CHAOTIC VIBRATIONS OF BEAMS, PLATES AND SHELLS

Jan Awrejcewicz
Technical University of Lodz
Department of Automatics and Biomechanics
Lodz, Poland
awrejcew@p.lodz.pl

Anton V. Krysko
Saratov State University,
Department of Mathematics,
410054, Saratov, Russia
tak@san.ru

Abstract
An iterative algorithm to solve efficiently one-sided interaction between two rectangular plates within the Kirchhoff hypothesis is proposed. Then a proof of convergence of this algorithm is given. The formulated theorem, proof and five remarks exhibit advantages of our proposed novel approach.

Key words
plate, Kirchhoff and Timoshenko clearance, partial differential equations.

1 Introduction

Non-linear dynamics of plates and shells is exhibited in various engineering structures (buildings, bridges, tanks) and machines such as flight-vehicles, power plants and various mechatronical devices.

In many engineering cases an interaction of multi-layer plates play an important role, and hence they require both careful mathematical modeling and analysis. The mentioned interaction is associated with one- and two-sided constraints creating a complex but challenging topic of research from the point of view of theory and application.

In reference [Krysko, Awrejcewicz 2005] complex vibrations of an Euler-Bernoulli beam with different types of non-linearities are considered. An arbitrary beam clamping is considered, and a deflection constraints (point barriers) are introduced in some beam points along its length. The influence of a constraint, as well as of the amplitude and frequency of excitation on the vibrations is analysed. Scenarios of the transition to chaos owing to the introduced non-linearities are reported.

In reference [Krysko et al., to appear] regular and chaotic vibrations together with bifurcations of flexible plate-strips with non-symmetric boundary conditions are investigated by the Bubnov-Galerkin method and a finite difference method of order $O\left(h^4\right)$. Special attention is paid to non-symmetric boundary conditions. Lyapunov exponents are estimated via Bennetin's method. Some new examples of routes from regular to chaotic dynamics, and within chaotic dynamics are illustrated and discussed. The phase transitions from chaos to hyperchaos, and a novel phenomenon of a shift from hyper chaos to another hyperhyper chaos is also reported.

In the work by Awrejcewicz et al. [to appear] a novel iteration procedure is proposed for dynamical problems, where in each time step a contacting plate zone is improved. Therefore, a zone and magnitude of a contact load is also improved. The effect of boundary conditions on externally driven vibrations of uncoupled two-layer plates, with the Kirchhoff hypothesis holding for each layer, is investigated.

Below, we consider mechanical one-sided interaction between two rectangular plates. It is assumed that the plates are thin and their stress-strain state is governed by classical Kirchhoff's theory supplemented by physical nonlinearities introduced by the theory of small elastic-plastic deformations. We assume, in addition, that a contact pressure (normal to stress surface) is significantly less in comparison to normal stresses measured in cross-sections of a plate, and that the plates slip in contact zones in a free way.

Note that the choice of the classical theory of plates is motivated by an observation that an influence of transversal shear deformation on the stress-strain state and on the distribution of contact pressure is essentially less important than that of transversal clamping in a contact zone. The latter factor plays a key role in our investigations.

2 Mathematical model

Differential equations governing behavior of contacting plates have the form

$$\begin{aligned} A_1 w_1(x,y) &= q_1(x,y) - q_k \psi \\ A_2 w_2(x,y) &= q_2(x,y) + q_k \psi \end{aligned}, \tag{1}$$

where: $w_i(x,y)$ – vector functions, $q_i(x,y)$ - vector of external load, i – plate number measured in direction of a positive normal axis. Contact pressure being proportional to transversal clamping $w_1 - h_1 - w_2$ in a contact zone is as follows

$$q_k(x,y) = k\frac{E}{h}(w_1 - h_1 - w_2)\,, \qquad (2)$$

and the function ψ has the form

$$\psi = [1 + sign(w_1 - h_1 - w_2)]/2\,, \qquad (3)$$

where h_1 denotes clearance between plates.

Observe that formula (2) holds for plates contact with the same values of k and h. Contact problems of the Kirchhoff theory of plates are associated with Winkler's type coupling between clamping and contact pressure.

If the initial plates distribution (clearance function h_1) and the load are such that there is no contact after deformation then $\psi \equiv 0$, and system (1) is uncoupled. Otherwise, system (1) contains coupled equations. Substituting (2) into (1) we have

$$A_1 w_1(x,y) = q_1(x,y) - k\frac{E}{h}(w_1 - h_1 - w_2)\psi\,,$$
$$A_2 w_2(x,y) = q_2(x,y) + k\frac{E}{h}(w_1 - h_1 - w_2)\psi\,. \qquad (4)$$

Eighth-order system (4) is further studied in the frame of the following boundary conditions:

a) ball-type support

$$w\big|_{\partial\Omega} = \frac{\partial^2 w}{\partial n^2}\Big|_{\partial\Omega} = 0\,;$$

b) stiff clamping

$$w\big|_{\partial\Omega} = \frac{\partial w}{\partial n}\Big|_{\partial\Omega} = 0\,.$$

Therefore, the mentioned equations define a physically and constructively nonlinear eighth-order problem.

