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DYNAMICS OF REINFORCED VISCOELASTIC PLATE – HOMOGENIZATION APPROACH

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Abstract: Oscillations and static bending deformation of a viscoelastic reinforced plate are considered. Analytical solutions are derived. An asymptotic technique, based on the homogenization method, is used for this purpose. In addition, a special perturbation approach is employed. An example is given for the purpose of illustration. The approximate analytical expressions are shown to adequately meet the requirements of optimal structural design.

1. Introduction

Reinforced plates and shells are described by partial differential equations with rapidly varying coefficients, and their stress-strain state may be represented as a sum of a slow and a fast varying components [1-3]. In many physical problems, some variables may vary rather slowly, while others change fast. It is natural to ask whether it would be appropriate first to study the overall structure at hand, neglecting its local distinctive features, and next to investigate the system locally.

The paper is structured as follows. Section 2 presents the governing relationships. Section 3 deals with the homogenization procedure in general. Solutions for the local problem and the boundary layer are given in Sections 4 and 5. Sections 6 is devoted to estimation of applicability of structurally orthotropic theory. Finally, a discussion and comments regarding the results obtained are given in Section 7.

2. Governing relationships and estimates

The derivation of the equilibrium motion equations for reinforced plates and shells taking into account discrete arrangement of ribs is the subject of numerous studies [4-8]. One can conclude on the basis of the corresponding results that a 3D theory of elasticity is needed for the correct description of the plate behaviour in the vicinity of the rib. Out of these narrow regions the results obtained for different contact hypotheses coincide in the width of the rib is not large compared to the thickness of the plate. Therefore, the line contact approximation will be explored further. The ribs themselves are treated in the framework of the Kirchhoff – Klebsch hypotheses. Viscoelasticity according to [9-11] is taken into account.

Consider oscillations of a rectangular plate ($0 \le x \le L_1$, $-L_2 \le y \le L_2$), supported by a regular array of N=2n+1 ribs. The stiffness extended along the x-direction. Each rib is symmetric with respect to the middle surface of the plate. Materials of plate and ribs are linear viscoelastic with instantaneous Young modulus E and Poisson coefficient ν. The governing equation of motion may be written as follows

$$\overline{D}\nabla^4\overline{W} + \overline{E_1 I}\Phi(y)\overline{W}_{xxxx} + [r + R\Phi(y)]\overline{W}_{tt} = 0 \quad (1)$$

where:

$$\Phi(y) \sum_{i=-0.5(N-1)}^{i=0.5(N-1)} \delta(y - ib); \quad b = \frac{2L_2}{N+1}; \quad \overline{D} = D\Gamma; \quad \overline{E_1 I} = E_1 I\Gamma; \quad \Gamma(\overline{W}) = \overline{W} +$$

$$-\int_{-\infty}^{t} G(t-\tau)\overline{W}(\tau)d\tau; \quad D = \frac{Eh^3}{12(1-\nu^2)};$$

and t is the time; $\overline{W}(x,y,t)$ is the normal displacement; E_1 is the rib Young modulus; h is the plate thickness; r, R are the plate and rib material density; I is the moment of rib cross section; δ(x) is Dirac delta-function; G(t - τ) is the kernel of relaxation velocity.

The boundary conditions, without loss of generality, can be written in the form

$$\overline{W} = \overline{W}_y = 0 \text{ for } y = \pm L_2, \quad (2)$$

$$\overline{W} = \overline{W}_{xx} = 0 \text{ for } x = 0, L_1 \text{ or} \quad (3)$$

$$\overline{W} = \overline{W}_x = 0 \text{ for } x = 0, L_1 \quad (4)$$

The condition of continuity and equilibrium are

$$\overline{W}^+ = \overline{W}^- = \overline{W}, \quad \overline{W}_y^+ = \overline{W}_y^-, \overline{W}_{yy}^+ = \overline{W}_{yy}^-,$$

$$\overline{D}\left(\overline{W}_{yyy}^+ - \overline{W}_{yyy}^-\right) = \overline{E_1 I}\,\overline{W}_{xxxx} + R\overline{W}_{tt} \quad (5)$$

where: $(\ldots)^{(\pm)} = \lim_{y \to ib \pm 0}(\ldots)$.

The torsion rigidity of the ribs is neglect for the thin stiffness.

