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Some bifurcations of a floating sliding region

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Abstract: The work focuses on the numerical analysis of bifurcations of a stick-slip solutions (termed sliding-standard solution as well) that result from a complex dynamics of a piece-wise smooth dynamical system represented by a block-on-belt model. The analysis of sliding solutions and conditions of their appearance has been done with the use of Filippov theory. Since a floating discontinuity boundary is introduced as a function of velocity of the belt, the investigated system can be divided into the two subsystems, which are separately and continuously solved in that two appropriate regions. To get an exact transition between solutions being defined in that regions, a kind of numerical procedure with the exact crossing set detection (with inclusion of detection of points lying precisely on crossing surface) has been introduced and implemented as well.

1. Introduction

Theory of bifurcations is presently not a new but the well known and carefully inspected branch of dynamics being still under deep considerations. Its the most particular problem is focused on a qualitative change in the nature of behaviour whilst a parameter value passes through any critical points. Fundamental scheme or process during which the selected analysed solution exposes double periodicity has been investigated in various ways. For instance, the nonsmooth aspects of either numerical or purely mathematical approaches can be matched by works [3,5]. Following the study of a buck converter exhibited by a nonlinear system, the period doubling (PD) bifurcation (diagnosed during periodically moving switching borders) leads to a dramatic loss of stability of the nominal operation state. The effect is there strongly undesirable so that, in addition, it reflects in a PD route to chaos. Making a little generalised simplification, the troublesome scenario could be automatically detected at initial stage just by an appearance of PD-bifurcation which is, at that time, significantly affecting the quality of circuits operation. The second cited work introduces a

PD route that is observed in a kind of dc-dc power converter being operated in the regime of the continuous conduction mode. The bifurcation's path of solutions is here processed by means of harmonic balance analysis but an exact condition for the PDs occurrence has been given in terms of solvability of a pair of algebraic equations.

The description of any miscellaneous bifurcations of states appearing in electronical circuits is now successively extended on the example of another dc-dc power converter [2]. The work gives an interesting classification of main forms of bifurcations possible to observe in piecewise-smooth (PWS) dynamical systems giving, at the end, an illustrative experimental application. The results cover with those numerically predicted and besides to the classical PD bifurcations other periodic solutions undergo more complex C-bifurcations. According to the systematization given, the attempt of consolidation of bifurcation theory for PWS systems constitutes one of the recent trends in that challenging subject.

In order to demonstrate a beneficial use or even any detrimental effects of bifurcation phenomena many interesting techniques are used. One of them, known as a stroboscopic view is widely and successfully used in investigations of stability of the switching electronic circuits to obtain PWS maps of a desired dimension. There is a particular analysis of various border collision (BC) bifurcations presented in [1], which is subject to the theoretical background provided. Including some applications that have been adopted in a practical way, the theory has became the proper tool for explanation of bifurcational schemes in PWS 2-dimensional maps.

A quite different type of switchings is also possible in any discontinuous mechanical systems. They are there created on the principle of the successive sequences of slippings and stickings of a physically specified vibrating masses. Let the paper [4] be a certain example of the non-smooth dynamics and the block-on-belt model with two degrees-of-freedom will reflect its mechanical realisation.

2. Theoretical background

Our analysis is based on a generic Filippov system of the form

$$\dot{x} = \begin{cases} f^{(1)}(x), & x \in S_1, \\ f^{(2)}(x), & x \in S_2, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, and

$$
\begin{aligned}
S_1 &= \{x \in \mathbb{R}^n : \quad H(x) < 0\}, \\
S_2 &= \{x \in \mathbb{R}^n : \quad H(x) > 0\}.
\end{aligned} \tag{2}
$$

H is a smooth scalar function with existing gradient $H_x(x) = \frac{\partial H(x)}{\partial x}$ on the discontinuity boundary

$$
\Sigma = \{x \in \mathbb{R}^n : \quad H(x) = 0\}, \tag{3}
$$

and $f^{(i)} : \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, 2$, are smooth functions.