Operator A_i (i = 1,2), for a physically nonlinear problem, has the following form [Krysko, 1976]:

$$A_i w_i(x,y) = \frac{\partial^2}{\partial x^2}\left(B_{11,i}\frac{\partial^2 w_i}{\partial x^2} + B_{10,i}\frac{\partial^2 w_i}{\partial y^2}\right) + \frac{\partial^2}{\partial y^2}\left(B_{10,i}\frac{\partial^2 w_i}{\partial x^2} + B_{11,i}\frac{\partial^2 w_i}{\partial y^2}\right) + 2\frac{\partial^2}{\partial x \partial y}\left((B_{11,i} - B_{10,i})\frac{\partial^2 w_i}{\partial x \partial y}\right)\,, \qquad (5)$$

where:

$$B_{mn,i} = \frac{1}{2}\left[\frac{E_{11,i}}{E_{01,i}} + (-1)^{n+1}\frac{E_{10,i}}{E_{00,i}} - E_{21,i} + (-1)^n E_{20,i}\right]$$

$$E_{mn,i} = \int_{a_i(x,y)}^{b_i(x,y)} \frac{E_i z^m}{1 + (-1)^n \lambda_i}\,dz\,,$$

$$(n = 0.1; \quad m = i = 1,2),$$

and where $z = a_i(x, y)$, $z = b_i(x, y)$, $(x, y)\in\Omega$ are the equations governing external plate surfaces, and allowing for introduction (during computations) the variated thickness of plates.

Owing to application of the changeable elasticity parameters method [Birger, 1951] $E_i(x, y, z, e_i)$, $\nu_i(x,y,z,e_i)$ are treated as parameters depending on the deformed plate states and they have the following form

$$E_i = \frac{9K_{1i}G_i}{3K_{1i} + G_i}\,, \quad \nu_i = \frac{1}{2}\frac{3K_{1i} - 2G_i}{3K_{1i} + G_i}\,, \qquad (6)$$

where K_{1i} = const. Recall, that in the theory of deformation, the shear modulus is

$$G_i = \frac{1}{3}\frac{\sigma_i^{(i)}(e_i^{(i)})}{e_i^{(i)}}\,, \qquad (7)$$

where $\sigma_i^{(i)}$, $e_i^{(i)}$ are the intensities of plate stress and strain intensities $(i = 1,2)$, respectively, and

$$e_{xz}^{(i)} = -z\frac{\partial^2 w_i}{\partial x^2}\,,\; e_{yy}^{(i)} = -z\frac{\partial^2 w_i}{\partial y^2}\,,\; e_{xy}^{(i)} = -2z\frac{\partial^2 w_i}{\partial x \partial y}\,. \qquad (8)$$

In (8), $e_{zz}^{(i)}$ are defined by flat stress condition state of the form ($\sigma_{zz} = 0$):

$$e_{zz}^{(i)} = -\frac{\nu_i}{1-\nu_i}(e_{xx}^{(i)} + e_{yy}^{(i)})\,. \qquad (9)$$

System (4) is solved by the method of variational iterations (MVI). In order to solve the constructively nonlinear problem (4) one may apply an iterative process allowing to solve only one equation from system (4) on each loading step. The same approach in the case of geometrically non-linear problems of shells has been proposed and applied in reference [Bochkarev, Krysko, 1981].

Application of the mentioned technique yields two times reduction of the system in the case of two layers package, and it yields n-times order reduction in the case of a package composed of n layers. Iterative procedure has the following form

$$A_1(w_1^{(n)}) + k\frac{E}{h}\psi_{(n-1)}w_1^{(n)} = q_1(x,y) + k\frac{E}{h}(h_1 + w_2^{(n-1)})\psi_{(n-1)},$$

$$A_2(w_2^{(n)}) + k\frac{E}{h}\psi_{(n-1)}w_2^{(n)} = q_2(x,y) + k\frac{E}{h}(w_1^{(n-1)} - h_1)\psi_{(n-1)}. \tag{10}$$

One has to attach to system (10) the corresponding boundary conditions for the *i*-th plate.

Let R^2 be the Euclidean surface with Descartes basis; $\Omega_i \in R^2$ is the area in this surface with its boundary $\partial\Omega_i$ (i = 1,2), $\overline{\Omega_i} = \Omega_i \bigcup \partial\Omega_i$ (x, y)$\in\Omega_i$, Ω^* is the sub-area of Ω_i, $\forall i$, $\Omega^* \subseteq \Omega_i$, and n_i is the external normal to $\partial\Omega_i$.

As it has been mentioned already, equations (10) with the associated boundary conditions will be solved using MVI [Krysko, 2000; Kantorovich, 1941]. For a fixed contacting zone both MVI and changeable elasticity parameters (CEP) procedures are applied with a successive improvement of a contacting zone by simple iterative procedure. Next, the solving procedure is repeated, i.e. we have three iterative procedures embedded in each other. Ordinary differential equations are reduced to algebraic ones by a finite difference method with accuracy of $O(\delta^2)$, where δ is the mesh step. Algebraic equations are solved by the Gauss procedure on each time step.

