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HOMOGENIZATION OF THE IRREGULAR CELL-TYPES CONSTRUCTIONS

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Abstract: We show, using two-scales method, how the method of homogenization can be applied even for non-periodic irregularities.

1. Introduction

Nowadays, cell-types materials and structures are often applied in mechanical engineering (composite structures, corrugated and perforated shells, complex structures of bridges). All of the mentioned construction elements possess regular (periodic) structure, although in many cases non-regular constructions can be more optimal. In general considerations two possibilities are distinguished: (i) a cell with different physical parameters, like for instance shells with different stiffness ribs, (ii) cells with non-constant step of homogenuity, i.e. non-regular support of ribs possessing the same stiffness.

Homogenization method is widely used for computation of regular but periodically non-homogenenous constructions. In the existing literature, there are also presented various ways for extension of this method into the case of first type irregularities [1]. In this work, a novel method for modification of the application of the homogenization procedure to second type irregularities is introduced.

2. Model and analysis

The mentioned novel approach will be classified briefly, with the use of a classical problem governed by the following equations [1]

$$\frac{d}{dx}\left[a\left(\frac{x}{\varepsilon}\right)\frac{du}{dx}\right]+b\left(\frac{x}{\varepsilon}\right)u=g(x), \qquad (1)$$

where $a(x/\varepsilon)$ is periodic with respect to *x*, and possesses period $\varepsilon\ (\varepsilon << 1)$.

If the non-homogenity is non-periodic, but it is often repeated, then the problem (1) can be cast into the form

$$\frac{d}{dx}\left\{a\left[\frac{f(x)}{\varepsilon}\right]\frac{du}{dx}\right\}+b\left[\frac{f(x)}{\varepsilon}\right]u=g(x)\,, \tag{2}$$

where $f(x)$ is a function of slow variability, $f'(x)\sim 1$. The step of non-homogeneity T can be approximately governed by the formula $T\approx\varepsilon\big(f(x)\big)$.

Applying the two-scales approach, two independent variables $\eta=x$ and $\xi=f(x)/\varepsilon$ are introduced, instead of x. Then, the derivative of x reads

$$\frac{d}{dx}=\frac{\partial}{\partial\eta}+\frac{f'(\eta)}{\varepsilon}\frac{\partial}{\partial\xi}\,, \tag{3}$$

i.e. instead of ODE, a PDE is obtained. Its solution is sought in the form of the following series

$$u=u_0(\eta,\xi)+\varepsilon u_1(\eta,\xi)+..., \tag{4}$$

where u_0, $u_{1,\,...}$ are periodic functions with respect to ξ and they have period 1.

Substituting (3), (4) into (2) and comparing the terms standing by the same powers of ε, the following recurrent set of equations is obtained

$$\begin{aligned}
&\frac{\partial}{\partial\xi}\left[a(\xi)\frac{au_0}{\partial\xi}\right]=0\,,\\
&f'(\eta)\frac{\partial}{\partial\xi}\left[a(\xi)\frac{au_1}{\partial\xi}\right]+\frac{\partial}{\partial\xi}\left[a(\xi)\frac{au_0}{\partial\xi}\right]=0\,,\\
&f'^2(\eta)\frac{\partial}{\partial\xi}\left[a(\xi)\frac{au_2}{\partial\xi}\right]+f'(\eta)\frac{\partial}{\partial\xi}\left[a(\xi)\frac{au_1}{\partial\eta}\right]+\\
&a(\xi)\frac{\partial}{\partial\eta}\left[f'(\eta)\frac{au_1}{\partial\xi}\right]+a(\xi)\frac{\partial^2u_0}{\partial\eta^2}+b(\xi)u_0=q(\eta).
\end{aligned} \tag{5}$$

The first equation of (5) yields $u_0=u_0(\eta)$, and the second equation of (5) reads

$$f'(\eta)\frac{\partial}{\partial\xi}\left[a(\xi)\frac{au_1}{\partial\xi}\right]=\frac{da}{d\xi}(\xi)\frac{du_0}{d\eta}\,.$$