By concatenating sliding solutions on Σ and the standard solutions in S_i, $i = 1, 2$, it is possible to find the desired general solutions of Eq. (1). In particular, the sliding solutions can be obtained with the briefly described Filippov convex method (see Fig. 1).

Let

$$
\sigma(x) = \langle H_x(x), f^{(1)}(x)\rangle \langle H_x(x), f^{(2)}(x)\rangle, \tag{4}
$$

be the definition of switch control function in which $\langle\cdot,\cdot\rangle$ denotes the standard scalar product in $\mathbb{R}^n$. The crossing set $\Sigma_c \subset \Sigma$ is defined by

$$
\Sigma_c = \{x \in \Sigma : \quad \sigma(x) > 0\}. \tag{5}
$$

The sliding set Σ_s is the complement to Σ_c in Σ, what allows to formulate the second condition on Σ:

$$
\Sigma_s = \{x \in \Sigma : \quad \sigma(x) \leq 0\}. \tag{6}
$$

The orbit of Eq. (1), in general, crosses Σ at points $x \in \Sigma_c$, while it slides on it when points $x \in \Sigma_s$.

The Filippov method is based on the convex combination $g(x)$ of the two vectors $f^{(i)}$ applied to each non-singular sliding point $x \in \Sigma_s$:

$$
g(x) = \lambda f^{(1)}(x) + (1 - \lambda) f^{(2)}(x), \tag{7}
$$

where

$$
\lambda = \frac{\langle H_x(x), f^{(2)}(x)\rangle}{\langle H_x(x), f^{(2)}(x) - f^{(1)}(x)\rangle}. \tag{8}
$$

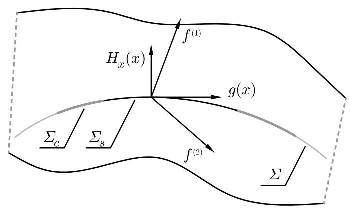

Figure 1. Graphical representation of the convex method.

Non-singularity of the sliding points will be assured if the denominator of λ will not be equal to zero.

The set of sliding points can be find through the solution of the smooth dynamical system of differential equations given in a $(n-1)$-dimensional domain of Σ_s:

$$\dot{x} = g(x), \quad x \in \Sigma_s, \tag{9}$$

If one of the vectors $f^{(i)}$ vanishes then the equilibrium of Eq. (7) is called boundary equilibrium. Otherwise, the boundary of a sliding region is a concatenation of sliding points, boundary equilibria and tangent points in which one of the vectors $f^{(i)}$ is tangent to Σ but the second one is nonzero. Therefore, the following definition of tangent points holds:

$$\langle H_x(x_T), f^{(i)} \rangle = 0. \tag{10}$$

In the point of view of a transient trajectory the visibility properties of the tangent points are very interesting. In particular, tangent points are visible if the orbit of $\dot{x} = f^{(1)}(x)$ starting from them belongs to S_1 for small $|t| \neq 0$. In other hand, the same points are invisible if the mentioned orbit belongs to S_2 (see Fig. 2).

The object of the work is a two degrees-of-freedom mechanical system consisting of a mass m moving on a belt and connected to a spring with nonlinear characteristics $k(x) = -k_1x + k_2x^3$ (see Fig. 3). The block-on-belt model consists of the mass m attached to inertial space by the spring k. Mass m vibrates on a driving belt that is moving at the velocity α. Between the mass and the belt dry friction occurs, of which friction force characteristics depends on velocity of the belt α and on the control parameter β, that have a direct influence on a shape of the static friction characteristics. Equations of motion of the considered dynamical system are given in non-dimensional form. In the first assumption, vibrating