Because of the discontinuities in the coefficient of (1) one should find its solution in the framework of the distribution theory [12]. Namely, the solution is defined as distribution w satisfying the integral identity

$$[w,z]+\int_{-L_2}^{L_2}\int_{0}^{L_1}[r+R\Phi(y)]\overline{W}_{tt}\,z\,dxdy=0, \tag{6}$$

$$[w,z]+\int_{-L_2}^{L_2}\int_{0}^{L_1}\left\{[\overline{D}+E_1 I\Phi(y)]\frac{\partial^2\overline{W}\partial^2 Z}{\partial x^2\partial x^2}+2\frac{\partial^2\overline{W}}{\partial x\partial y}\frac{\partial^2 Z}{\partial x\partial y}+\frac{\partial^2\overline{W}\partial^2 Z}{\partial y^2\partial y^2}\right\}dxdy,$$

where: z(x,y) is smooth function satisfying condition (2), (3) or (2), (4).

The problem (6), (2), (3) (or (6), (2), (4)) has a countable real spectrum $\{\lambda_k\}$ (k = 1, 2, ...). The corresponding eigenfunctions w_k are also real and form orthogonal basis in sense of a scalar product [w, z].

The following Ansatz is used

$$\overline{W}=w(x,y)\exp(i\Lambda t),$$

where: $\Lambda = \omega + ia$, ω is the frequency and a is the damping factor, and the following relation is applied

$$\int_{-\infty}^{t}G(t-\tau)e^{i\Lambda\tau}d\tau=e^{i\Lambda t}C,$$

where: $C=\int_{0}^{t}G(\theta)e^{i\Lambda\theta}d\theta.$

Then equation (1) and conditions (5) may be reduced to the following dimensionless form

$$\nabla^4 W+\alpha\Phi(\varphi)W_{\xi\xi\xi\xi}-[1-\lambda\rho\Phi(\varphi)]W=0, \tag{7}$$

$$W^+=W^-=W,\quad W^+_{\eta_1\eta_1}=W^-_{\eta_1\eta_1},$$

$$-W^+_{\eta_1\eta_1\eta_1}+W^-_{\eta_1\eta_1\eta_1}=\varepsilon\left(\alpha\frac{\partial^4}{\partial\xi^4}-\lambda\rho\right)W,$$

where: $\nabla^4=\frac{\partial^4}{\partial\xi^4}+2\frac{\partial^4}{\partial\xi^2\partial\eta_1^2}+\frac{\partial^4}{\partial\eta_1^4}$; $\xi=\frac{x}{2L_2}$, $\eta_1=\frac{y}{2L_2}$;

$$\alpha=\frac{E_1 I}{Db};\ \varphi=\frac{y}{b};\ \rho=\frac{R}{rb};\ \Phi(\varphi)=\sum_{i=-0.5(N-1)}^{i=0.5(N-1)}\delta(\varphi-1),$$

$$\lambda=\frac{16r}{D(1-C)}\Lambda^2 L_2^4;\ (\ldots)^{(\pm)}=\lim_{\varphi\to-\pm 0}(\ldots);\ \varepsilon=\frac{b}{2L_2}.$$

For a real reinforced plates $\varepsilon << 1$, α and ρ are of the order of ε^{-1}.

The conventional approaches for reinforced plates are efficient in two opposite limiting cases. A large number of ribs is the first limit. This limit is analysed using the structurally orthotropic theory (SOT). A small number of ribs is the second limit. The corresponding technique is based on the separation of plate into elastic panels between the ribs in the complex with proper compatibility conditions on the rib lines. However, the case of a finite number of ribs is extremely important for applications. SOT can be correctly used for estimation in the low-frequency region of frequencies and displacements. At the same time the application of SOT is not correct for transverse shear forces and the description of bending moments. Unfortunately, exploring the technique corresponding to the small number of ribs limit is not efficient here. To overcome these difficulties a homogenization method is used.

3. Homogenization procedure

An explanation of the problem stated above is important for both theoretical and computational considerations. Due to the complexity of its structure, any kind of calculation is difficult to perform for a reinforced plate. An approximation of the problem at hand by a "homogenized" one is therefore desirable. The method used here is a variant of the multiscaling technique. It is well-known that this is a general method applicable to a wide range of problems. The problems are characterised by having two physical processes, each with its own scales, and with the two processes acting simultaneously. "Slow" ($\eta = \eta_1$) and "fast" (φ) variables will be used. Then derivative $\frac{\partial}{\partial \eta_1}$ has the form

$$\frac{\partial}{\partial\eta_1}=\frac{\partial}{\partial\eta}+\varepsilon^{-1}\frac{\partial}{\partial\varphi}. \qquad (8)$$

