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CONTINUOUS APPROXIMATION FOR DISCRETE MEDIA WITH LINEAR RELAXATION

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Abstract: An n-mass damped oscillator with linear relaxation governed by 1D differential-difference equation is analyzed. Two points Padé´approximation is applied yielding new mathematical models.

1. Introduction

Construction of an accurate continuous model for discrete media is an important topic in various fields of science [1-8]. We deal with a 1D differential-difference equation governing the behavior of a n-mass oscillator with linear relaxation. It is known that a string-type approximation is justified for low part of frequency spectra of a continuous model, but for free and forced oscillations a solution of a discrete model and of a wave equation can be quite different. The difference operator makes analysis difficult due to its non-local form. Approximate equations can be gained by replacing the difference operators via a local derivative operator. Although the application of a model with derivative of more than second order improves the continuous model, a higher order of approximated differential equation seriously complicates a solution of continuous problem. It is known that accuracy of the approximation can dramatically increase using Padé approximation [2-8]. In this report, one- and two-point Padé approximation for clarifying of structural damping models proposed and analyzed in [9-12] are applied.

Many different phenomenological theories are used to describe energy dissipation in oscillating elastic bodies. Usually used viscous damping models which produce uniform damping rates are often inadequate for describing of real behavior of elastic structures. In [9-11] was proposed a new phenomenological dissipation model for beams, where the damping is assumed to be proportional to the bending rate of the beam

$$my_{tt}(x,t) - \delta y_{t\,xx}(x,t) + c_1 y_{xxxx}(x,t) = 0. \tag{1}$$

For many cases equation (1) gives solutions where the damping rate increases with frequency. We will analyze applicability of equation (1) using 1D discrete media.

2. String model

We deal with a n-mass damped oscillator. The governing equations of motion follow

$$my_{jtt} + \gamma y_{jt} = c(y_{j+1} - 2y_j + y_{j-1}) + s_j(t), \quad j = 0,1,...,n, \tag{2}$$

$$y_0 = y_n = 0, \tag{3}$$

where: $y_j(t)$ is the displacement of the *j*-th point; $s_j(t)$ is the external force acting on *j*-th point; γ the coefficient of linear relaxation; m is the mass; *c* is the rigidity.

The following initial conditions are applied

$$y_j(t) = y_{jt}(t) = 0 \quad \text{for} \quad t = 0. \tag{4}$$

Usually for large values of *n* the string-like continuous approximation to the above discrete problem is applied:

$$my_{tt}(x,t) + \gamma y_t(x,t) = ch^2 y_{xx}(x,t) + s(x,t), \tag{5}$$

$$y(0,t) = 0, \quad y(l,t) = 0, \tag{6}$$

$$y(x,0) = y_t(x,0) = 0, \tag{7}$$

where: $l = (n+1)h$.

Function $s(x, t)$ represents a continuous approximation of the function of discrete argument $s_j(t)$. Observe that it is defined with an accuracy to any arbitrary function which equals zero in nodal points $x = j$, $j = 0,1,2,...,n$. For this reason, from a set of interpolating functions one may choose, say, the most smoothened function owing to filtration of fast oscillating terms. This problem has been solved in reference [2], where the following function has been applied

$$s(x,t) = \sum_{k=1}^{n} s_k(t) \frac{\sin \frac{k\pi x}{h}}{\pi kh} .$$

The following relation between discrete and continuous systems holds

$$y_j(t) = y(jh,t), \quad j = 0,1,...,n.$$

Difference operator in system (2) can be represented by the following pseudo-differential operator

$$4c \sin^2\left(-\frac{ih}{2}\frac{\partial}{\partial x}\right) y . \tag{8}$$

The following Maclaurin series is applied

$$\sin^2\left(-\frac{ih}{2}\frac{\partial}{\partial x}\right) = -(\frac{h^2}{4}\frac{\partial^2}{\partial x^2} + \frac{h^4}{48}\frac{\partial^4}{\partial x^4} + \frac{h^6}{1440}\frac{\partial^6}{\partial x^6} + ...) \ . \tag{9}$$

Taking into account the first term in the series (9), the string-like continuous approximation (5) is obtained. Taking into account the three first terms, the following higher order approximation is found

$$m\frac{\partial^2 y}{\partial t^2} + \gamma\frac{\partial y}{\partial t} = ch^2\left(\frac{\partial^2 y}{\partial x^2} + \frac{h^4}{12}\frac{\partial^4 y}{\partial x^4} + \frac{h^6}{360}\frac{\partial^6 y}{\partial x^6}\right) + s(x,t) \ . \tag{10}$$

Initial conditions here and further have the form (7).

The problem associated with boundary conditions requires a more subtle analysis. Note that boundary conditions for n-mass oscillator (3) are transited automatically into boundary conditions for string (6). However, if a continuous approximation has a relatively high order (10), one has to define $y_k(t)$ for both $k < 0$ and for $k > n+1$. If we choose $y_k(t)$ for $k < 0$ and $y_k(t)$ for $k > n+1$ to satisfy periodicity translation conditions, then boundary conditions must keep the translation symmetry ($y_{-1}(t) = -y_1(t)$, etc.). Finally, the following boundary conditions associated with equation (10) are obtained

$$y = y_{xx} = y_{xxxx} = 0 \quad \text{for} \quad x = 0, l. \tag{11}$$

Previously we used the development of a pseudo-differential operator into the Taylor series. However, more effective results may be obtained using Padé approximation. In [3-6] the continuous models are constructed using one-point Padé approximation. Let us take into account only two terms of expansion (9). Then one has the following continuous model

$$m\frac{\partial^2 y}{\partial t^2} + \gamma\frac{\partial\ y}{\partial t} = ch^2\left(\frac{\partial^2 y}{\partial x^2} + \frac{h^4}{12}\frac{\partial^4 y}{\partial x^4}\right) + s(x,t) \ , \tag{12}$$

$$y = y_{xx} = 0 \quad \text{for} \quad x = 0, l \ . \tag{13}$$

The Padé approximation of the truncated series (9), when one takes into account only two terms of expansion, has the form

$$\frac{\partial^2}{\partial x^2} \Bigg/ \left(1 - \frac{h^2}{12}\frac{\partial^2}{\partial x^2}\right) \ .$$