3 Proof of convergence of iterative algorithms

We are going to prove the convergence of iterative algorithms used to solve contact problems of freely coupled plates within Kirchhhoff's hypotheses. Next, we consider 3D plates construction consisting of contacting plates prescribed by Kirchhoff's hypotheses, i.e. system (4) with an account of (5) has the following form

$$\begin{aligned} &A_1(w_1(x,y)) + k\frac{E}{h}w_1\psi(x,y) = \\ &q_1 + k\frac{E}{h}(w_2 + h_1)\psi(x,y), \\ &A_2(w_2(x,y)) + k\frac{E}{h}w_2\psi(x,y) = \\ &q_2(1-\psi(x,y)) + k\frac{E}{h}(w_1 - h_1)\psi(x,y), \end{aligned} \tag{11}$$

with the associated boundary conditions

$$w_i\Big|_{\partial\Omega_i} = \frac{\partial w_i}{\partial n_i}\Big|_{\partial\Omega_i} = 0 \quad \text{(i = 1,2)}, \tag{12}$$

and with the function defining the contacting plates zone Ω^* of the form

$$\psi(x,y) = \begin{cases} 1, & (x,y)\in\Omega^*, \\ 0, & (x,y)\notin\Omega^*, \end{cases}$$

where $q_1(x, y)$, $q_2(x, y)$ are the functions of external loads acting on the first and second contacting plate, respectively; $\|\cdot\|_A$ is the norm in the normalized space A; $(\cdot,\cdot)_B$ denotes scalar product in the Hilbert space B (notation of functional spaces corresponds to that used in reference [Ladyzhenskaya, Uraltseva, 1973]).

In order to solve problems (11) and (12), the following iterative algorithm is applied

$$\begin{aligned} &A_1(w_1^{(n+1)}(x,y)) + k\frac{E}{h}w_1^{(n+1)}\psi(x,y) = \\ &q_1 + k\frac{E}{h}(w_2^{(n)} + h_1)\psi(x,y) \\ &A_2(w_2^{(n+1)}(x,y)) + k\frac{E}{h}w_2^{(n)}\psi(x,y) = \\ &q_2(1-\psi(x,y)) + k\frac{E}{h}(w_1^{(n)} - h_1)\psi(x,y) \end{aligned} \tag{13}$$

$$w_1^{(n+1)}\Big|_{\partial\Omega_1} = \frac{\partial w_1^{(n+1)}}{\partial n_1}\Big|_{\partial\Omega_1} = 0,$$

$$w_2^{(n+1)}\Big|_{\partial\Omega_2} = \frac{\partial w_2^{(n+1)}}{\partial n_2}\Big|_{\partial\Omega_2} = 0. \tag{14}$$

The following theorem holds.

Theorem. Let Ω_i, (i = 1,2) be bounded areas with the boundaries $\partial\Omega_i$ that satisfy Sobolev's embedding theorem conditions [Ladyzhenskaya, Uraltseva, 1973], let Ω^* be the measurable space, $q_i(x, y) \subseteq L_2(\Omega_i)$ and let there be real constants $c_i > 0$, $D_i > 0$, such that

$$D_1\|\Delta(w_i)\|^2_{L_2(\Omega^*)} \le \Big(B_{11,i}\frac{\partial^2 w_i}{\partial x^2} + B_{10,i}\frac{\partial^2 w_i}{\partial y^2}, \frac{\partial^2 w_i}{\partial x^2}\Big)_{L_2(\Omega_2)} + \Big(B_{10,i}\frac{\partial^2 w_i}{\partial x^2} + B_{11,i}\frac{\partial^2 w_i}{\partial y^2}, \frac{\partial^2 w_i}{\partial y^2}\Big)_{L_2(\Omega_2)} + \Big[\Big(B_{11,i} + B_{10,i}\Big)\frac{\partial^2 w_i}{\partial x\partial y}, \frac{\partial^2 w_i}{\partial x\partial y}\Big]_{L_2(\Omega_2)} \le c_i\|\Delta(w_i)\|^2_{L_2(\Omega_i)}$$

Then

1) $\forall n$, $w_i^{(n)} \in W_2^4(\Omega_i)\bigcap \dot{W}_2^2(\Omega_i)$, i = 1,2;
2) There are functions $w_i^*(x,y) \in \dot{W}_2^2(\Omega_i)$, i = 1,2, being solutions of problems (11), (12), and $\lim_{n\to\infty}\|w_i^{(n)} - w_i^*\|_{W_2^2(\Omega_i)} = 0$.

Proof. Only fundamental proof steps will be given. The conclusion of the first theorem follows from the theory of solvability of elliptic equations [Ladyzhenskaya, Uraltseva, 1973] assuming that initial approximations $w_i^0 \in L_2(\Omega_i)$, i = 1,2.

