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TWO ANALYTICAL CRITERIONS OF CHAOTIC THRESHOLDS FOR DUFFING TYPE SELF-EXCITED OSCILLATOR WITH DRY FRICTION

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Abstract: In this work a Duffing type self-excited oscillator with dry friction and harmonically driven is analyzed. Two Mielnikov's criterions associated with decomposition of two homoclinic orbits occurred in the non-perturbed system are formulated. The obtained analytical results are in agreement with the earlier numerical investigations.

1. Introduction

The considered system has been already studied in references [1, 2], where using the Mielnikov's method [4] the fundamental relations defining one Melnikov's function and one homoclinic chaos threshold for mechanical systems with dry friction have been formulated. The numerical investigations of the studied system have been reported in [3], where the analytical results given in references [1, 2] have been confirmed for a certain set of parameters. Furthermore, the numerical results reported in reference [3] indicate a possibility of generation of a second homoclinic bifurcation, which appears for very small values of of the exciting amplitude. We are aimed to derive analytical conditions for numerically discovered additional homoclinic bifurcation.

2. Analyzed system

We consider harmonically driven self-excited Duffing oscillator shown in Fig. 1.

Its dynamics is governed by the following equation

$$\ddot{x} - ax + bx^3 = \varepsilon\left(\gamma\cos\omega t - \delta\dot{x} - T\left(\dot{x} - v_*\right)\right),$$

where the friction force is defined in the following way

$$T\left(\dot{x} - v_*\right) = T_0\,\mathrm{sgn}\left(\dot{x} - v_*\right) - \alpha\left(\dot{x} - v_*\right) + \beta\left(\dot{x} - v_*\right)^3.$$

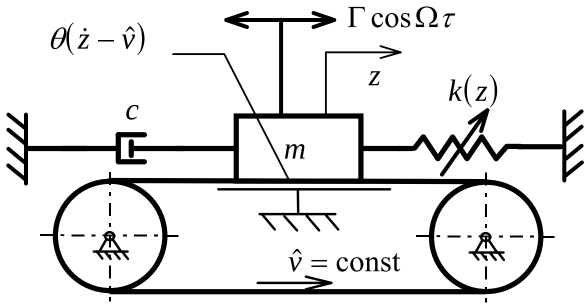

Fig. 1. Scheme of the investigated system with dry friction

For $\varepsilon = 0$ one gets the following autonomous system

$$\ddot{x} - ax + bx^3 = 0 . \tag{1}$$

Equilibria of this system are yielded by the relation

$$bx_0^3 - ax_0 = 0 .$$

There are three equilibrium positions: $x_{01} = 0$ and $x_{02,3} = \pm\sqrt{a/b}$. The first is saddle, whereas two others are centers. The system (1) possesses the constant value of full energy of the form

$$\frac{\dot{x}^2}{2} - \frac{ax^2}{2} + \frac{bx^4}{4} = \frac{C}{2} ,$$

where C is constant. Phase trajectory reads

$$\dot{x} = \pm\sqrt{C + ax^2 - \frac{bx^4}{2}} .$$

The following homoclinic trajectory is associated with saddle (for $C = 0$)

$$\dot{x} = \pm x\sqrt{a - \frac{bx^2}{2}} ,$$

and a solution to this differential equation follows

$$x_0\left(t\right) = \pm\sqrt{\frac{2a}{b}}\,\mathrm{sech}\left(\sqrt{a}t\right) . \tag{2}$$

Its differentiation yields

$$\dot{x}_0\left(t\right) = \mp a\sqrt{\frac{2}{b}}\,\mathrm{sech}\left(\sqrt{a}t\right)\mathrm{tgh}\left(\sqrt{a}t\right) . \tag{3}$$