It gives

$$\frac{\partial u_1}{\partial\xi}=-\frac{1}{f'(\eta)}\frac{du_0}{d\eta}+\frac{c_1(\eta)}{a(\xi)}\,, \tag{6}$$

and the constant $c_1(\eta)$ is defined by the periodicity condition of the function u_1 with respect to ξ

$$c_1(\eta) = -\bar{a}\frac{du_0}{d\eta}, \quad \bar{a} = \left[\int_0^1 a^{-1} d\xi\right]^{-1}.$$

Removing the derivative $\frac{\partial u_1}{\partial \xi}$ from the third equation of (5), one gets

$$f'^2(\eta)\frac{\partial}{\partial \xi}\left(a\frac{\partial u_2}{\partial \xi}\right) + f'(\eta)\frac{\partial}{\partial \xi}\left(a\frac{\partial u_1}{\partial \xi}\right) + \bar{a}\frac{d^2 u_0}{d\eta^2} + bu_0 = q. \tag{7}$$

Let us apply the homogenization procedure $\int_0^1 (...)d\xi$ to the equation (7). First two terms are zero due to the periodicity of functions $f'^2(\eta)\frac{\partial}{\partial \xi}\left(a\frac{\partial u_2}{\partial \xi}\right)$, $f'(\eta)\frac{\partial}{\partial \xi}\left(a\frac{\partial u_1}{\partial \eta}\right)$ with respect to ξ, and finally one gets

$$\bar{a}\frac{d^2 u_0}{d\eta^2} + \bar{b}u_0 = q(\eta), \tag{8}$$

where:

$$\bar{b} = \int_0^1 b(\xi)d\xi.$$

The following boundary condition is applied to the equation (8):

$$u_0 = 0 \quad \text{for} \quad \eta = 0, L. \tag{9}$$

Note that the non-regularity of non-homogenuity is not exhibited by the equation (1). Non-regularity is accounted through the computation of the first correction to the to the homogenized solution, which is defined by the equation (6):

$$u_1 = f'(\eta)^{-1}\int\left(a^{-1}\hat{a} - 1\right)d\xi\frac{du_0}{d\eta} + c_2(\eta), \tag{10}$$

where c_2 is found from the boundary condition.

It is worth noticing that many cell-type constructions are governed by differential equations with non-smooth coefficients. In what follows, we illustrate the computation of such constructions, using an example of cylindrical shell with non-constant steps of support.

For an axially symmetric case, the equation governing equilibrium state of cylindrical shell reads

$$w^{IV} + \left\{\beta + \gamma\sum_{k=1}^{N}\delta\left[f(x) - kl\right]\right\}w = P(x), \tag{11}$$

where:

$$\beta=\frac{3(1-\nu^2)}{R^2h^2}, \quad p=\frac{q}{D}, \quad \gamma=-\frac{EF}{DR^2}, \quad D=\frac{Eh^3}{12(1-\nu^2)} .$$

and R,h,l denote shell radius and thickness, and distance between supports, respectively; E is Young modulus; ν is Poisson coefficient; F is area of the support cross-section; δ is Dirac function .

Introducing variable $\eta = f(x)$, when $x = f^{-1}(\eta)$, the equation (11) is recast into the form

$$P^4(w)+\left[\beta+\gamma\sum_{k=1}^{N}\delta(\eta-kl)\right]wy=Q(\eta), \tag{12}$$

$$P^4(w)=\frac{d}{d\eta}\left\{\frac{1}{\varphi(\eta)}\frac{d}{d\eta}\left[\frac{1}{\varphi(\eta)}\frac{d}{d\eta}\left(\frac{1}{\varphi(\eta)}\frac{dw}{d\eta}\right)\right]\right\},$$

where

$$\varphi(\eta)=\left[f^{-1}(\eta)\right]_{\eta}^{1}, \; Q=\varphi G\left[f^{-1}(\eta)\right].$$