The conclusion of the second theorem proves an existence of a general solution to problems (11) and (12) in the space $\dot{W}_2^2(\Omega_1)\times\dot{W}_2^2(\Omega_2)$ and a strong convergence of a sequence of approximating solutions $\{w_i^{(n)}\}$ to exactly one w_i^* with respect to the space norm $\dot{W}_2^2(\Omega_i)$, i = 1,2. In order to prove the second conclusion the following operations should be applied:

1) Functions $w_i^{(n)}$ (i = 1,2) are computed from each of equations (13);
2) First equation of the obtained system is multiplied by $(w_1^{(n+1)} - w_1^{(n)})$, whereas second equation is multiplied by $(w_2^{(n+1)} - w_2^{(n)})$;
3) Again, the first equation is integrated in the space $\Omega_1(\Omega_2)$. As a result, after application of the Green formula one gets

$$\Big(B_{10,1}\frac{\partial^2(w_1^{(n+1)}-w_1^{(n)})}{\partial x^2}+B_{10,1}\frac{\partial^2\left(w_1^{(n+1)}-w_1^{(n)}\right)}{\partial y^2},\ \frac{\partial^2\left(w_1^{(n+1)}-w_1^{(n)}\right)}{\partial x^2}\Big)_{L_2(\Omega_2)}+$$
$$+\Big(B_{10,1}\frac{\partial^2(w_1^{(n+1)}-w_1^{(n)})}{\partial x^2}+B_{11,1}\frac{\partial^2\left(w_1^{(n+1)}-w_1^{(n)}\right)}{\partial y^2},\ \frac{\partial^2\left(w_1^{(n+1)}-w_1^{(n)}\right)}{\partial x^2}\Big)_{L_2(\Omega_2)}+$$
$$+\Big(\left[B_{11,1}-B_{10,1}\right]\frac{\partial^2\left(w_1^{(n+1)}-w_1^{(n)}\right)}{\partial x\partial y},\ \frac{\partial^2\left(w_1^{(n+1)}-w_1^{(n)}\right)}{\partial x\partial y}\Big)_{L_2(\Omega_1)}+$$
$$+\frac{kE}{h}\Big(\psi(x,y)\left(w_1^{(n+1)}-w_1^{(n)}\right),\ \left(w_1^{(n+1)}-w_1^{(n)}\right)\Big)_{L_2(\Omega_1)}=$$
$$\frac{kE}{h}\Big(\psi(x,y)\left(w_2^{(n)}-w_2^{(n-1)}\right),\ \left(w_1^{(n+1)}-w_1^{(n)}\right)\Big)_{L_2(\Omega_1)},\qquad(15)$$

$$\Big(B_{11,2}\frac{\partial^2\left(w_2^{(n+1)}-w_2^{(n)}\right)}{\partial x^2}+B_{10,2}\frac{\partial^2\left(w_2^{(n+1)}-w_2^{(n)}\right)}{\partial y^2},\ \frac{\partial^2\left(w_1^{(n+1)}-w_1^{(n)}\right)}{\partial x^2}\Big)_{L_2(\Omega_2)}+\Big(B_{10,2}\frac{\partial^2\left(w_2^{(n+1)}-w_2^{(n)}\right)}{\partial x^2}+$$
$$B_{11,2}\frac{\partial^2\left(w_2^{(n+1)}-w_2^{(n)}\right)}{\partial y^2},\ \frac{\partial^2\left(w_2^{(n+1)}-w_2^{(n)}\right)}{\partial y^2}\Big)_{L_2(\Omega_2)}+\Big(\left[B_{11,2}-B_{10,2}\right]\frac{\partial^2\left(w_2^{(n+1)}-w_2^{(n)}\right)}{\partial x\partial y},$$
$$\frac{\partial^2\left(w_2^{(n+1)}-w_2^{(n)}\right)}{\partial x\partial y}\Big)_{L_2(\Omega_2)}+$$
$$+\frac{kE}{h}\Big(\psi(x,y)\left(w_2^{(n+1)}-w_2^{(n)}\right),\ \left(w_2^{(n+1)}-w_2^{(n)}\right)\Big)_{L_2(\Omega_2)}=$$
$$=\frac{kE}{h}\Big(\psi(x,y)\left(w_1^{(n)}-w_1^{(n-1)}\right).\qquad(16)$$

Owing to the definition of functions $\psi(x, y)$, equations (15) and (16) assume the form

$$\Big(B_{11,1}\frac{\partial^2\left(w_1^{(n+1)}-w_1^{(n)}\right)}{\partial x^2}+B_{10,1}\frac{\partial^2\left(w_1^{(n+1)}-w_1^{(n)}\right)}{\partial y^2},$$
$$\frac{\partial^2\left(w^{(n+1)}-w^{(n)}\right)}{\partial x^2}\Big)_{L_2(\Omega_1)}+\Big(B_{10,1}\frac{\partial^2\left(w_1^{(n+1)}-w_1^{(n)}\right)}{\partial x^2}+$$
$$B_{11,1}\frac{\partial^2\left(w_1^{(n+1)}-w_1^{(n)}\right)}{\partial y^2},$$
$$\frac{\partial^2\left(w_1^{(n+1)}-w_2^{(n)}\right)}{\partial y^2}\Big)_{L_2(\Omega_1)}+\Big(\left[B_{11,1}-B_{10,1}\right]\frac{\partial^2\left(w_1^{(n+1)}-w_1^{(n)}\right)}{\partial x\partial y},$$
$$\frac{\partial^2\left(w_1^{(n+1)}-w_1^{(n)}\right)}{\partial x\partial y}\Big)_{L_2(\Omega_1)}+\frac{kE}{h}\left\|\left(w_1^{(n+1)}-w_1^{(n)}\right)\right\|^2_{L_2(\Omega^*)}=$$
$$\frac{kE}{h}\Big(\left(w_2^{(n)}-w_2^{(n-1)}\right),\ \left(w_1^{(n+1)}-w_1^{(n)}\right)\Big)_{L_2(\Omega^*)},\qquad(17)$$

