*Proceedings of the 23rd Southeastern Conference on Theoretical and Applied Mechanics*
*Mayagüez, Puerto Rico, May 21 – 23, 2006*

# Dynamics of contacting bodies with impacts, wear and heat generated by friction

*Jan Awrejcewicz, Yuriy Pyryev*
*Department of Automatics and Biomechanics, Technical University of Łódź, 1/15 Stefanowskiego St., 90-924 Łódź, Poland*

## Abstract

In this work the model of a contact system with heat and wear generated by friction and possible impacts is studied. The methods and mathematical models of such systems applied so far contribute only partially to the description of complex dynamics. First, the analysis of contacting dynamic models omits tribological processes on a contact body surface. Second, the mentioned models do not include either the body inertia or impact phenomena usually appearing within the body clearance. We contribute to the problem by matching both phenomena, which improves modeling of dynamic behavior of contacting bodies. Analysis of both stick-slip and slip-slip motion exhibited by the system is performed (impact-less behavior of this model has been already studied by the authors [1-3]).

## 1. Introduction

Attention is focused on modeling of non-linear dynamics of two bodies consisting of a stiff bush with clearance $2\Delta_\varphi$ (see Fig. 1). The bush is coupled with housing by springs with stiffness $k_2$ and is mounted on the rotating thermoelastic shaft 1. The following assumptions are taken: (i) the shaft rotates with such angular velocity $\Omega$ that centrifugal forces can be omitted; (ii) non-linear kinetic friction occurs between the bush and the shaft; (iii) heat is generated on the contacting surface $R = R_1$ due to friction; (iv) heat transfer between contacting bodies is governed by Newton's law.

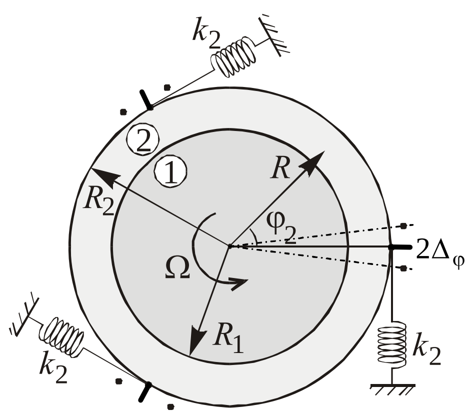


Figure 1: Analyzed system.

## 2. Equations for rotational movement of an absolutely rigid bush

Let axis $Z$ be a cylinder axis. The equilibrium state of the moments of forces with respect to the shaft axis yields

$$B_2\ddot{\varphi}_2(t) + k_2 R_2^2 \varphi_2(t) = f(V_r) 2\pi R_1^2 P(t)\,, \quad |\varphi_2(t)| < \Delta_\varphi\,,\ \dot{\varphi}_2(t) \neq \Omega \tag{1}$$

$$\ddot{\varphi}_2(t) = 0\,, \quad |\varphi_2(t)| < \Delta_\varphi\,,\ \dot{\varphi}_2(t) = \Omega\,, \tag{2}$$

$$\dot{\varphi}_2^+ = -k\dot{\varphi}_2^-,\ |\varphi_2| = \Delta_\varphi,\ \dot{\varphi}_2^-\varphi_2 > 0\ ,(3)$$

where: $V_r = R_1\Omega - R_1\dot{\varphi}_2(t)$ relative velocity of the contact bodies, $k$ is the coefficient of restitution, $\dot{\varphi}_2^-$ ($\dot{\varphi}_2^+$) is the bush velocity just before (after) impact, $B_2$ is the moment of inertia of the bush for a length unit, $f(V_r)$ is the kinematic friction coefficient depending on relative velocity, $P(t)$ is the contact pressure. Let the initial conditions be

$$\varphi_2(0) = \varphi_2^\circ,\ \dot{\varphi}_2(0) = \omega_2^\circ\ . \tag{4}$$

The relation approximating the curve $f(V_r)$ has the following form

$$f(V_r) = \mathrm{sgn}(V_r)F(|V_r|),\ F(V_r) = \begin{cases} F_0 - \kappa V_r, & 0 < V_r \le V_{\min} \\ F_0 - \kappa V_{\min}, & V_{\min} < V_r \end{cases}, \tag{5}$$

where $F_0$, $\kappa$, $V_{\min}$ are the constant coefficients.

