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# CHAOS IN A THREE DEGREES-OF-FREEDOM MECHANICAL SYSTEM: EXPERIMENTAL AND NUMERICAL INVESTIGATIONS

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## ABSTRACT

*Chaotic dynamics of a triple physical pendulum is studied. The introduced mathematical model of non-linear ODEs is solved numerically and the obtained results are compared with those yielded by mesurement equipment of the experimantal rig. Parameters of the model are identified by the minimization of the sum of squares of deviations between the signal from the simulation and the signal obtained from the experiment. A good agreement between results from the experiment and from simulation is shown in a few examples.*

## KEYWORDS:

Experimental investigations, model identification triple pendulum.

## 1 INTRODUCTION

It is well known, that simple pendulum harmonically excited can exhibit regular or irregular behavior, the classical bifurcations (like saddle-node, symmetry breaking and period-doubling bifurcations), coexisting attractors, etc. [1-5].

It can be noticed also an interest focused on experimental investigation of either simple or coupled mechanical and electronic setup of pendulums. As the single or double pendulum in their different forms are quite often a subject of experiment [3,6,7], the triple physical pendulum is rather rarely presented in literature as experimental model. For example in the work [8] the triple pendulum excited by horizontal harmonic motion of the pendulum frame is presented with a few examples of chaotic attractors.

The present paper is a part of larger project of investigations [9,10], where the triple physical pendulum with rigid limiters of motion was analyzed numerically.

## 2 EXPERIMENTAL INVESTIGATIONS

The experimental rig of the triple physical pendulum consists of the following subsystems: pendulum, driving movement subsystem of the pendulum, damping subsystem, and the measurement subsystem.

Analog signals incoming from measuring devices (angle sensors) are processed in LabView measurement software. The modular LabView measurement package is presently widely used in industry as complete programmable set of test instruments which are, in particular, in cooperation with the well developed block-diagram building software. Dynamic data acquisition is made with the use of the following test instruments: chassis PXI-1011, SCXI-1125 module installed in the chassis, the terminal block SCXI-1313 (high-attenuator), which are in cooperation with the PCI-6052E PC computer's card.

The LabView software environment offers the complete library of numerical and mathematical tools which allow processing of experimental data. Blocks are connected by lines of various colors and pattern in the environment and represent some pre-defined application procedures (reading and writing to channel inputs and outputs, numerical analysis). The series of measured data are possible to be stored in text files and then showed on any waveforms graphs.

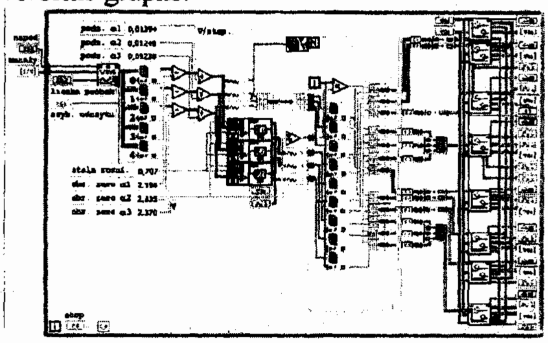

Figure 1. Main block diagram in the LabView scheme.

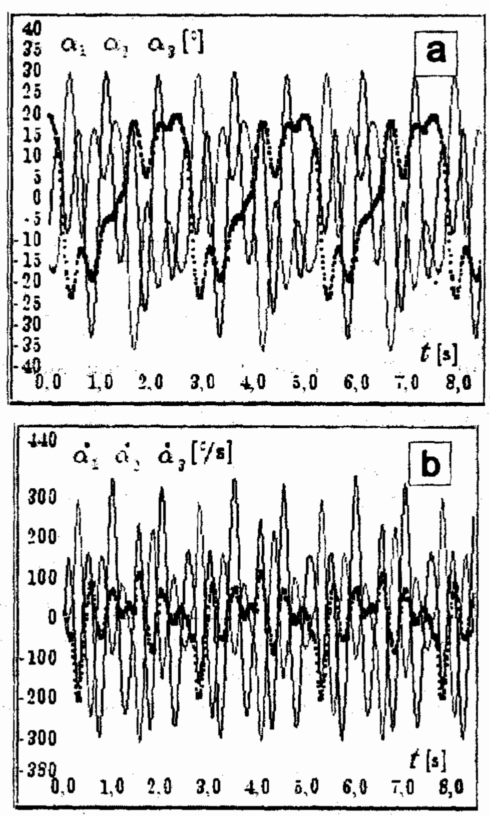


Figure 2. Time histories (a) and velocity dependencies (b) of the pendulum's links.

voltage

The main diagram of the measurement-calculating blocks is presented in Figure 1. This diagram has been used to measure all signals incoming from angle sensors installed directly on the three pendulum links. The signals are then stored as data in the floating point representation in computer's memory.

