + 2ie^{iT_0}\frac{dB}{dT_1} - 2ie^{-iT_0}\frac{d\overline{B}}{dT_1} = 0$$

In the above equations the arguments of A and B are omitted in order to shorten the notation.

Removal of secular terms in equations (9), requires

$$\frac{dA(T_1)}{dT_1} = 0 \quad \text{and} \quad \frac{dB(T_1)}{dT_1} = 0\,. \tag{10}$$

The solution of (9) is then

$$x_1 = -\frac{3}{2}\frac{B^2}{\Omega^2 - 4}e^{2iT_0} + \frac{1}{2\Omega^2}B\overline{B} - \frac{3}{2}\frac{\overline{B}^2}{\Omega^2 - 4}e^{-2iT_0}$$
$$\theta_1 = -\frac{-2 - 3\Omega + 2\Omega^2}{\Omega\left(-4 + \Omega^2\right)}ABe^{i(\Omega+1)T_0} - \frac{2 - 3\Omega - 2\Omega^2}{\Omega\left(-4 + \Omega^2\right)}\overline{A}Be^{i(-\Omega+1)T_0} + \tag{11}$$
$$-\frac{2 - 3\Omega - 2\Omega^2}{\Omega\left(-4 + \Omega^2\right)}A\overline{B}e^{-i(1-\Omega)T_0} - \frac{-2 - 3\Omega + 2\Omega^2}{\Omega\left(-4 + \Omega^2\right)}\overline{A}\,\overline{B}e^{-i(\Omega+1)T_0}$$

The above solution becomes singular when the internal resonance occurs i.e. when $\Omega = 2$. In order to treat this resonance case, we can introduce the new parameter σ according to:

$$\varepsilon\sigma = \Omega - 2 \tag{12}$$

The equations (9) can be written now as follows:

$$\frac{\partial^2 x_1}{\partial t_0^2} + \Omega^2 x_1 + \left(\frac{3}{2}e^{-i\sigma T1}B^2 + 2i\Omega\frac{dA}{dT_1}\right)e^{iT_0\Omega} - \left(\frac{3}{2}e^{i\,\sigma T_1}\overline{B}^2 + 2i\Omega\frac{d\overline{A}}{dT_1}\right)e^{-iT_0\Omega} - B\,\overline{B} = 0$$
$$\frac{\partial^2 \theta_1}{\partial t_0^2} + \theta_1 + \left(e^{iT_1\sigma}\left(-A\,\overline{B} + 2\,\Omega\,A\,\overline{B}\right) + 2i\frac{dB}{dT_1}\right)e^{iT0} + \left(e^{iT_1\sigma}\left(-\overline{A}\,B - 2\,\Omega\,\overline{A}\,B\right) - 2i\frac{d\overline{B}}{dT1}\right)e^{-iT_0} \tag{13}$$
$$e^{iT_0(-1-\Omega)}\left(-\overline{A}\,\overline{B} - 2\Omega\,\overline{A}\,\overline{B}\right) + e^{iT_0(1+\Omega)}\left(-A\,B - 2\,\Omega\,A\,B\right) = 0$$

Removal of secular terms from (13) requires:

$$\frac{3}{2}e^{-i\sigma T_1}B^2 + 2i\Omega\frac{dA}{dT_1} = 0$$
$$e^{iT_1\sigma}\left(-A\overline{B} + 2\Omega A\overline{B}\right) + 2i\frac{dB}{dT_1} = 0 \tag{14}$$

In order to present the solution of (14) in more familiar form:

$$x = a\cos(\Omega t + \alpha)$$
$$\theta = b\cos(t + \beta) \quad , \tag{15}$$

the following substitution can be made [2]:

$$A \to \frac{a}{2} e^{i\alpha}, \qquad B \to \frac{b}{2} e^{i\beta} \tag{16}$$

The introduced amplitudes a and b and phases α and β are functions of T_1.

With the use of the above substitution (16), the equations (14) lead to expressions of derivatives of the sought functions looked for:

$$\begin{aligned}
\frac{da}{dT_1} &= \frac{3}{8\Omega} b^2 \sin(\gamma) \\
\frac{d\alpha}{dT_1} &= \frac{3}{8\Omega} \frac{b^2}{a} \cos(\gamma) \\
\frac{db}{dT_1} &= \frac{2\Omega - 1}{4} ab \sin(\gamma) \\
\frac{d\beta}{dT_1} &= \frac{2\Omega - 1}{4} a \cos(\gamma)
\end{aligned} \tag{17}$$

where $\gamma = T_1 \sigma + \alpha(T_1) - 2\beta(T_1)$.