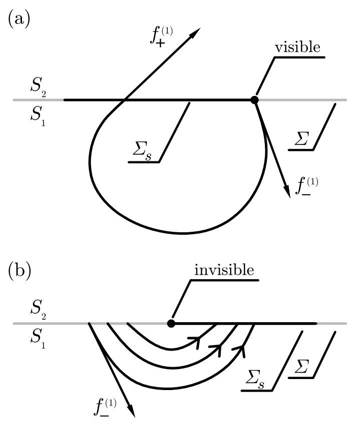

Figure 2. Visible (a) and invisible (b) tangent points.

mass m and its connection with driving belt provide the first two equations of our system. The next two used in next section, will describe small dynamic changes in parameters α and β. The model depicted in Fig. 3 can be described by the set of two non-dimensional

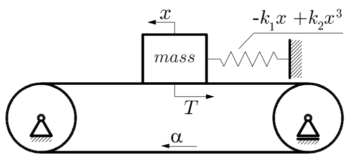

Figure 3. A 2-DOF mechanical system.

equations:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_1 - x_1^3 - T, \end{cases} \tag{11}$$

where T is a friction force (see Fig. 4) expressed by the following characteristics

$$T = \frac{\beta \operatorname{sgn}(x_2 - \alpha)}{1 + |x_2 - \alpha|}. \tag{12}$$

Transforming Eq. (11) to the form given in Eq. (1) we get the following generic planar

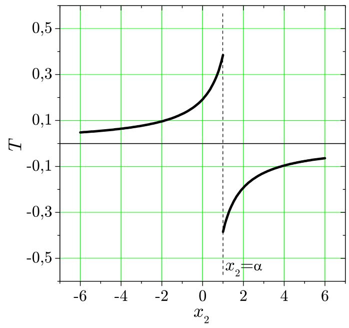

Figure 4. Friction force model.

Filippov system

$$\dot{x} = f(x) = \begin{bmatrix} x_2, \; -x_1^3 + x_1 + \dfrac{\beta}{\alpha - x_2 + 1} \\ x_2, \; -x_1^3 + x_1 + \dfrac{\beta}{\alpha - x_2 - 1} \end{bmatrix}, \tag{13}$$

where $x = [x_1, \, x_2]^T$ and $f = \left[f^{(1)}, \, f^{(2)}\right]^T$.

The discontinuity boundary Σ separating the two regions S_1 and S_2 is in this case defined in the following way

$$\Sigma = \{x \in \mathbb{R}^2 : \, H(x) = x_2 - \alpha = 0\}, \tag{14}$$

with $H_x(x) = [0, \, 1]^T$, and

$$\begin{aligned} S_1 &= \{x \in \mathbb{R}^2 : \, H(x) = x_2 - \alpha < 0\}, \\ S_2 &= \{x \in \mathbb{R}^2 : \, H(x) = x_2 - \alpha > 0\}. \end{aligned} \tag{15}$$

The switch control function $\sigma(x)$ given in Eq. (4) have now the form

$$\sigma(x) = f_2^{(1)}(x) f_2^{(2)}(x), \tag{16}$$

where $f_2^{(1)}$ and $f_2^{(2)}$ are the second components of $f^{(1)}$ and $f^{(2)}$ of Eq. (13).

3. Numerical simulation and results

Assume that $x_\alpha = [x_1, x_2 = \alpha]^T$ and let the switch control function (16) on the boundary will be as follows

$$\begin{aligned} \sigma(x_\alpha) &= (x_1(x_1^2 - 1) - \beta)(x_1(x_1^2 - 1) + \beta) \\ &= x_1^2(x_1^2 - 1)^2 - \beta^2. \end{aligned} \tag{17}$$

Some particular cases of Eq. (17) can be distinguished if it will be possible to find

$$\begin{aligned} D_{\alpha_0} &= \frac{d\sigma(x_{\alpha_0})}{dx} = 0, \\ \sigma(x_{\alpha_0}) &= 0. \end{aligned} \tag{18}$$