In Figure 1 in the main loop running until the break of measurements appears the following general blocks are observed (starting from the left hand side): Wave -for measurement of data from hardware channels; Array Element -for simple extraction of the measured data appearing in measurement channels as a waveform generated with a sample period; Constant, Substract, Divide - mathematical transformations for signals scaling in degrees; Derivative, IR Filter -numerical transformations of series of the stored data for the velocity of the three pendulum's links calculation and the filtering process of them, respectively; For - conditional loop for data preparing to draw; Bitmap Graph -graphical interfaces with the optional parameters of graphs.

The previously described process of data acquisition has been used in measurements on the pendulum's real laboratory rig. Graphs which were made thanks to graphical interface of the diagram presented in Figure 1 are presented in Figures 2-4.
It visible in Figure 2 that the pendulum's links motion is 3-, 4-, and 5-periodic as well. The amplitude of the first link is about $20^\circ$, while the amplitudes of the others is at the same value about $30^\circ$. Changes of angular velocity of the pendulum's links are visible in Figure 2b. It can be observed that the first link rotates with the velocity above three times less then the next two. A comparison of the time changes of both angle and angular velocity of the pendulum's links presented in Figure 2 brings an natural conclusion, the same angles of rotation of links 2 and 3 have another values of angular velocities.

On the basis of location characteristics presented in Figure 3 the sets of relative location of the appropriate links are possible to find. For example, if the first pendulum's link was observed in the zero angle position, the third of them could be observed only in positions of $+5^\circ$ and $-15^\circ$ (see Figure 3). In such a way and on the basis of various combinations it possible to determine other location compositions of links of the investigated pendulum.

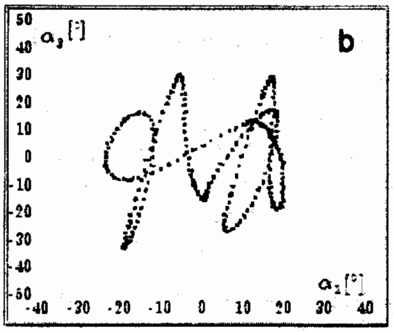


Figure 3. Location characteristics of the pendulum's links.

An experimental phase space projection on the phase planes of pendulum's links has been shown in Figure 4. According to the earlier conclusions, pendulum's links rotate in a periodical manner what is exactly proved by the loops visible in the figure.

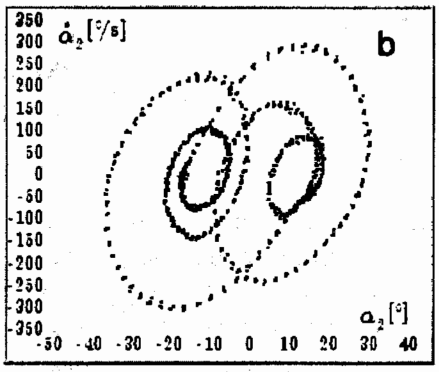


Figure 4. Phase space projections on the planes $\dot{\alpha}_1(\alpha_1)$, $\dot{\alpha}_2(\alpha_2)$, $\dot{\alpha}_3(\alpha_3)$.

## 3 MATHEMATICAL MODEL

As a physical model of the real system presented in previous section, we use the system of three rotationally coupled rigid bodies moving in the vacuum, in the gravitational field of acceleration $g$, shown in Figure 5. The pendulum moves in the plane and its position is determined by three angles $\psi_i$ $(i=1,2,3)$. In the joints $O_i$ the viscous damping with the coefficient $c_i$ $(i=1,2,3)$ is present. It is assumed that the mass centers of the links lie on the lines including the joints and one of the principal central inertia axes ($z_{ci}$) of each link is perpendicular to the pendulum movement plane. The masses of the pendulums are $m_i$ and the moments of inertia with respect to the axes $z_{ci}$ are $J_{zi}$ $(i=1,2,3)$, respectively.