Formulas (2) and (3) define the parametric equations of two homoclinic orbits $\left(x_0\left(t\right), \dot{x}_0\left(t\right)\right)$ associated with the saddle type equilibrium. The Mielnikov [4] function is defined in the following way

$$M(t_0)=\int_{-\infty}^{\infty}\Big\{\dot{x}_0(t)\Big[\gamma\cos\omega(t+t_0)-\delta\dot{x}_0(t)-T_0\operatorname{sgn}(\dot{x}_0(t)-v_*)+$$

$$+\alpha(\dot{x}_0(t)-v_*)-\beta(\dot{x}_0(t)-v_*)^3\Big]\Big\}dt,$$

and after some transformations one obtains

$$M(t_0)=\int_{-\infty}^{\infty}\Big[\gamma\dot{x}_0(t)\cos\omega(t+t_0)-T_0\dot{x}_0(t)\operatorname{sgn}(\dot{x}_0(t)-v_*)+$$

$$-\beta\dot{x}_0^4(t)+3\beta v_*\dot{x}_0^3(t)+(\alpha-3\beta v_*^2-\delta)\dot{x}_0^2(t)+v_*(\beta v_*^2-\alpha)\dot{x}_0(t)\Big]dt.$$

Using (2) for the sign "+" and (3) for the sign "-" one gets

$$M_+(t_0)=I_1+I_2+I_3+I_4+I_5+I_6\,, \tag{4}$$

where:

$$I_1=-\gamma a\sqrt{\frac{2}{b}}\int_{-\infty}^{\infty}\operatorname{sech}(\sqrt{a}t)\operatorname{tgh}(\sqrt{a}t)\cos\omega(t+t_0)dt\,,$$

$$I_2=T_0a\sqrt{\frac{2}{b}}\int_{-\infty}^{\infty}\operatorname{sech}(\sqrt{a}t)\operatorname{tgh}(\sqrt{a}t)\operatorname{sgn}(\dot{x}_0(t)-v_*)dt\,,$$

$$I_3=-\frac{4\beta a^4}{b^2}\int_{-\infty}^{\infty}\operatorname{sech}^4(\sqrt{a}t)\operatorname{tgh}^4(\sqrt{a}t)dt\,,$$

$$I_4=-3\cdot\left(\frac{2}{b}\right)^{\frac{3}{2}}\beta v_*a^3\int_{-\infty}^{\infty}\operatorname{sech}^3(\sqrt{a}t)\operatorname{tgh}^3(\sqrt{a}t)dt\,,$$

$$I_5=\frac{2a^2(\alpha-3\beta v_*^2-\delta)}{b}\int_{-\infty}^{\infty}\operatorname{sech}^2(\sqrt{a}t)\operatorname{tgh}^2(\sqrt{a}t)dt\,,$$

$$I_6=-v_*(\beta v_*^2-\alpha)a\sqrt{\frac{2}{b}}\int_{-\infty}^{\infty}\operatorname{sech}(\sqrt{a}t)\operatorname{tgh}(\sqrt{a}t)dt\,.$$

Relation I_1 is defined in the following way

$$I_1=I_{11}+I_{12}\,, \tag{5}$$

where:

$$I_{11}=-\gamma a\sqrt{\frac{2}{b}}\cos\omega t_0\int_{-\infty}^{\infty}\operatorname{sech}(\sqrt{a}t)\operatorname{tgh}(\sqrt{a}t)\cos\omega t dt\,,$$

$$I_{12}=\gamma a\sqrt{\frac{2}{b}}\sin\omega t_0\int_{-\infty}^{\infty}\operatorname{sech}(\sqrt{a}t)\operatorname{tgh}(\sqrt{a}t)\sin\omega t dt\,.$$

In order to compute I_{11} observe that *hyperbolic secans* and *cosinus* are even functions, whereas a *hyperbolic tangens* is an odd function. Therfore their product is an odd function, and its integral from $-\infty$ to ∞ is $I_{11} = 0$.