Therefore, if

$$D_{\alpha_0} = 2x_1(x_1^2 - 1)(3x_1^2 - 1) = 0, \tag{19}$$

then we obtain following set of roots

$$x_{\alpha_0 i} = \left\{0, \pm 1, \pm\frac{\sqrt{3}}{3}\right\}, \tag{20}$$

for $i = 1 \ldots 5$. Substituting x_{α_0} given in Eq. (20) for x_1 indicated in Eq. (17) and then

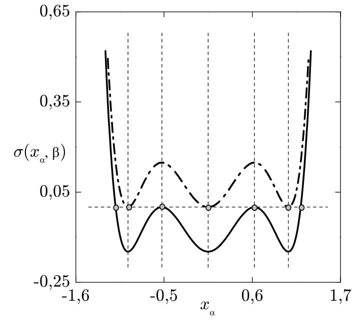

Figure 5. Tangent points (circles) on the crossing of function $\sigma(x_\alpha, \beta)$ with x_α horizontal axis for $\beta = \beta_1$ (dash-dot line) and $\beta = \beta_{2,3}$ (solid line).

solving it with respect to β we get the set of boundary values $\beta_i = [0, \pm\frac{2\sqrt{3}}{9}]$ for $i = 1 \ldots 3$. It is worth noticing, that for $\beta = \beta_1$ the sliding segment vanish and degenerates to the three tangent points $x_{\alpha_0 i}$ for $i = 1 \ldots 3$ while for $\beta = \beta_{2,3}$ it covers the maximum region of the discontinuity boundary beginning at $-\frac{2\sqrt{3}}{3}$, finishing at $\frac{2\sqrt{3}}{3}$, and including by the way points $x_{\alpha_0 i}$ for $i = 4, 5$. The explained above sliding segment bifurcation scenario has been illustrated in Fig. 5.

The introduced in Eq. (16) control function $\sigma(x)$ multiplies the two terms $f_2^{(1)}$ and $f_2^{(2)}$ which can be used to find tangent points of $f^{(1)}$ in S_1 and $f^{(2)}$ in S_2, respectively (see Eq. (10)). Having noticed that at characteristic values of β_i the whole set of tangent points

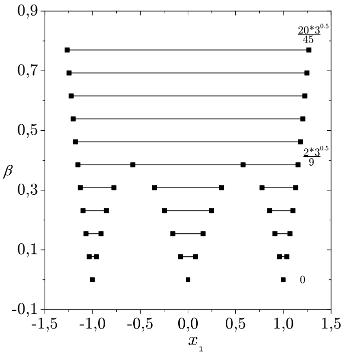

Figure 6. Bifurcation diagram of the tangent points characterised by solution to the Eq. (10) for the control parameter $\beta \in \left[\beta_1, \frac{20\sqrt{3}}{45}\right]$.

is $x_T = x_{\alpha_0}$, the above derivations are used to determine a curve of tangent points (visible in Fig. 6) when the control parameter β varies from β_1 to $\frac{20\sqrt{3}}{45}$.

Let us now define a set of sliding points which are in agreement with Eq. (9). Recalling Eq. (7) we have

$$g(x) = \lambda \begin{bmatrix} x_2 \\ f_2^{(1)}(x) \end{bmatrix} + (1-\lambda) \begin{bmatrix} x_2 \\ f_2^{(2)}(x) \end{bmatrix}, \tag{21}$$

The derived earlier σ_2, H_x, $f^{(1)}$, and $f^{(2)}$ are used in order to determine the convex method parameter λ (see Eq. (8))

$$\lambda = -\frac{(\alpha - x_2 + 1)(x_{12}(\alpha - x_2 - 1) - \beta)}{2\beta}, \tag{22}$$

where $x_{12} = x_1(x_1^2 - 1)$.

By putting λ given in Eq. (22) to (21) we get the differential equation of sliding solutions on Σ_s, which is smooth on 1-dimensional sliding intervals of Σ_s

$$\dot{x} = g(x) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}. \tag{23}$$

The velocity vector of motion $\dot{x}$ on the estimated line is composed only of one non-vanishing component x_2, therefore the mass m goes transversely on the (x_1, x_2)-plane until the boundary of sliding region is achieved.