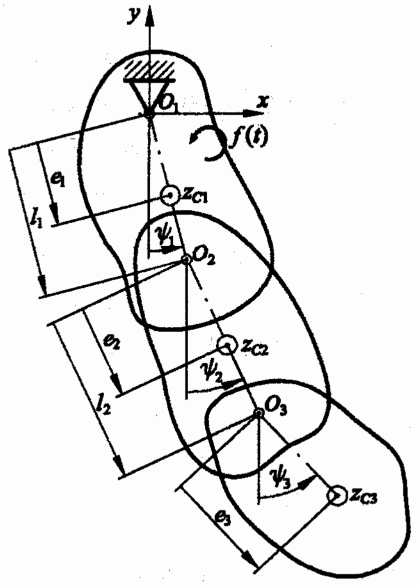


Figure 5. Model of the triple physical pendulum.

The first pendulum is externally excited by the square-shape moment of the force $f(t)$ with the amplitude $q$, angular velocity $\omega$ and with the initial phase $\phi_0$, as shown in Figure 6.

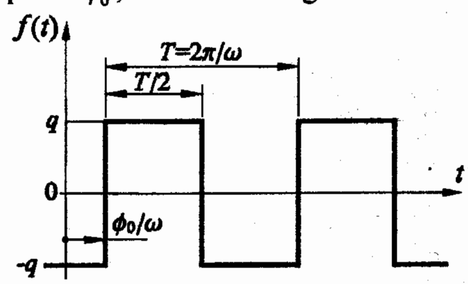


Figure 6. The external forcing function.

The governing equations of the system presented in Figure 5 follow

$$\mathbf{M}(\psi)\ddot{\psi} + \mathbf{N}(\psi)\dot{\psi}^2 + \mathbf{C}\dot{\psi} + \mathbf{p}(\psi) = \mathbf{f}_e(t) \qquad (1)$$

Where:

$$\mathbf{M}(\psi) = \begin{bmatrix} B_1 & N_{12}\cos(\psi_1-\psi_2) & N_{13}\cos(\psi_1-\psi_3) \\ N_{12}\cos(\psi_1-\psi_2) & B_2 & N_{23}\cos(\psi_2-\psi_3) \\ N_{13}\cos(\psi_1-\psi_3) & N_{23}\cos(\psi_2-\psi_3) & B_3 \end{bmatrix},$$

$$\mathbf{N}(\psi)= \begin{bmatrix} 0 & N_{12}\sin(\psi_1-\psi_2) & N_{13}\sin(\psi_1-\psi_3) \\ -N_{12}\sin(\psi_1-\psi_2) & 0 & N_{23}\sin(\psi_2-\psi_3) \\ -N_{13}\sin(\psi_1-\psi_3) & -N_{23}\sin(\psi_2-\psi_3) & 0 \end{bmatrix},$$

$$\mathbf{C}=\begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}, \quad \mathbf{p}(\psi)=\begin{Bmatrix} M_1\sin\psi_1 \\ M_2\sin\psi_2 \\ M_3\sin\psi_3 \end{Bmatrix},$$

$$\mathbf{f}_e(t)=\begin{Bmatrix} f(t) \\ 0 \\ 0 \end{Bmatrix}, \quad \psi=\begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{Bmatrix}, \quad \dot\psi=\begin{Bmatrix} \dot\psi_1(19) \\ \dot\psi_2(20) \\ \dot\psi_3 \end{Bmatrix}, \tag{2}$$

$$\ddot\psi=\begin{Bmatrix} \ddot\psi_1 \\ \ddot\psi_2 \\ \ddot\psi_3 \end{Bmatrix}, \quad \dot\psi^2=\begin{Bmatrix} \dot\psi_1^2 \\ \dot\psi_2^2 \\ \dot\psi_3^2 \end{Bmatrix}$$

The parameter vector of the pendulum is

$$\mu=[B_1,B_2,B_3,N_{12},N_{13},N_{23},M_1,M_2,M_3, c_1,c_2,c_3], \tag{3}$$

where

$$\begin{aligned} B_1 &= J_{z1}+e_1^2 m_1 + l_1^2(m_2+m_3), \\ B_2 &= J_{z2}+e_2^2 m_2 + l_2^2 m_3, \\ B_3 &= J_{z3}+e_3^2 m_3, \\ N_{12} &= m_2 e_2 l_1 + m_3 l_1 l_2, \\ N_{13} &= m_3 e_3 l_1, \\ N_{23} &= m_3 e_3 l_2, \\ M_1 &= m_1 g e_1 + (m_2+m_3) g l_1, \\ M_2 &= m_2 g e_2 + m_3 g l_2, \\ M_3 &= m_3 g e_3. \end{aligned} \tag{4}$$

Note that the parameters of the external forcing ($q,\omega,\phi$) are treated separately. For more details on the triple pendulum equations and their derivation see works [9,10].