The solution of boundary value problem (7) is represented in the form of a formal expansion

$$W(x,y)=\left[W_{00}(\xi,\eta)+\varepsilon W_{01}(\xi,\eta)+\varepsilon^2 W_{02}(\xi,\eta)+\ldots\right]+$$
$$\varepsilon^3\left[W_1(\xi,\eta,\varphi)+\varepsilon W_2(\xi,\eta,\varphi)+\varepsilon^2 W_3(\xi,\eta,\varphi)+\ldots\right] \qquad (9)$$

$$\lambda=\varepsilon^{-1}\lambda_0+\lambda_1+\varepsilon\lambda+\ldots.$$

It is assumed that $W_j(x,\eta,\varphi + 1) = W_j(x,\eta,\varphi)$, $j = 1,2, \ldots$.

Substituting series (9) into boundary value problem (7), taking into account relation (8) and splitting it with respect to powers of ε, one obtains a recurrent sequence of boundary value problems

$$
\begin{aligned}
&W_{1\varphi\varphi\varphi\varphi} + \Pi_0 W_{00} = 0,\\
&W_{2\varphi\varphi\varphi\varphi} + \Pi_0 W_{01} = \Pi_1 W_1 - \nabla^4 W_{00},\\
&W_{3\varphi\varphi\varphi\varphi} + \Pi_0 W_{02} = \Pi_1 W_2 + \Pi_2 W_1 - \nabla^4 W_{01},\\
&W_{4\varphi\varphi\varphi\varphi} + \Pi_0 W_{03} = \Pi_1 W_3 + \Pi_2 W_2 + \Pi_3 W_1 - \nabla^4 W_{02},\\
&W_{5\varphi\varphi\varphi\varphi} + \Pi_0 W_{04} = \sum_{i=1}^{3} \Pi_i W_{5-1} - \nabla^4 (W_{03} + W_1),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&W_{k\varphi\varphi\varphi\varphi} + \Pi_0 W_{0k-1} = \sum_{i=1}^{3} \Pi_i W_{k-1} + \nabla^4 (W_{0k-2} + W_{k-4}),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
\tag{10}
$$

$$\Pi_0 = \alpha\Phi \frac{\partial^4}{\partial\xi^4} - \lambda_0 [1 - \rho\Phi(\varphi)],\ \Pi_1 = -4 \frac{\partial^4}{\partial\varphi^3 \partial\eta},$$

$$\Pi_2 = -6 \frac{\partial^4}{\partial\varphi^2 \partial\eta^2} - 2 \frac{\partial^4}{\partial\varphi^2 \partial\xi^2}, \Pi_3 = \Pi_{31} + \Pi_{30} = -\left(\frac{\partial^4}{\partial\varphi\partial\eta\partial\xi^2} + \frac{\partial^4}{\partial\varphi\partial\eta^3} \right) - \alpha\Phi \frac{\partial^4}{\partial\xi^4}.$$

Upon splitting, the boundary conditions take the following form

for $\xi = 0, d$: $W_{0i} = -W_{i-2}$, $W_{0i\xi\xi} = -W_{i-2\xi\xi}$, (11)

or $W_{01} = -W_{i-2}$, $W_{0i\xi} = -W_{i-2\xi}$, (12)

for $\eta = \pm 0.5$: $W_{0i} = -W_{i-2}$; $W_{0i\eta} = -W_{i-1\varphi} - W_{i-2\eta}$. (13)

Here $i = 0,1,\ldots$; $W_i = 0$ for $i \le 0$; $d = \frac{L_1}{2L_2}$.

Consider the following homogenization operator

$$\hat{(\ldots)} = \frac{1}{N+1} \int_{-0.5(N+1)}^{0.5(N+1)} (\ldots) d\varphi. \tag{14}$$

One must takes into account that $\hat{W}_{0i} = W_{0i}$, $\hat{\Phi} = 1$, $\Pi_1 \hat{W}_i = \Pi_2 \hat{W}_i = \Pi_{31} \hat{W}_i = 0$.