## 3. Thermoelastic shaft

In the analyzed case, the inertial terms in the equation of motion can be omitted and the problem may be considered as a quasi-static one. For axially symmetric stress of the shaft, the governing equations belong to the theory of thermal stresses for an isotropic body, formulated by Nowacki [4]. Applying cylindrical coordinates one gets the system

$$\frac{\partial^2 U(R,t)}{\partial R^2} + \frac{1}{R}\frac{\partial U(R,t)}{\partial R} - \frac{1}{R^2}U(R,t) = \alpha_1\frac{1+\nu_1}{1-\nu_1}\frac{\partial T_1(R,t)}{\partial R}, \tag{6}$$

$$\frac{\partial^2 T_1(R,t)}{\partial R^2} + \frac{1}{R}\frac{\partial T_1(R,t)}{\partial R} = \frac{1}{a_1}\frac{\partial T_1(R,t)}{\partial t},\ 0 < R < R_1,\ 0 < t < t_c \tag{7}$$

with the attached mechanical

$$U(0,t) = 0,\ U(R_1,t) = -U_0 h_U(t) + U^w(t),\ 0 < t < t_c, \tag{8}$$

and thermal boundary conditions

$$\lambda_1\frac{\partial T_1(R_1,t)}{\partial R} + \alpha_T T_1(R_1,t) = (1-\eta)f(V_r)V_r P(t), \tag{9}$$

$$R\left.\frac{\partial T_1(R,t)}{\partial R}\right|_{R\to 0} = 0,\ 0 < t < t_c, \tag{10}$$

and with the following initial conditions

$$T_1(R,0)=0\,,\ \ 0<R<R_1\,. \tag{11}$$

Velocity of the bush wear is proportional to a certain power of friction force. According to Archard's assumption [5] we have

$$\dot{U}^w(t)=K^w\left|V_r(t)\right|P(t)\,. \tag{12}$$

Radial stress $\sigma_R(R,t)$ in the cylinder may be found with the use of the radial displacement $U(R,t)$ and temperature $T_1(R,t)$ by the application of the following formula

$$\sigma_R(R,t)=\frac{E_1}{1-2\nu_1}\left[\frac{1-\nu_1}{1+\nu_1}\frac{\partial U(R,t)}{\partial R}+\frac{\nu_1}{1+\nu_1}\frac{U(R,t)}{R}-\alpha_1 T_1(R,t)\right]. \tag{13}$$

The following notation is used: $P(t)=-\sigma_R(R_1,t)$ - contact pressure; $E_1$ - Young's modulus; $\nu_1$ - Poisson's ratio; $a_1$ - thermal diffusivity, $\alpha_1$ - thermal expansion coefficient; $\lambda_1$ - thermal conductivity; $K^w$ - wear constant, $t_c$ - time of contact.

Integrating equation (5), with (7) and (12) taken into account, the contact pressure is determined:

$$P(t)=\frac{2E_1\alpha_1}{1-2\nu_1}\frac{1}{R_1^2}\int_0^{R_1}T_1(\xi,t)\xi d\xi+\frac{E_1}{(1-2\nu_1)(1+\nu_1)R_1}\left[U_0 h_U(t)-U^w(t)\right] \tag{14}$$

## 4. Solution Algorithm

Let us introduce the following dimensionless parameters:

$$\tau=\frac{t}{t_*}\,,\ \tau_c=\frac{t_c}{t_*}\,,\ r=\frac{R}{R_1}\,,\ \varphi(\tau)=\frac{\varphi_2}{\Delta_\varphi}\,,\ p=\frac{P}{P_*}\,,\ \theta=\frac{T_1}{T_*}\,,\ u^w=\frac{U^w}{U_0}\,,\ \alpha_0=\frac{f_0}{F_0}\,,\ \omega_1=\frac{\Omega t_*}{\Delta_\varphi}=\frac{1}{\gamma\sqrt{\alpha_0}}\,,$$

$$\mu_0=\frac{1-\alpha_0}{\alpha_0}\,,\ \eta_0=\frac{V_0-V_{\min}}{V_0}\,,\ \varepsilon=\mu_0\gamma\sqrt{\alpha_0}=\frac{\mu_0}{\omega_1}\,,\ \gamma=\sqrt{\frac{2\pi R_1^2 P_* F_0\Delta_\varphi}{B_2\Omega^2}}\,,\ \omega_0^2=\frac{k_2 R_2^2 t_*^2}{B_2}\,,$$

$$Bi=\frac{\alpha_T R_1}{\lambda_1}\,,\ k^w=\frac{K^w\Delta_\varphi E_1}{(1-2\nu_1)(1+\nu_1)}\,,\ \gamma_1=\frac{(1-\eta)E_1\alpha_1 R_1^2 f_0\Delta_\varphi}{\lambda_1(1-2\nu_1)t_T}\,,\ \tilde{\omega}=\frac{t_*}{t_T}\,,$$

$$f(V_0)=f_0\,,\ h_U(\tau)=h_U(t_*\tau) \tag{15}$$

where:

$$t_*=\sqrt{\frac{B_2\Delta_\varphi}{f_0 2\pi R_1^2 P_*}}\,,\ V_*=\frac{R_1\Delta_\varphi}{t_*}\,,\ P_*=\frac{E_1U_0}{(1-2\nu_1)(1+\nu_1)R_1}\,,\ T_*=\frac{U_0}{\alpha_1(1+\nu_1)R_1}\,,\ t_T=\frac{R_1^2}{a_1}\,,\ V_0=\Omega R_1\,.$$