On the other hand I_{12}, including the integral with the same limits as earlier, can be found from integral tables, and one gets

$$I_{12} = \pi a \gamma \omega \sqrt{\frac{2}{b}} \sin \omega t_0 \operatorname{sech}\left(\frac{\pi\omega}{2\sqrt{a}}\right).$$

Substituting obtained in the above relations to (5) one gets

$$I_1 = \pi a \gamma \omega \sqrt{\frac{2}{b}} \sin \omega t_0 \operatorname{sech}\left(\frac{\pi\omega}{2\sqrt{a}}\right). \tag{6}$$

Before computing I_2 some necessary relations will be derived. Consider first the following relation

$$\dot{x}_0(t)\operatorname{sgn}\left(\dot{x}_0(t) - v_*\right) = \begin{cases} -\dot{x}_0(t), & \dot{x}_0 < v_* \\ \dot{x}_0(t), & \dot{x}_0 > v_* \end{cases},$$

which defines the under integral function depended on the parameter v_*. Note that the under integral expression sign change occurs when a velocity of mass *m* is equal to the belt velocity v_*. Using (3) one gets

$$v_* = \dot{x}_0(t) = \mp a\sqrt{\frac{2}{b}} \operatorname{sech}\left(\sqrt{a}t\right) \operatorname{tgh}\left(\sqrt{a}t\right).$$

After some transformations one obtaines

$$x^2 - x + \frac{bv_*^2}{2a^2} = 0, \tag{7}$$

where the following relation has been applied

$$x = \operatorname{sech}^2 \sqrt{a}t$$

Equation (7) is a second order polynomial with the determinant $\Delta = \left(a^2 - 2bv_*^2\right)/a^2$ responsible for its roots number For $\Delta < 0$; $\left(|v_*| > a/\sqrt{2b}\right)$ it has no real roots. If one assumes that $\Delta \geq 0$; $\left(|v_*| \leq a/\sqrt{2b}\right)$, then equation (7) has two following solutions

$$x_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{bv_*^2}{2a^2}}.$$

Therefore one obtains

$$\operatorname{sech}\sqrt{a}t_{1,2} = \sqrt{\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{bv_*^2}{2a^2}}}\,, \tag{8}$$

and

$$\begin{aligned} t_1 &= \frac{1}{\sqrt{a}} \operatorname{arcsech} \sqrt{\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{bv_*^2}{2a^2}}}\,, \\ t_2 &= \frac{1}{\sqrt{a}} \operatorname{arcsech} \sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{bv_*^2}{2a^2}}}\,. \end{aligned} \tag{9}$$

Observe that computation of integral I_2 depends on the determinant Δ. First we consider $\Delta < 0$. Recall that in this case equation (7) does not have real roots. Since the under integral function is odd, hence

$$I_2 = -T_0 a \sqrt{\frac{2}{b}} \int_{-\infty}^{\infty} \operatorname{sech}\left(\sqrt{a}t\right) \operatorname{tgh}\left(\sqrt{a}t\right) dt = 0\,.$$

In the second case, i.e. for $\Delta \geq 0$ one obtains

$$I_2 = T_0 a \sqrt{\frac{2}{b}} \left[\int_{-\infty}^{t_1} \operatorname{sech}\left(\sqrt{a}t\right) \operatorname{tgh}\left(\sqrt{a}t\right) dt - \int_{t_1}^{t_2} \operatorname{sech}\left(\sqrt{a}t\right) \operatorname{tgh}\left(\sqrt{a}t\right) dt + \int_{t_2}^{\infty} \operatorname{sech}\left(\sqrt{a}t\right) \operatorname{tgh}\left(\sqrt{a}t\right) dt \right],$$

and after integration

$$\begin{aligned} I_2 &= -T_0 \sqrt{\frac{2a}{b}} \left[\operatorname{sech}\left(\sqrt{a}t\right)\Big|_{-\infty}^{\tau_{1*}} - \operatorname{sech}\left(\sqrt{a}t\right)\Big|_{\tau_{1*}}^{\tau_{2*}} + \operatorname{sech}\left(\sqrt{a}t\right)\Big|_{\tau_{2*}}^{\infty} \right] \\ &-2T_0 \sqrt{\frac{2a}{b}} \left(\operatorname{sech}\left(\sqrt{a}t_1\right) - \operatorname{sech}\left(\sqrt{a}t_2\right) \right). \end{aligned}$$