$$\Big(B_{11,2}\frac{\partial^2\left(w_2^{(n+1)}-w_2^{(n)}\right)}{\partial x^2}+B_{10,2}\frac{\partial^2\left(w_2^{(n+1)}-w_2^{(n)}\right)}{\partial y^2},$$
$$\frac{\partial^2\left(w_1^{(n+1)}-w_1^{(n)}\right)}{\partial x^2}\Big)_{L_2(\Omega_2)}+\Big(B_{10,2}\frac{\partial^2\left(w_2^{(n+1)}-w_2^{(n)}\right)}{\partial x^2}+$$
$$B_{11,2}\frac{\partial^2\left(w_2^{(n+1)}-w_2^{(n)}\right)}{\partial y^2},$$
$$\frac{\partial^2\left(w_2^{(n+1)}-w_2^{(n)}\right)}{\partial y^2}\Big)_{L_2(\Omega_2)}+$$
$$\frac{kE}{h}\left\|\left(w_2^{(n+1)}-w_2^{(n)}\right)\right\|^2_{L_2(\Omega^*)}=\frac{kE}{h}\Big(\left(w_1^{(n)}-w_1^{(n-1)}\right),$$
$$\left(w_2^{(n+1)}-w_2^{(n)}\right)\Big)_{L_2(\Omega^*)}\qquad(18)$$

Applying Young's inequality [Ladyzhenskaya, Uraltseva, 1973], equations (17), (18) and the theorem condition, we have

$$D_1\left\|\Delta\left(w_1^{(n+1)}-w_1^{(n)}\right)\right\|^2_{L_2(\Omega^*)}+\frac{kE}{h}\left\|\left(w_1^{(n+1)}-w_1^{(n)}\right)\right\|^2_{L_2(\Omega^*)}\le$$
$$\le\frac{kE}{2h}\left\|\left(w_2^{(n)}-w_2^{(n-1)}\right)\right\|^2_{L_2(\Omega^*)}+\frac{kE}{2h}\left\|\left(w_1^{(n+1)}-w_1^{(n)}\right)\right\|^2_{L_2(\Omega^*)},$$
$$D_2\left\|\Delta\left(w_2^{(n+1)}-w_2^{(n)}\right)\right\|^2_{L_2(\Omega_2)}+\frac{kE}{h}\left\|\left(w_2^{(n+1)}-w_2^{(n)}\right)\right\|^2_{L_2(\Omega^*)}\le$$

$$\leq \frac{kE}{h}\left\| \left(w_1^{(n)} - w_1^{(n-1)}\right)\right\|^2_{L_2(\Omega^*)} + \frac{kE}{2h}\left\| \left(w_2^{(n+1)} - w_2^{(n)}\right)\right\|^2_{L_2(\Omega^*)}.$$

and finally we get

$$D_1\left\| \Delta\left(w_1^{(n+1)} - w_1^{(n)}\right)\right\|^2_{L_2(\Omega^*)} + \frac{kE}{2h}\left\| \left(w_2^{(n)} - w_2^{(n-1)}\right)\right\|^2_{L_2(\Omega^*)} \leq \frac{kE}{2h}\left\| \left(w_2^{(n+1)} - w_2^{(n)}\right)\right\|^2_{L_2(\Omega^*)}, \quad (19)$$

$$D_2\left\| \Delta\left(w_1^{(n+1)} - w_1^{(n)}\right)\right\|^2_{L_2(\Omega^*)} + \frac{kE}{2h}\left\| \left(w_2^{(n+1)} - w_2^{(n)}\right)\right\|^2_{L_2(\Omega^*)} \leq \frac{kE}{2h}\left\| \left(w_1^{(n)} - w_1^{(n-1)}\right)\right\|^2_{L_2(\Omega^*)}. \quad (20)$$

Inequalities (19), (20) are rewritten in the form

$$D_1\left\| \Delta\left(w_1^{(n+1)} - w_1^{(n)}\right)\right\|^2_{L_2(\Omega^*)} + \frac{kE}{2h}\left\| \left(w_1^{(n+1)} - w_1^{(n)}\right)\right\|^2_{L_2(\Omega^*)} \leq \leq (1-\alpha)\frac{kE}{2h}\left\| \left(w_2^{(n)} - w_2^{(n-1)}\right)\right\|^2_{L_2(\Omega^*)} + \alpha\frac{kE}{2h}\left\| \left(w_2^{(n)} - w_2^{(n-1)}\right)\right\|^2_{L_2(\Omega^*)}, \quad (21)$$

$$D_2\left\| \Delta\left(w_2^{(n+1)} - w_2^{(n)}\right)\right\|^2_{L_2(\Omega_2)} + \frac{kE}{2h}\left\| \left(w_2^{(n+1)} - w_2^{(n)}\right)\right\|^2_{L_2(\Omega^*)} \leq \leq (1-\alpha)\frac{kE}{2h}\left\| \left(w_1^{n} - w_1^{(n-1)}\right)\right\|^2_{L_2(\Omega^*)} + \alpha\frac{kE}{2h}\left\| \left(w_1^{(n)} - w_1^{(n-1)}\right)\right\|^2_{L_2(\Omega^*)}, \quad (22)$$

where $\alpha \in R^1$, $0 < \alpha < 1$.