The dimensionless equations governing the system dynamics have the form

$$\ddot{\varphi}(\tau)+\omega_0^2\varphi(\tau)=\operatorname{sgn}(\omega_1-\dot{\varphi})\Psi(\dot{\varphi})p(\tau)\,,\ \left|\varphi(\tau)\right|<1\,,\ \dot{\varphi}(\tau)\neq\omega_1\,, \tag{16}$$

$$\ddot{\varphi}(\tau)=0\,,\ \left|\varphi(\tau)\right|<1\,,\ \dot{\varphi}(\tau)=\omega_1\,,$$

$$\dot{\varphi}^+=-k\dot{\varphi}^-\,,\ \left|\varphi\right|=1\,,\ \dot{\varphi}^-\varphi>0\,,$$

$$\varphi(0)=x\,,\ \dot{\varphi}(0)=y\,,$$

where:

$$\Psi(\dot{\varphi})=\begin{cases}1+\varepsilon\omega_1\eta_0, & \dot{\varphi}<\omega_1\eta_0,\ \omega_1(2-\eta_0)<\dot{\varphi}\\ 1+\varepsilon\dot{\varphi}, & \omega_1\eta_0<\dot{\varphi}<\omega_1\\ 1+2\varepsilon\omega_1-\varepsilon\dot{\varphi}, & \omega_1<\dot{\varphi}<\omega_1(2-\eta_0)\end{cases}. \tag{17}$$

In order to solve the motion equations (inclusion) one needs to know contact pressure $p(\tau)$ and wear $u^w(\tau)$:

$$p(\tau)=h_U(\tau)-u^w(\tau)+\int_0^1\theta(\xi,\tau)\,\xi\,d\xi\,,\ u^w(\tau)=k^w\int_0^\tau\left|\omega_1-\dot{\varphi}(\tau)\right|p(\tau)d\tau\,. \tag{18}$$

The one-dimensional transient heat conduction equation under consideration takes the following dimensionless form

$$\frac{\partial^2\theta(r,\tau)}{\partial r^2}+\frac{1}{r}\frac{\partial\theta(r,\tau)}{\partial r}=\frac{1}{\tilde{\omega}}\frac{\partial\theta(r,\tau)}{\partial\tau}\,, \tag{19}$$

whereas the thermal boundary conditions are

$$\left[\frac{\partial\theta(r,\tau)}{\partial r}+Bi\theta(r,\tau)\right]_{r=1}=\gamma_1\tilde{\omega}^{-1}\,\Psi\left(\dot{\varphi}(\tau)\right)\left|\omega_1-\dot{\varphi}(\tau)\right|p(\tau)\,,\ r\frac{\partial\theta(r,\tau)}{\partial r}\bigg|_{r\to0}=0 \tag{20}$$

and initial conditions are as follows

$$\theta(r,0)=0\,. \tag{21}$$

Applying an inverse Laplace transformation [1, 6], our nonlinear problem governed by Eqs. (19), (20) and (21) is reduced to the following integral equation

$$p(\tau)=h_U(\tau)-u^w(\tau)+2\gamma_1\tilde{\omega}^{-1}\int_0^\tau\dot{G}_p(\tau-\xi)\Psi\left(\dot{\varphi}(\xi)\right)\left|\omega_1-\dot{\varphi}(\xi)\right|p(\xi)d\xi, \tag{22}$$

which yields both dimensionless pressure $p(\tau)$ and velocity $\dot{\varphi}(\tau)$. The temperature is defined by the following formula

$$\theta(r,\tau)=\gamma_1\tilde{\omega}^{-1}\int_0^{\tau}\dot{G}_\theta(r,\tau-\xi)\Psi\left(\dot{\varphi}(\xi)\right)\left|\omega_1-\dot{\varphi}(\xi)\right|p(\xi)d\xi, \tag{23}$$