The following is easily obtained from equations (10) – (13) by applying the homogenization operator defined by (14)

$$\Pi_{00} W_{00} = \left[\alpha \frac{\partial^4}{\partial \xi^4} - \lambda_0 (1+\rho) \right] W_{00} = 0, \tag{15}$$

$$\Pi_{00} W_{01} = -\nabla^4 W_{00}, \tag{16}$$

$$\Pi_{00} W_{02} = -\nabla^4 W_{00}, \tag{17}$$

$$\Pi_{00} W_{03} = \Pi_{30} \hat{W}_1 - \nabla^4 W_{02}, \tag{18}$$

....

$$\Pi_{00} W_{0k} = \Pi_{30} \hat{W}_{k-2} - \nabla^4 \left(W_{0k-1} + \hat{W}_{k-3} \right), \tag{19}$$

....

for $\xi = 0, d: W_{0i} = -\hat{W}_{i-2}, \quad W_{0i\xi\xi} = -\hat{W}_{i-2\xi\xi},$, (20)

or $W_{01} = -\hat{W}_{i-2}, \quad W_{0i\xi} = -\hat{W}_{i-2\xi},$ (21)

for $\eta = \pm 0.5: W_{0i} = -\hat{W}_{0i-2}, \quad W_{0i\eta} = -\hat{W}_{i-2\eta},$ (22)

where: $i = 0,1,\ldots$; $\hat{W}_i = 0$ for $i \le 0$.

Equations (15) and (16) are combined to yield

$$D_1 W_{0xxxx} + 2DW_{0xxyy} + DW_{0yyyy} - \Lambda_1 W_0 = 0.$$

Here $D_1 = D + \frac{E_1 I}{b}; \quad \Lambda_1 = \Lambda^2 \left(r + \frac{R}{b} \right).$

$D_1 W_{0xxxx} + 2DW_{0xxyy} + DW_{0yyyy} - \Lambda_1 W_0 = 0$,

where $D_1 = D + E_1 I/b$; $\Lambda_1 = \Lambda^2(r + R/b)$.

4. Local solution

Using boundary value problems (15) – (22), one can obtain a homogenized ("global") solution. However it is very important to calculate the local component W_i ($i \ge 1$) of the initial solution of the problem as well. Following boundary value problems exists for the functions W_i

$$W_{1\varphi\varphi\varphi\varphi} = \Pi_{01} W_{00} = \left[\alpha \frac{\partial^4}{\partial \xi^4} - \lambda_0 \rho \right] W_{00} \tag{23}$$

$$W_{2\varphi\varphi\varphi\varphi} = \Pi_1 W_1 + \Pi_{01} W_{01}, \tag{24}$$

....

$$W_{k\varphi\varphi\varphi\varphi} = \sum_{i=1}^{3} \Pi_i \overline{W}_{k-i} - \nabla^4 \overline{W}_{k-4} + \Pi_{01} W_{0k-1}, \tag{25}$$

....

for φ=± k (k = 0,1,…,0.5(N+1)) $W_i = W_{i\varphi} = 0$, (26)

where: $\overline{(\ldots)} = (\ldots) - (\hat{\ldots})$.

The conditions of compatibility are automatically satisfied. Using equation (23) and boundary conditions (26), one obtains

$$W_1 = \frac{1}{24} \Pi_{01} W_0 F_4(\varphi),$$

where: $F_4(\varphi)$ is a periodical function, $F_4(\varphi) = \varphi^2(\varphi - 1)^2$ for $0 \le \varphi \le 1$.

For the governing variables,

$$W_1 = \frac{1}{24b}\left(\frac{E_1 I}{b}\frac{\partial^4}{\partial x^4} - \frac{R}{b}\Lambda^2\right) W_0 y^2 (y-b)^2. \tag{27}$$

The expression for W_i for i >1 may be written as follows

$$W_1 = \frac{1}{24}\Pi_{01} W_{0i-1} F_4(\varphi) + \sum_{j=0}^{3} c_j^{(i)} \varphi^j + \iiiint \left[\sum_{j=1}^{3} \Pi_j \overline{W}_{i-j} - \nabla^4 W_{i-4}\right] d\varphi d\varphi d\varphi d\varphi.$$

The functions $c_j^{(i)}(\xi,\eta)$ satisfy boundary conditions (26).

7. Concluding remarks

A new approximate solution of the viscoelastic problem for a reinforced plate has been developed. The homogenization procedure and the averaging approach have made it possible to obtain an analytical form. The method can also be advantageously used for analysis of viscoelastic shells with periodic structures.

It is very important that obtained solution takes into account the real arrangement of the ribs.

Some problems in the theory of plates and shells, for which solutions were found at an initial level, are closed in certain aspects to those proposed above. However, a number of difficulties arise when studying reinforced plated and shells which can not be overcome at the "intuitive level". These relate especially to dynamic and non-linear problems with realistic boundary conditions. It is also not clear a priori which terms in the initial equations have to be remain during the subsequent simplification. These difficulties can be overcome if we construct a grounded asymptotic procedure only. So, above proposed solution is important from the engineering standpoint.

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