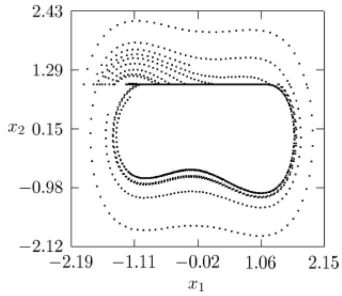

Figure 7. Trajectories in $x_2(x_1)$ phase plane surrounding invisible tangent point $-\frac{2\sqrt{3}}{3}$ for $\beta = \frac{2\sqrt{3}}{9}$, and starting from various $[x_{10,i}, x_2 = \alpha]$ initial conditions.

There are many trajectories presented in Fig. 7 of which plotting has been stopped when they were attracted to the discontinuity boundary. It can be observed that starting far to the left from the invisible tangent point, some of trajectories make big loops on the phase plane. For the biggest loop, in particular, there are visible (to the right) points before and after crossing, and additionally, the used numerical algorithm have properly calculated the solution point on Σ_c-line. Continuing, when an other trajectory crosses Σ_s-plane (in this case it is from $-\frac{2\sqrt{3}}{3}$ to $\frac{2\sqrt{3}}{3}$) then it starts sliding until the visible tangent point is reached. All of the visible trajectories approach the same limit cycle when the time goes to infinity.

Bifurcation diagram presented in Fig. 8a is a quantitative continuation of the observations described earlier. It is shown on it that points in which mass m begins sliding when starting from various positions at the same velocity of the belt ($x_2 = \alpha$). Concluding a little change in starting position x_{10} provides rapid displacement of landing points placement at which sliding phase begins. It is very interesting that very close to the extremely stable loops which before landing on Σ_s-plane make a few loops surrounding it, exist some initial conditions ((x_1, x_2) points) at which a transient trajectory is almost unstable (see Fig. 8b). In this situation the convergence process takes very long time which corresponds to 85 loops until the poor-stable trajectory lands on Σ_s-plane.

4. Conclusion

On the basis of the presented results, the introduced two degrees-of-freedom block-on-belt dynamical system is an interesting object which is very useful in analysis of discontinuous

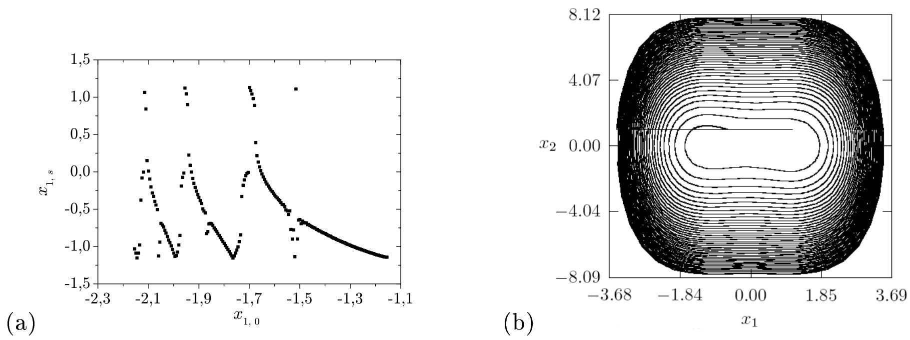

Figure 8. (a) Bifurcation diagram of the landing points $x_{1,s}$ (as a function of initial condition $x_{1,0}$) at which sliding on the discontinuity boundary begins; (b) phase trajectory starting from point $(-\frac{2\sqrt{3}}{3} - 0.36, \alpha)$ for $\beta = \frac{2\sqrt{3}}{9}$ of which number of Σ_c-plane crossings is 85.

bifurcations. Special attention has been paid on analysis of sliding segments bifurcations, but there were also observed some interesting remarks associated with the investigations of influence of initial conditions on the phase trajectory convergence to the sliding region at discontinuity.

5. References

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