where

$$\left\{G_p(\tau),\,G_\theta(1,\tau)\right\}=\frac{\{0.5,\,1\}}{Bi}-\sum_{m=1}^{\infty}\frac{\left\{2Bi,\,2\mu_m^2\right\}}{\mu_m^2(Bi^2+\mu_m^2)}e^{-\mu_m^2\tilde{\omega}\tau}\ , \tag{24}$$

and $\mu_m$ are the roots of the characteristic equation

$$BiJ_0(\mu)-\mu J_1(\mu)=0\ . \tag{25}$$

Note that the investigated problem has been transformed to the set of nonlinear differential equation (16), integral equation (22) describing angular velocity $\dot{\varphi}(\tau)$, and contact pressure $p(\tau)$. Temperature is defined by (23). A numerical analysis of the problem is performed using Runge-Kutta method by taking into account the following asymptotes

$$G_\theta(1,\tau)\approx 2\sqrt{\tau\tilde{\omega}/\pi}\ ,\ G_p(\tau)\approx\tilde{\omega}\tau\ ,\ \tau\to 0\ . \tag{26}$$

## 5. Analysis

First the case of bush vibrations without tribological processes is studied ($\gamma_1=0$, $k^w=0$). For this case we have $p(\tau)=h_U(\tau)$. Our system governed by equations (16) may exhibit four different periodic motions. Namely:
(i) Periodic orbit with one impact, where a stick does not appear (Figure 2a);
(ii) Periodic orbit with one impact, where a stick-slip occurs;
(iii) Periodic orbit with two impacts, where a slip of the contacting bodies occurs;
(iv) Periodic orbit with two impacts, where stick-slip appears.

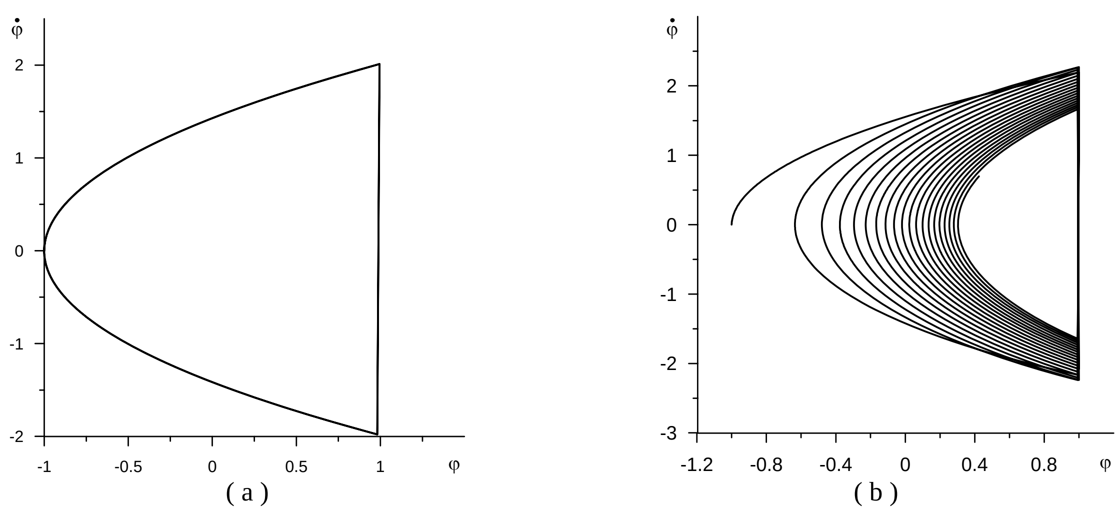


Figure 2: Phase trajectory of bush movement in the absence of heat expansion $\gamma_1=0$ (a) and $\gamma_1=0.2$ (b).

Below, we illustrate conditions of the first type periodic orbit occurrence, i.e. with one impact during a period. We assume that $\mu_0 << 1$ and $\omega_0^2 << 1$, and $\eta_0 \le -1$. It means that that the system works in the interval ( $0 < V_0 < V_{\min}$ ), where the kinetic friction coefficient decreases with respect to the relative velocity of the bodies. The value of parameter $k$ has been detected with the accuracy of $o(\varepsilon^2)$ and $o(\omega_0^4)$, where a periodic motion may occur. It is

$$k = 1 - (2\tau_{20}/3)\varepsilon, \ \tau_{20} = \sqrt{2(1-x)}, \ x \in (-1,1), \tag{27}$$