Next, we apply the Friedrichs inequality [Awrejcewicz, Krysko, 2003] in the following form. One may find a constant $c_i \in R^1$ such that $\forall f(x,y) \in \dot{W}_2^2(\Omega_i)$, i = 1,2 and the following inequalities hold

$$\|f\|^2_{L_2(\Omega^*)} \leq c_i\|\Delta f\|^2_{L_2(\Omega^*)} \leq c_i\|\Delta f\|^2_{L_2(\Omega)}, \text{ i = 1,2,} \quad (23)$$

under assumption that $\Omega^* \subseteq \Omega_i$ and that it satisfies conditions of application of Friedrichs inequality. Owing to (23), the inequalities (21) and (22) have the following form

$$D_1\left\| \Delta\left(w_1^{(n+1)} - w_1^{(n)}\right)\right\|^2_{L_2(\Omega_i)} + \frac{kE}{2h}\left\| \left(w_1^{(n+1)} - w_1^{(n)}\right)\right\|^2_{L_2(\Omega^*)} \leq \leq (1-\alpha)\frac{kE}{2h}\left\| \left(w_2^{(n)} - w_2^{(n-1)}\right)\right\|^2_{L_2(\Omega^*)} + \frac{\alpha C_2 kE}{D_2 2h} D_1\left\| \Delta\left(w_1^{(n+1)} - w_1^{(n)}\right)\right\|^2_{L_2(\Omega)}, \quad (24)$$

$$D_2\left\| \Delta\left(w_2^{(n+1)} - w_2^{(n)}\right)\right\|^2_{L_2(\Omega_2)} + \frac{kE}{2h}\left\| \left(w_2^{(n+1)} - w_2^{(n)}\right)\right\|^2_{L_2(\Omega^*)} \leq (1-\alpha)\frac{kE}{2h}\left\| \left(w_1^{n} - w_1^{(n-1)}\right)\right\|^2_{L_2(\Omega^*)} + \frac{\alpha C_1 kE}{D_1 2h} D_1\left\| \Delta\left(w_1^{(n+1)} - w_1^{(n)}\right)\right\|^2_{L_2(\Omega)}. \quad (25)$$

We choose α in inequalities (24) and (25) from the following condition

$$\frac{\alpha C_2 kE}{2hD_2} < 1, \quad \frac{\alpha C_1 kE}{2hD_1} < 1, \quad 0 < \alpha < 1,$$

and we take

$$\rho = \max\left\{(1-\alpha), \frac{\alpha C_2 kE}{2hD_2}, \frac{\alpha C_1 kE}{2hD_1}\right\}.$$

It follows immediately that $0 < \rho < 1$.

Now, inequalities (24) and (25) can have the following form

$$D_1\left\| \Delta\left(w_1^{(n+1)} - w_1^{(n)}\right)\right\|^2_{L_2(\Omega_1)} + \frac{kE}{2h}\left\| \left(w_1^{(n+1)} - w_1^{(n)}\right)\right\|^2_{L_2(\Omega^*)} \leq \leq \rho\left\{ D_2\left\| \Delta\left(w_2^{(n)} - w_2^{(n-1)}\right)\right\|^2_{L_2(\Omega_2)} + \frac{kE}{2h}\left\| \left(w_2^{(n)} - w_2^{(n-1)}\right)\right\|^2_{L_2(\Omega^*)} \right\}, \quad (26)$$

$$D_2\left\| \Delta\left(w_2^{(n+1)} - w_2^{(n)}\right)\right\|^2_{L_2(\Omega_2)} + \frac{kE}{2h}\left\| \left(w_2^{(n+1)} - w_2^{(n)}\right)\right\|^2_{L_2(\Omega^*)} \leq \leq \rho\left\{ D_1\left\| \Delta\left(w_1^{(n)} - w_1^{(n-1)}\right)\right\|^2_{L_2(\Omega_1)} + \frac{kE}{2h}\left\| \left(w_1^{(n)} - w_1^{(n-1)}\right)\right\|^2_{L_2(\Omega^*)} \right\}. \quad (27)$$