Taking into account (9) one obtains

$$I_2 = 2T_0 \sqrt{\frac{2a}{b}} \left(\sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{bv_*^2}{2a^2}}} - \sqrt{\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{bv_*^2}{2a^2}}} \right).$$

Finally, the earlier considerations yield

$$I_2 = \begin{cases} 2T_0 \sqrt{\frac{2a}{b}} \left(\sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{bv_*^2}{2a^2}}} - \sqrt{\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{bv_*^2}{2a^2}}} \right) & \textit{for} \quad v_* < \frac{a}{\sqrt{2b}} \\ 0 & \textit{for} \quad v_* \geq \frac{a}{\sqrt{2b}} \end{cases}, \tag{10}$$

On the other hand, the integral I_3 reads

$$I_3 = -\frac{4a^{\frac{7}{2}}\beta}{35b^2}\left(6+\cosh\left(2\sqrt{a}t\right)\right)\text{sech}^2\left(\sqrt{a}t\right)\text{tgh}^5\left(\sqrt{a}t\right)\Bigg|_{-\infty}^{\infty},$$

and hence

$$I_3 = -\frac{16a^{\frac{7}{2}}\beta}{35b^2}. \tag{11}$$

Furthermore

$$I_4 = I_6 = 0, \tag{12}$$

because one should compute an integral of the odd function in the limits from $-\infty$ to ∞. Integral of I_5 reads

$$I_5 = \frac{2a^{\frac{5}{2}}}{3b}\left(\alpha - 3\beta v_*^2 - \delta\right)\text{tgh}^3\left(\sqrt{a}t\right)\Bigg|_{-\infty}^{\infty},$$

and hence

$$I_5 = \frac{4a^{\frac{5}{2}}}{3b}\left(\alpha - 3\beta v_*^2 - \delta\right). \tag{13}$$

Substituting relations (6), (10), (11), (12) and (13) to (4) one obtains

$$M_+(t_0) = \pi a\gamma\omega\sqrt{\frac{2}{b}}\sin\omega t_0\,\text{sech}\left(\frac{\pi\omega}{2\sqrt{a}}\right) - \frac{16a^{\frac{7}{2}}\beta}{35b^2} + \frac{4a^{\frac{5}{2}}}{3b}\left(\alpha - 3\beta v_*^2 - \delta\right) +$$
$$+\begin{cases} 2T_0\sqrt{\frac{2a}{b}}\left(\sqrt{\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{bv_*^2}{2a^2}}} - \sqrt{\frac{1}{2}-\sqrt{\frac{1}{4}-\frac{bv_*^2}{2a^2}}}\right) & \text{for} \quad v_* < \frac{a}{\sqrt{2b}} \\ 0 & \text{for} \quad v_* \geq \frac{a}{\sqrt{2b}} \end{cases}.$$

Stable and unstable manifolds intersection condition has the following form

$$\pi\gamma\omega\sqrt{\frac{2}{b}}\,\text{sech}\left(\frac{\pi\omega}{2\sqrt{a}}\right) > \left| -\frac{16a^{\frac{5}{2}}\beta}{35b^2} + \frac{4a^{\frac{3}{2}}}{3b}\left(\alpha - 3\beta v_*^2 - \delta\right) + \right.$$
$$\left. +\begin{cases} 2T_0\sqrt{\frac{2}{ab}}\left(\sqrt{\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{bv_*^2}{2a^2}}} - \sqrt{\frac{1}{2}-\sqrt{\frac{1}{4}-\frac{bv_*^2}{2a^2}}}\right) & \text{for} \quad v_* < \frac{a}{\sqrt{2b}} \\ 0 & \text{for} \quad v_* \geq \frac{a}{\sqrt{2b}} \end{cases} \right|.$$