## 4 PARAMETER IDENTIFICATION

In order to identify the model parameters $\mu$, firstly we lead to the experimental stand the external forcing (input signal) of the known parameters: $q=2$ Nm, $f_r=0.4$ Hz and $\phi_0=0.163$ rad (where $f_r=\omega/2\pi$). Then the response (output signal) $\tilde\psi_i(t)$ ($i=1,2,3$) of the real system starting from zero initial conditions ($\dot{\tilde\psi}_i(0)=0$, $\tilde\psi_i(0)=0$, $i=1,2,3$) is measured on time interval $t\in[0,t_e]$ as the following time series

$$\{\tilde\psi_i^{(0)},\tilde\psi_i^{(1)},\tilde\psi_i^{(2)},\dots,\tilde\psi_i^{(N)}\}, \quad (i=1,2,3), \tag{5}$$

where

$$\tilde\psi_i^{(k)}=\tilde\psi_i(k\Delta t), \quad k=0,1,2,\dots,N, \quad \Delta t=\frac{t_e}{N},$$

and where $t_e=100$ s and $N=10^4$ ($\Delta t=0.01$ s) were taken.

We want to find such a vector $\mu$, for which the model matches the real system most of all in the sense of the least deviation between the output signals from the model and the real pendulum. Therefore we define the following criterion-function of matching the output signals

$$F(\mu)=\sum_{i=1}^{3}\sum_{k=0}^{N}(\psi_i^{(k)}-\tilde\psi_i^{(k)})^2 \tag{6}$$

where

$$\psi_i^{(k)}=\psi_i(k\Delta t)$$

is the output signal of the model for the same input signal and initial conditions, obtained from numerical simulation. The global minimum of $F(\mu)$ is now the solution of the problem.

The MATLAB environment for evaluation and minimum searching of the function $F(\mu)$ is used. For minimum finding the simplex search method is used that does not use numerical or analytic gradients, and finds a local minimum of a scalar function, starting at an initial estimate. First we roughly estimate the parameter vector and use it as a starting point $\mu_0$ in the procedure to find $\mu_{opt}$ with the following elements

$$\begin{aligned} &B_1=0.2705\,\text{kgm}^2, && B_2=0.1547\,\text{kgm}^2, \\ &B_3=0.0480\,\text{kgm}^2, && N_{12}=0.1176\,\text{kgm}^2, \\ &N_{13}=0.0776\,\text{kgm}^2, && N_{23}=0.0452\,\text{kgm}^2, \\ &M_1=11.020\,\text{kgm}^2/\text{s}^2, && M_2=5.6508\,\text{kgm}^2/\text{s}^2, \\ &M_3=3.7383\,\text{kgm}^2/\text{s}^2, && c_1=0.1203\,\text{kgm}^2/\text{s}, \\ &c_2=0.0025\,\text{kgm}^2/\text{s}, && c_3=0.0028\,\text{kgm}^2/\text{s}. \end{aligned} \tag{7}$$

which minimize the function $F(\mu)$ at least locally and, as shown in the next section, gives a good agreement between the model and the real system.

## 5 NUMERICAL RESULTS

Here some examples of numerical simulations of the triple pendulum model together with corresponding experimental results from the real object are presented. The model simulations are performed for the optimal parameter vector $\mu_{opt}$ (7), zero initial conditions ($\psi_i(0)=0, \dot{\psi}_i(0)=0, i=1,2,3$), for the excitation amplitude $q = 2$ Nm and the initial phase $\phi_0 = 0.163$rad. The experimental results are obtained for the same input signal and zero initial conditions.

Figures 7-9 show periodic solutions obtained numerically (black line) compared with the experimental data (grey line) for the excitation frequency $f_r$ equal to 0.4, 0.6 and 0.8Hz, respectively. In order to eliminate a transient motion, solutions are restricted to the time interval $t \in [200s, 300s]$. As it is seen, the results from the model match the results from the object very well.