Solving the above equations lets to obtain the information about modulations of amplitudes $a(T_1)$ and $b(T_1)$, and phases $\alpha(T_1)$ and $\beta(T_1)$ respectively.

3. Examples

3.1 *Autonomous vibrations in the vicinity of the internal resonance*

The set (17) was solved numerically by Runge-Kutta method. The modulations of amplitudes and phases are shown in the Figs. 2-5. The admitted parameters: $\Omega = 2, \varepsilon = 0.1$ and initial conditions $a(0) = 0.1, b(0) = 0.1, \alpha(0) = 0, \beta(0) = 0$.

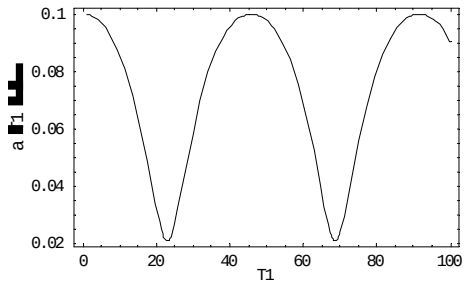

Fig. 2. Amplitude modulation of elastic vibrations (x)

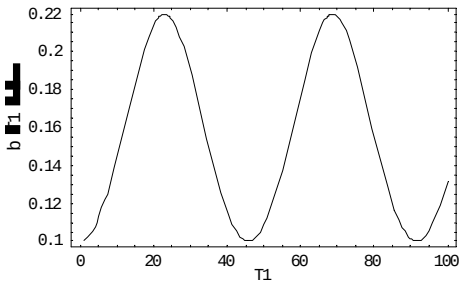

Fig. 3. Amplitude modulation of swing vibrations (θ)

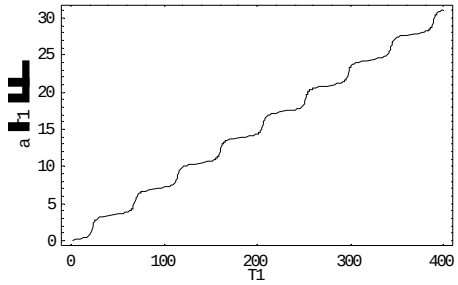

Fig. 4. Phase modulation of elastic vibrations (x)

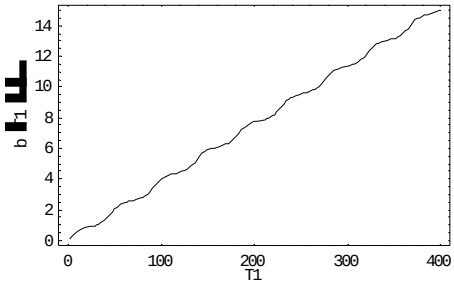

Fig. 5. Phase modulation of swing vibrations (θ)

The slow-changing process shown in the Figs. above, clearly indicates the energy exchange between the modes of vibrations. The time history for both co-ordinates with the same parameters and initial conditions as before is presented in Fig. 6.

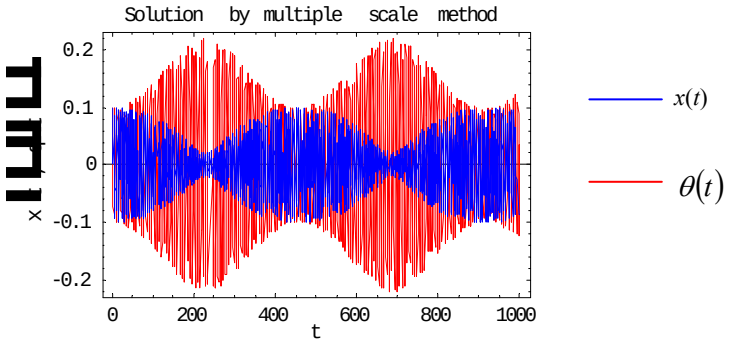

Fig. 6. Deflections of the pendulum as a function of time, $\Omega = 2, \varepsilon = 0.1$.