and the estimated period follows

$$T = 2\tau_{20} - (1/12)\tau_{20}^3\omega_0^2 . \tag{28}$$

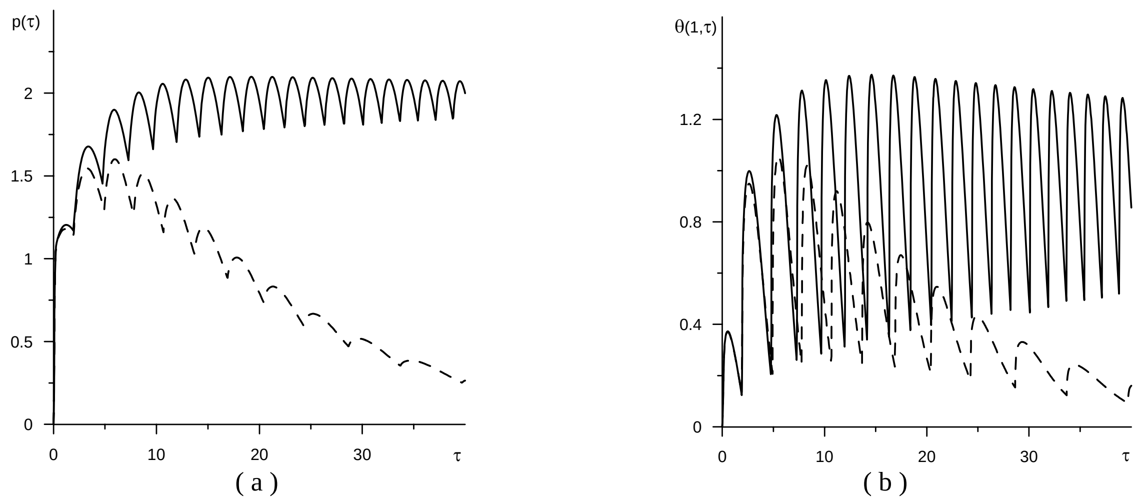


Figure 3: Behavior of dimensionless contact pressure $p(\tau)$ (a), dimensionless contact temperature $\theta(1,\tau)$ (b) versus dimensionless time $\tau$ for different values of $k^w$ ( $\gamma_1 = 0.2$ ); solid curves: $k^w = 0$, dashed curves: $k^w = 0.01$.

In the case of a periodic motion with two impacts, the coefficient of restitution has the following estimated value

$$k = 1 - \frac{\tau_{20}^3 - \tau_{10}^3}{3(2+\tau_{10}^2)}\varepsilon, \ \tau_{10} = \sqrt{-(1+x)}, x \in (-\infty, -1), \tag{29}$$

and the associated initial conditions for realization of this motion with the accuracy of $o(\varepsilon^2)$ and $o(\omega_0^4)$ are

$$\varphi(0) = -1, \ \dot{\varphi}(0) = \tau_{10} + \frac{\tau_{10}}{24}\left(8\tau_{10}\varepsilon + (12 + 3\tau_{10}^2)\omega_0^2\right). \tag{30}$$

A numerical analysis has been carried out for the following dimensionless parameters: $\varepsilon = 0.01$, $\omega_1 = 2.5$, $\omega_0^2 = 0.02$, $\eta_0 = -2$ (case (i)). For $x = -1$ one obtains $k = 0.987$ from (28), and the corresponding bush phase trajectory has been shown in Figure 2a.

Observe that the occurrence of frictional heat generation ( $\gamma_1 = 0.2$ ) causes an increase of the contact pressure (see Figure 3a) and periodic motion vanishes. Time evolution of the dimensionless temperature of the contacting surfaces is shown in Figure 3b. The following parameters have been taken during numerical computations: $Bi = 10$, $k^w = 0$, $\tilde{\omega} = 0.1$.

The contact pressure decreases (see dashed curves in Figure 3a), when wear occurs ( $k^w = 0.01$ ), and the contacting temperature decreases too (see the dashed curve in Figure 3b).

## 6. Concluding Remarks

We have proposed a novel model of vibrations of the bush-shaft system with inclusion of both impacts and tribological processes occurring on the contact surface. A similar system, however without impacts, has been studied earlier by the authors in references [1-3]. The occurrence of self-excited vibrations in a more simplified system with a gap (without tribological processes and springs) has been also analyzed in reference [7].
Applying the Laplace transformation, our problem has been reduced to that of the system of one non-linear differential equation and one second-order Volterra integral equation with respect to the contact pressure. A kernel of the latter equation is the function of the sliding velocity. We have estimated analytically the restitution coefficient for which a periodic motion occurs assuming small slope of friction characteristics. We have shown, among the others, various periodic motions exhibited by the analyzed system and we have verified numerically the theoretical considerations.

## 7. Acknowledgement

This work has been financially supported by the Polish Ministry of Education and Science for the years 2005-2008 (grant No. 4 TO7C 044 29).

## 8. References

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