Now we are going to compute the Mielnikov function defined by (2) for the sign "+" and (3) for the sign"-". In this case one gets

$$M_-(t_0) = -I_1 - I_2 + I_3 - I_4 + I_5 - I_6. \tag{14}$$

Substituting relations (6), (10), (11), (12) and (13) to (14) one obtains

$$M_-\left(t_0\right) = -\pi a\gamma\omega\sqrt{\frac{2}{b}}\sin\omega t_0 \operatorname{sech}\left(\frac{\pi\omega}{2\sqrt{a}}\right) - \frac{16a^{\frac{7}{2}}\beta}{35b^2} + \frac{4a^{\frac{5}{2}}}{3b}\left(\alpha - 3\beta v_*^2 - \delta\right) +$$

$$-\begin{cases} 2T_0\sqrt{\frac{2a}{b}}\left(\sqrt{\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{bv_*^2}{2a^2}}} - \sqrt{\frac{1}{2}-\sqrt{\frac{1}{4}-\frac{bv_*^2}{2a^2}}}\right) & for \quad v_* < \frac{a}{\sqrt{2b}} \\ 0 & for \quad v_* \ge \frac{a}{\sqrt{2b}} \end{cases}.$$

Second condition of stable and unstable manifolds intersection follows

$$\pi\gamma\omega\sqrt{\frac{2}{b}}\operatorname{sech}\left(\frac{\pi\omega}{2\sqrt{a}}\right) > \left| -\frac{16a^{\frac{5}{2}}\beta}{35b^2} + \frac{4a^{\frac{3}{2}}}{3b}\left(\alpha - 3\beta v_*^2 - \delta\right) + \right.$$

$$\left. -\begin{cases} 2T_0\sqrt{\frac{2}{ab}}\left(\sqrt{\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{bv_*^2}{2a^2}}} - \sqrt{\frac{1}{2}-\sqrt{\frac{1}{4}-\frac{bv_*^2}{2a^2}}}\right)\right| & for \quad v_* < \frac{a}{\sqrt{2b}} \\ 0 & for \quad v_* \ge \frac{a}{\sqrt{2b}} \end{cases}.$$

The obtained so far Mielnikov's criterions can be summanted in the following way

$$\pi\gamma\omega\sqrt{\frac{2}{b}}\operatorname{sech}\left(\frac{\pi\omega}{2\sqrt{a}}\right) > \left| -\frac{16a^{\frac{5}{2}}\beta}{35b^2} + \frac{4a^{\frac{3}{2}}}{3b}\left(\alpha - 3\beta v_*^2 - \delta\right) + \right.$$

$$\left. \pm\begin{cases} 2T_0\sqrt{\frac{2}{ab}}\left(\sqrt{\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{bv_*^2}{2a^2}}} - \sqrt{\frac{1}{2}-\sqrt{\frac{1}{4}-\frac{bv_*^2}{2a^2}}}\right)\right| & for \quad v_* < \frac{a}{\sqrt{2b}} \\ 0 & for \quad v_* \ge \frac{a}{\sqrt{2b}} \end{cases}. \qquad (15)$$

Fig. 2 shows thresholds of chaos in the $\left(\gamma, v_*\right)$ plane obtained from relation (15) for $a = b = 1$, T_0, $\alpha = 0.1$, $\beta = 0.2$, $\delta = 0.15$, $\omega = 1$.

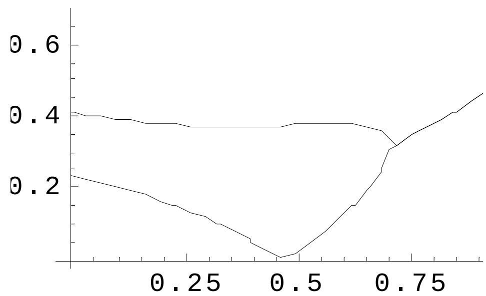

Fig. 2. Thresholds of chaos in the $\left(\gamma, v_*\right)$ plane
for $a = b = 1$, T_0, $\alpha = 0.1$, $\beta = 0.2$, $\delta = 0.15$, $\omega = 1$.

3. Conclusions

Application of the Melnikov's method to discontinuous self-excited Duffing oscillator harmonically driven and taking into account two homoclinic orbits allowed to get two different analytical criterions for chaos occurrence associated with destruction of two homoclinic orbits in the system without pertutrbation. The obtained results are in full agreement with the results obtained in reference [3] for relatively small belt velocities v_* .

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