The perturbative solution was confirmed numerically by Runge-Kutta method. The time history on Fig. 6 confirms the auto-parametric resonance phenomenon and transfer of energy within the system connected with it. Analytical results obtained by the multi-scale perturbation method were confirmed numerically . Correctness of the numerical solution was controlled by the constant of total non-dimensional energy

$$E = \frac{1}{2}\left[\dot{x}^2 + (1+x)^2\dot{\theta}^2\right] + (1+x)(1-\cos\theta) + \frac{1}{2}\Omega^2 x^2 \tag{18}$$

3.2 *Forced vibrations in the vicinity of the internal resonance*

The weak periodic force acting along the pendulum was introduced. The equations of motion are now:

$$\ddot{x}+\Omega^2 x=\varepsilon\left(\dot{\theta}^2-\frac{1}{2}\theta^2\right)+\varepsilon f\cos(\omega t)$$
$$\ddot{\theta}+\theta=\varepsilon\left(x\theta-2\dot{x}\dot{\theta}\right) \tag{19}$$

f is the dimensionless amplitude and ω frequency of the force.

The method described above was applied here in the same manner. After removal of the secular terms we can obtain the solution of the first order in the form

$$x_1=f\frac{e^{iT0\omega}+e^{-iT_0\omega}}{2\left(\omega^2-\Omega^2\right)}+\frac{3}{2}\frac{B^2}{4-\Omega^2}e^{2iT_0}+\frac{1}{\Omega^2}B\overline{B}+\frac{3}{2}\frac{\overline{B}^2}{4-\Omega^2}e^{-2iT_0}$$
$$\theta_1=\frac{1+2\Omega}{\Omega(2+\Omega)}ABe^{i(\Omega+1)T_0}+\frac{-1+2\Omega}{\Omega(-2+\Omega)}\overline{A}Be^{i(-\Omega+1)T_0}+ \tag{20}$$
$$+\frac{-1+2\Omega}{\Omega(-2+\Omega)}A\overline{B}e^{-i(1-\Omega)T_0}-\frac{1+2\Omega}{\Omega(2+\Omega)}\overline{A}\overline{B}e^{-i(\Omega+1)T_0}$$

The above solutions suggest that the resonances are possible

- the primary for $\omega=\Omega$

- the internal for $\Omega=2$.

The time history for chosen parameters are presented in Figs. 7-10. The initial conditions were admitted as before $\theta(0)=0.1, \frac{\partial\theta}{\partial t}(0)=0, x(0)=0.1, \frac{\partial x}{\partial t}(0)=0$ and $f=1$.

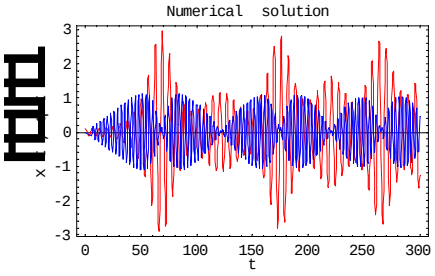

Fig. 7. Vibrations in resonance, $\Omega=\omega=2$, primary and internal resonance

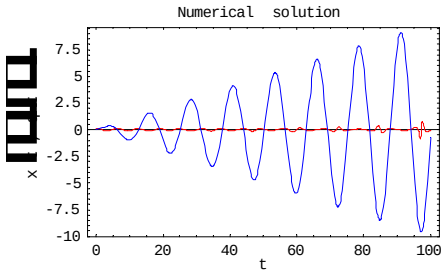

Fig. 8. Vibrations in resonance, $\Omega=\omega=0.5$, primary resonance

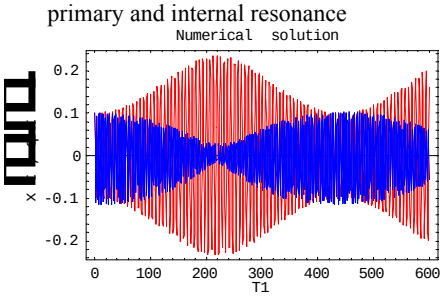

Fig. 9. Vibrations in resonance, $\Omega=2,\ \omega=4$, subharmonical resonance

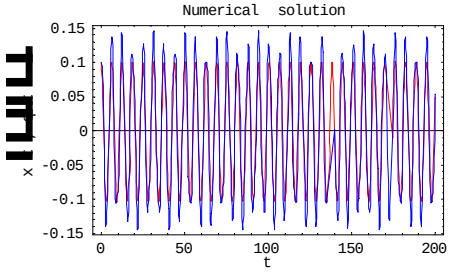

Fig. 10. Vibrations beyond resonance, $\Omega=1,\ \omega=2.3$

4. Conclusions

The transformations within the multiple-scale-method were carried out automatically with the use of a procedure written in *Mathematica.*

The results show that both quantitative and qualitative analyses of nonlinear dynamical systems can be made by the multi-scale-method in time domain. The method allow to recognize the parameters of the system with respect to the occurring resonance.

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