Matching inequalities (26) and (27) yields

$$\left\{ D_1\left\| \Delta\left(w_1^{(n+1)} - w_1^{(n)}\right)\right\|^2_{L_2(\Omega_1)} + D_2\left\| \Delta\left(w_2^{(n+1)} - w_2^{(n)}\right)\right\|^2_{L_2(\Omega_2)} + + \frac{kE}{2h}\left\| \left(w_1^{(n+1)} - w_1^{(n)}\right)\right\|^2_{L_2(\Omega^*)} + \frac{kE}{2h}\left\| \left(w_2^{(n+1)} - w_2^{(n)}\right)\right\|^2_{L_2(\Omega^*)} \right\} \leq \leq \rho\left\{ D_1\left\| \Delta\left(w_1^{(n)} - w_1^{(n-1)}\right)\right\|^2_{L_2(\Omega_1)} + D_2\left\| \Delta\left(w_2^{(n)} - w_2^{(n-1)}\right)\right\|^2_{L_2(\Omega_1)} + + \frac{kE}{2h}\left\| \left(w_1^{(n)} - w_1^{(n-1)}\right)\right\|^2_{L_2(\Omega^*)} + \frac{kE}{2h}\left\| \left(w_2^{(n)} - w_2^{(n-1)}\right)\right\|^2_{L_2(\Omega^*)} \right\} \quad (28)$$

or

$$\left\|\overline{w}^{(n+1)} - \overline{w}^{(n)}\right\|^2_H \leq \rho\left\|\overline{w}^{(n)} - \overline{w}^{(n-1)}\right\|^2_H, \quad (29)$$

where $\overline{w}^{(n)} = (w_1^{(n)}, w_2^{(n)})$, H is the normalized space with the norm equivalent to that of space $\dot{W}_2^2(\Omega_1) \times \dot{W}_2^2(\Omega_2)$ and defined by left (or right) part of inequality (28).

Inequality (29) yields

$$\left\|\overline{w}^{(n+1)}-\overline{w}^{(n)}\right\|_H \le \rho\left\|\overline{w}^{(n)}-\overline{w}^{(n-1)}\right\|_H ,$$

$$\rho_1=\sqrt{\rho}<1,$$

$$\left\|\overline{w}^{(n+1)}-\overline{w}^{(n)}\right\|_H \le \theta\rho_1^n ,$$

where

$$\theta=\left\|\overline{w}^{(1)}-\overline{w}^{0}\right\|_H .$$

Hence, for any natural ρ one obtains

$$\left\|\overline{w}^{(n+p)}-\overline{w}^{(n)}\right\|_H \le \left\|\overline{w}^{(n+p)}-\overline{w}^{(n+p-1)}\right\|_H + \left\|\overline{w}^{(n+p-1)}-\overline{w}^{(n+p-2)}\right\|_H + \quad (30)$$

$$+\left\|\overline{w}^{(n+1)}-\overline{w}^{(n)}\right\|_H \le \theta\rho_1^{(n+p-1)}+\theta\rho_1^{(n+p-2)}+...+\theta\rho_1^{(n)}=$$

$$\frac{\theta(\rho_1^{(n)}+\rho_1^{(n+p)})}{1-\rho_1}\le\frac{\theta\rho_1^{(n)}}{1-\rho_1}.$$

Inequality (30) implies fundamentality of the sequence $\left\{\overline{w}^{(n)}\right\}$, and owing to completeness of H, also convergence $\left\{\overline{w}^{(n)}\right\}$ in H to the function $\overline{w}^*=(w_1^*,w_2^*)$ holds.

Therefore, owing to equivalence of the norms in spaces H and $\dot{W}_2^2(\Omega_1)\times\dot{W}_2^2(\Omega_2)$, one gets

$$\lim_{n\to\infty}\left\|w_i^n-w_i^*\right\|_{\dot{W}_2^2(\Omega_i)}=0,\ \ i = 1,2. \quad (31)$$

Systems (13) and (14) can be presented in the following integral form

$$\left(B_{11,1}\frac{\partial^2 w_1^{(n+1)}}{\partial x^2}+B_{10,1}\frac{\partial^2 w_1^{(n+1)}}{\partial y^2},\frac{\partial^2\varphi_1}{\partial x^2}\right)_{L_2(\Omega_1)}+\left(B_{10,1}\frac{\partial^2 w_1^{(n+1)}}{\partial x^2}+B_{11,1}\frac{\partial^2 w_1^{(n+1)}}{\partial y^2},\frac{\partial^2\varphi_1}{\partial y^2}\right)_{L_2(\Omega_1)}+$$

$$\left(\left[B_{11,1}-B_{10,1}\right]\frac{\partial^2 w_1^{(n+1)}}{\partial x\partial y}\frac{\partial^2\varphi_1}{\partial x\partial y}\right)_{L_2(\Omega_1)}+$$

$$+\frac{kE}{h}\left(\psi(x,y)w_1^{(n+1)},\varphi_1\right)_{L_2(\Omega_1)}=\left(q_1\varphi_1\right)_{L_2(\Omega_1)}+\frac{kE}{h}\left(\psi(x,y)\left[w_1^{(n)}+h_1\right],\varphi_1\right)_{L_2(\Omega^*)}, \quad (32)$$

$$\left(B_{11,2}\frac{\partial^2 w_1^{(n+1)}}{\partial x^2}+B_{10,2}\frac{\partial^2 w_2^{(n+1)}}{\partial y^2},\frac{\partial^2\varphi_2}{\partial x^2}\right)_{L_2(\Omega_2)}+\left(B_{10,2}\frac{\partial^2 w_2^{(n+1)}}{\partial x^2}+B_{11,2}\frac{\partial^2 w_2^{(n+1)}}{\partial y^2},\frac{\partial^2\varphi_2}{\partial y^2}\right)_{L_2(\Omega_2)}+$$