It leads to the following continuous model

$$\left(1 - \frac{h^2}{12}\frac{\partial^2}{\partial x^2}\right)(my_{tt} + \gamma y_t) - ch^2 y_{xx} = c\left(1 - \frac{h^2}{12}\frac{\partial^2}{\partial x^2}\right)s(x,t) \tag{14}$$

with boundary conditions (6).

In equation (14) one has term $-\delta y_{txx}$, but also some additional terms.

Owing to references [5-8] it is clear that the application of two-point Padé approximation is more efficient than the one-point Padé approximation. In order to construct a two-point Padé approximation, two limiting points are required. In our case one of the mentioned points is defined by equation (5). The second limiting case is provided by the following consideration. Namely, $(n+1)-th$ natural frequencies of a discrete system and its continuous approximation should coincide [5-8]. It leads to equation $\omega_k = \alpha_{n+1} = 2\sqrt{c/m}$, and the sought operator is

$$ch^2 \frac{\partial^2}{\partial x^2} \Bigg/ \left(1-\alpha^2 h^2 \frac{\partial^2}{\partial x^2}\right),$$

where: $\alpha^2 = 0.25 - \pi^{-2}$.

Hence a continuous approximation is governed by the equation

$$\left(1-\alpha^2 h^2 \frac{\partial^2}{\partial x^2}\right)(my_{tt} + \gamma y_t) - ch^2 y_{xx} = \left(1-\alpha^2 h^2 \frac{\partial^2}{\partial x^2}\right) s(x,t) \tag{15}$$

with the boundary conditions (6).

3. Beam model

Now transversal vibrations of the mass chain are analysed.

The governing equations read

$$m\frac{\partial^2 y_i}{\partial t^2} + \gamma \frac{\partial\ y_i}{\partial t} + c\left(6y_i - 4y_{i+1} - 4y_{i-1} + y_{i+2} + y_{i-2}\right) = s_j(t),\ j = 1,2,...,n \tag{16}$$

with boundary conditions (3).

A typical continuous approximation of the system (16) reads

$$m\frac{\partial^2 y(x,t)}{\partial t^2} + \gamma \frac{\partial\ y(x,t)}{\partial t} + c_1 \frac{\partial^4 y}{\partial x^4} = s(x,t)\,,\quad c_1 = ch^4\,, \tag{17}$$

with boundary conditions (13).

The obtained continuous approximation can be improved. For this purpose one may formally replace a difference operator in equation (16) with a pseudo-differential operator in the following way

$$m\frac{\partial^2 y}{\partial t^2} + \gamma \frac{\partial\ y}{\partial t} + 16c\sin^4\left(-\frac{ih}{2}\frac{\partial}{\partial x}\right) y = 0\,.$$

The pseudo-differential operator can be developed into the Maclaurin series of the form

$$16\sin^4\left(-\frac{ih}{2}\frac{\partial}{\partial x}\right)=\frac{h^4\partial^4}{\partial x^4}+\frac{h^6}{6}\frac{\partial^6}{\partial x^6}+\frac{h^8}{80}\frac{\partial^8}{\partial x^8}+\ldots \quad (18)$$

Taking into account only the first term in expression (18), one obtains the classical continuous approximation (16). However, taking into account the first three terms, the obtained approximation reads

$$m\frac{\partial^2 y}{\partial t^2}+\gamma\frac{\partial\ y}{\partial t}=c_1\frac{\partial^4 y}{\partial x^4}\left(1+\frac{h^2}{6}\frac{\partial^2}{\partial x^2}+\frac{h^4}{80}\frac{\partial^4}{\partial x^4}\right)y+s(x,t)\ ,$$

with boundary conditions

$$y=y_{xx}=y_{xxxx}=y_{xxxxxx}=0 \quad \text{for} \quad x=0,l.$$

A transformation of the first two terms of the series (18) into the Padé approximant yields the following result:

$$h^4\frac{\partial^4}{\partial x^4}\left(1+\frac{h^2}{6}\frac{\partial^2}{\partial x^2}\right)\sim h^4\frac{\partial^4}{\partial x^4}\frac{1}{1-\frac{h^2}{6}\frac{\partial^2}{\partial x^2}}\ .$$

Hence, a continuous approximation reads

$$c_1\frac{\partial^4 y}{\partial x^4}+\left(1-\frac{h^2}{6}\frac{\partial^2}{\partial x^2}\right)(m\frac{\partial^2 y}{\partial t^2}+\gamma\frac{\partial\ y}{\partial t}-s(x,t))=0\ . \quad (19)$$

Two-point Padé approximants leads to the following equation

$$c_1\frac{\partial^4 y}{dx^4}+\left(1-\beta^2\frac{\partial^2}{\partial x^2}\right)(m\frac{\partial^2 y}{dt^2}+\gamma\frac{\partial\ y}{\partial t}-s(x,t))=0, \quad (20)$$

where: $\beta^2=h^2\left(\pi^4-16\right)/\left(16\pi^4\right)$.

Comparing of equations (19), (20) with equation (1) shows presence of additional terms.

4. Conclusions

Proposed in [9-11] phenomenological model of energy dissipations needs further improvement, but the proposed idea can be justified.

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