Moreover, the numerical simulation shows existence of a chaotic region near the frequency $f_r = 0.7$Hz. It is in a good agreement with the obtained experimental results.

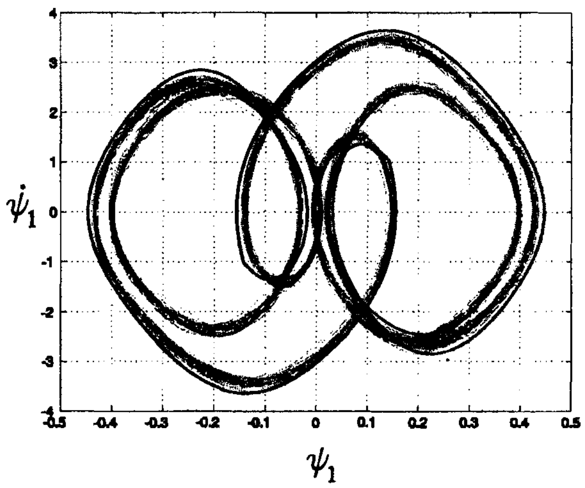


Figure 7. The experimental (grey line) and numerical (black line) results for $f_r = 0.4$ Hz.

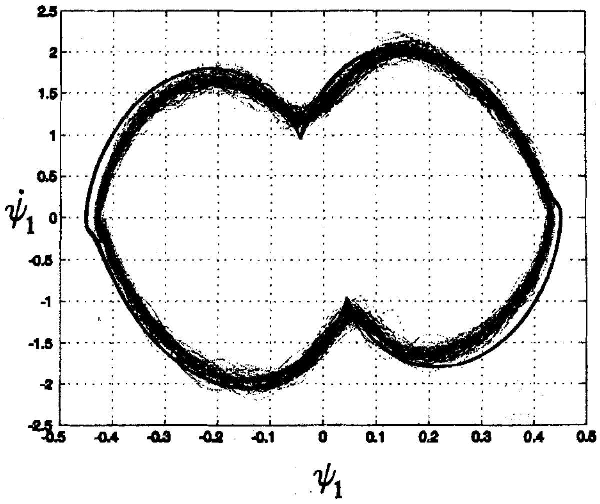


Figure 8. The experimental (grey line) and numerical (black line) results for $f_r = 0.6$ Hz.

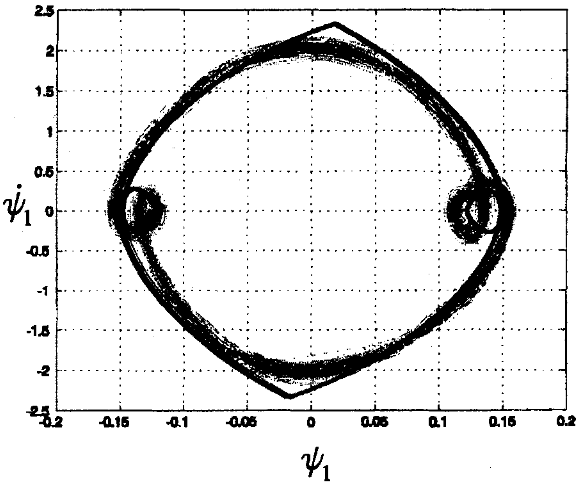


Figure 9. The experimental (grey line) and numerical (black line) results for $f_r = 0.8$ Hz.

## 6 CONCLUDING REMARKS

In the paper an experimental stand of the triple physical pendulum and the corresponding mathematical model have been presented. The pendulum is a rich source of nonlinear dynamics phenomena. Furthermore, it can be used for modeling of a number of real objects being observed in nature and engineering. One can mention for example about arms of robots and cranes. A subtle reconstruction of that pendulum's experimental rig can be made by adding a few obstacles bounding the rotational motion of its last link. Such an operation will provide a possibility for modeling of the piston -connecting rod - crankshaft system moving in a cylinder with backlash.

The model parameters identification have been performed by the minimization of the scalar criterion-function in the form of the sum of squares of deviations between the numerical and experimental series. It should be noted, that the parameter vector $\mu_{opt}$, which minimizes the function (29), is optimal in the sense of best matching of output signals of the model and the real object. Therefore, it unnecessarily matches best the real physical parameters of the object, that can be computed from (27). As shown in section 5, the numerical simulation of the model for parameters $\mu_{opt}$, can be successfully used for the prediction of behavior of the real triple physical pendulum.

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