$$\left(\left[B_{11,2}-B_{10,2}\right]\frac{\partial^2 w_2^{(n+1)}}{\partial x\partial y}\frac{\partial^2\varphi_2}{\partial x\partial y}\right)_{L_2(\Omega_2)}+$$

$$+\frac{kE}{h}\left(\psi(x,y)w_2^{(n+1)},\varphi_2\right)_{L_2(\Omega_2)}=\left(q_2\left[1-\psi(x,y)\right],\varphi_2\right)_{L_2(\Omega_2)}+$$

$$+\frac{kE}{h}\left(\psi(x,y)\left[w_1^{(n)}+h_1\right],\varphi_1\right)_{L_2(\Omega^*)}, \quad (33)$$

$$\forall\varphi_1\in\dot{W}_2^2(\Omega_1),\ \ \forall\varphi_2\in\dot{W}_2^2(\Omega_2).$$

Owing to condition (21), one may apply a limiting transition for n $\to\infty$ in equations (22) and (23) to get

$$\left(B_{11,1}\frac{\partial^2 w_1^*}{\partial x^2}+B_{10,1}\frac{\partial^2 w_1^*}{\partial y^2},\frac{\partial^2\varphi_1}{\partial x^2}\right)_{L_2(\Omega_1)}+\left(B_{10,1}\frac{\partial^2 w_1^*}{\partial x^2}+B_{11,1}\frac{\partial^2 w_1^*}{\partial y^2},\frac{\partial^2\varphi_1}{\partial y^2}\right)_{L_2(\Omega_1)}+$$

$$\left(\left[B_{11,1}-B_{10,1}\right]\frac{\partial^2 w_1^*}{\partial x\partial y}\frac{\partial^2\varphi_1}{\partial x\partial y}\right)_{L_2(\Omega_1)}+$$

$$+\frac{kE}{h}\left(\psi(x,y)w^*,\varphi_1\right)_{L_2(\Omega_1)}=\left(q_1\varphi_1\right)_{L_2(\Omega_1)}+\frac{kE}{h}\left(\psi(x,y)\left[w_2^*+h_1\right],\varphi_1\right)_{L_2(\Omega^*)}, \quad (34)$$

$$\left(B_{11,2}\frac{\partial^2 w_1^*}{\partial x^2}+B_{10,2}\frac{\partial^2 w_1^*}{\partial y^2},\frac{\partial^2\varphi_1}{\partial x^2}\right)_{L_2(\Omega_2)}+\left(B_{10,2}\frac{\partial^2 w_1^*}{\partial x^2}+B_{11,2}\frac{\partial^2 w_1^*}{\partial y^2},\frac{\partial^2\varphi_1}{\partial y^2}\right)_{L_2(\Omega_2)}+$$

$$\left(\left[B_{11,2}-B_{10,2}\right]\frac{\partial^2 w_1^*}{\partial x\partial y}\frac{\partial^2\varphi_1}{\partial x\partial y}\right)_{L_2(\Omega_2)}+$$

$$+\frac{kE}{h}\left(\psi(x,y)w_2^*,\varphi_2\right)_{L_2(\Omega_2)}=\left(q_2\left[1-\psi(x,y)\right],\varphi_2\right)_{L_2(\Omega_2)}+\frac{kE}{h}\left(\psi(x,y)\left[w_1^*+h_1\right],\varphi_1\right)_{L_2(\Omega^*)}, \quad (35)$$

which proves the second conclusion of the theorem.

Remark 1. Convergence of the iterative algorithm is easily extended into the case of other boundary conditions. For instance, the proposed scheme of the proof remains valid when one or two contacting plates are ball-type supported.

Remark 2. The proposed iterative algorithm can be applied in the problems of contacting plates taking into account physical nonlinearities (matching with the method of elastic solution of Iliushin causes that the scheme of proof remains valid).

Remark 3. Convergence of the proposed iterative algorithm can be extended into solution of contact problems of 3D constructions composed of freely coupled plates within Timoshenko type hypotheses for each plate, as well as for combined models, i.e. when one of the layers is associated with the Kirchhoff's hypothesis, and for other one the Timoshenko assumptions hold.

Remark 4. In relations (1), (4), (10), (11) and (13), $q_i = q_i^0 - \frac{y}{g} h \frac{\partial^2 w_i}{\partial t^2} - \varepsilon_i h \frac{\partial w_i}{\partial t}$ denotes intensity of the given external loads and inertia forces acting on the i-th plate are taken into account according to d'Alembert's principle; ε_i is the coefficient of the mechanical property medium; $g-$ acceleration of gravity; $\gamma-$ unit material gravity coefficient; $t-$ time.

Remark 5. The proposed iterative procedure is carried out on each time step.

4 Conclusions

This paper is basically devoted to a rigorous proof of convergence of the iterative algorithm applied to solve contact problems of freely coupled plates within Kirchhoff's hypotheses. The main effort of our research is focused on the formulation of a theorem and its proof. It is expected that the given theorem may play an important role in various problems of plates and shells dynamics with both changeable contacting zones and elasticity of parameters.

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