Note that the coefficients of the equation (12) are periodic and non-smooth. If N is large enough, then one may apply an homogenization procedure to solve the equation (12). The equilibrium equation for the shell deformation between successive frames reads

$$P^4(w)+\beta\varphi w=Q, \tag{13}$$

whereas the conditions for transition through a frame has the form

$$w^+=w^-; \quad \frac{dw^+}{d\eta}=\frac{dw^-}{d\eta}; \quad (w_{\eta\eta})^+=(w_{\eta\eta})^-, \tag{14}$$

$$(w_{\eta\eta\eta})^+-(w_{\eta\eta\eta})^-=\gamma\varphi w\big|_{\eta=kl}, \tag{15}$$

where: $(...)^+,(...)^-$ are right and left limiting values calculated in the point $\eta = kl$, respectively.

The boundary conditions at shell ends $\eta = 0,l$ are taken in the form

$$w=\frac{dw}{d\eta}=0 . \tag{16}$$

Introducing the small parameter $\varepsilon = N^{-1}$ and fast (slow) variable $\xi=\frac{\eta}{\varepsilon}$ (η), one gets

$$\frac{d}{d\eta}=\frac{\partial}{d\eta}+\varepsilon^{-1}\frac{\partial}{\partial\xi} . \tag{17}$$

Displacement w is sought in the form

$$w=w_0(\eta)+\varepsilon^4 w_1(\eta,\xi)+\dots, \tag{18}$$

where: $w_i\,(i=1,2,...)$ are periodic functions of ξ with the period L.

Substituting (17), (18) into (13), (15) and carrying out the asymptotical splitting with respect to ε powers, the following equations are obtained

$$\frac{1}{\varphi^3}\frac{\partial^4 w_1}{\partial\xi^4}+P^4(w_0)+\beta\varphi w_0=Q\ , \tag{19}$$

$$\left(w_1,\frac{\partial w_1}{\partial\xi},\frac{\partial^2 w_1}{\partial\xi^2}\right)_{\xi=0}=\left(w_1,\frac{\partial w_1}{\partial\xi},\frac{\partial^2 w_1}{\partial\xi^2}\right)_{\xi=L}\ , \tag{20}$$

$$\left.\frac{\partial^3 w_1}{\partial\xi^3}\right|_{\xi=L}-\left.\frac{\partial^3 w_1}{\partial\xi^3}\right|_{\xi=0}=\varphi^3\gamma N w_0\ , \tag{21}$$

$$w_{\eta=0,l}=\left.\frac{dw_0}{d\eta}\right|_{\eta=0,L}=0\ . \tag{22}$$

Integrating the equation (19), one has

$$w_1=\varphi^3\left[Q-P^4(w_0)-\beta\varphi w_0\right]\frac{\xi^4}{24}+C_1(\eta)\xi^3+C_2(\eta)\xi^2+C_3(\eta)\xi+C_4(\eta)\ . \tag{23}$$

Taking $C_4=0$ and defining C_1-C_3 from the condition (20), one gets

$$w_1=\varphi^3\left[Q-P^4(w_0)-\beta\varphi w_0\right]\frac{\xi^4(\xi-L)^2}{24}\ . \tag{24}$$

Substituting (24) into (21) , the following equation for the determination of w_0 is obtained

$$P^4(w_0)+\left(\beta\varphi+\frac{\gamma N}{L}\right)w_0=Q\ . \tag{25}$$

Coming back to the variable x, the equation (25) takes the form

$$w_0^{IV}+\left(\beta+\frac{\gamma N}{L}f'(x)\right)w_0=G(x)\ . \tag{26}$$

It is worth noticing that the derived equation (26) describes a beam lying on continuous elastic foundation with changeable stiffness deflection. Therefore, non-regular properties of support are already included in an averaged solution. A correction term accounting the support discretness is defined through the formula (24).

3. Concluding remarks

Equation governing behaviour of a beam lying on continuous elastic foundation with changeable stiffness deflection is derived. The presented approach can be also applied in other engineering oriented research.

References

1. Awrejcewicz J., Andrianov I.V., Manevitch L.I., Asymptotic Approaches in Nonlinear Dynamics: New Trends and Applications, Springer-Verlag, Berlin, 1998.

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