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ANALYTICAL PREDICTION OF STICK-SLIP CHAOS IN R^4

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Abstract

We consider two coupled oscillators with negative Duffing type stiffness which are self (due to friction) and externally (harmonically) excited. The Melnikov-Gruendler approach is used to define the Melnikov's function including smooth and stick-slip chaotic behaviour. Next, we present a threshold curve obtained for selected values of parameters.

INTRODUCTION

It is needless to say that a prediction of chaos in an analytical way in non-smooth objects modelled as systems in R^4 plays a crucial role for both theoretical and applicable reasons. A key role of research carried out in this direction plays the paper by Awrejcewicz and Holicke [1], where a chaotic threshold for both smooth and stick-slip chaotic behaviour in one degree-of-freedom system with friction has been obtained using directly the Melnikov's technique [5]. On the other hand, it was impossible to extend directly the original Melnikov's method devoted to analysis of an analytic system in R^2. Therefore, we have applied the Gruendler extension of the Melnikov's method to R^4, which is further referred as the Melnikov-Gruendler approach. However, in the cited Gruendler's work [4] again an emphasis of C^2 systems is given. In contrary, in our research we extend the results obtained earlier (see [1]) to R^4. Although we do not give a rigorous definitions and proofs of a C^n vector field on R^n, but we show the computations of related integrals yielding a being sought chaotic threshold defined by the approproate Melnikov's function. Furthermore, a reduction of the obtained Melnikov integrals to those associated with previously considered one degree-of-freedom mechanical system indicate a validity of our approach.

THE MELNIKOV-GRUENDLER FUNCTION

The analysed mechanical object consists of two stiff bodies with the masses m coupled via linear and nonlinear springs in the way shown in Figure 1. Note that when the system is autonomous, i.e. $\Gamma = 0$, the self-excited oscillations appear, which are generated by frictional characteristics. The latter ones possess a decreasing part versus a relative velocity between both bodies and the tape moving with a constant velocity w. Although this problem belongs to classical ones and has been studied by vast number of researchers, an attempt to formulate threshold for chaos occurence in the analytical way failed. In what follows we show how to solve this problem using the Melnikov technique applied to our discontinuous system. It is also recommended to be familiar with the reference [1], where a similar like approach has been applied to predict chaos in a similar like system, but with one degree-of-freedom.

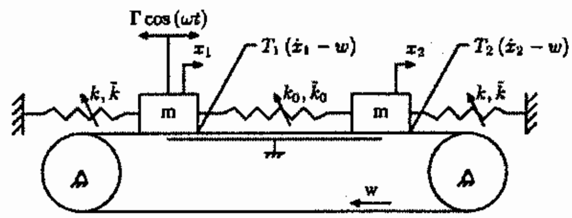

Figure 1: The analysed system.

Dynamics of our system is governed by the following ODEs:

$$\begin{cases} \dot{x}_1 = & p_1/m \\ \dot{p}_1 = & kx_1 - \tilde{k}x_1^3 + k_0(x_1 - x_2) - \tilde{k}_0(x_1 - x_2)^3 \\ & +\varepsilon_1 \Gamma \cos(\omega t) - \varepsilon_2 T_1(p_1/m - w) \\ \dot{x}_2 = & p_2/m \\ \dot{p}_2 = & kx_2 - \tilde{k}x_2^3 - k_0(x_1 - x_2) + \tilde{k}_0(x_1 - x_2)^3 \\ & -\varepsilon_3 T_2(p_2/m - w) \end{cases} \tag{1}$$

where:

$$T_i(p_i/m - w) = T_{i0}\,\mathrm{sgn}(p_i/m - w) - B_{i1}(p_i/m - w) + B_{i2}(p_i/m - w)^3 \tag{2}$$

and w is the tape velocity, whereas $B_{11}, B_{12}, B_{21}, B_{22}, T_{10}, T_{20}$ are the friction coefficients.

Introducing the following scaling

$$t \to t\sqrt{\frac{k}{m}}, \quad x = x_1\sqrt{\frac{\tilde{k}}{k}}, \quad u = p_1\sqrt{\frac{\tilde{k}}{mk^2}}, \tag{3}$$

$$y = x_2\sqrt{\frac{\tilde{k}}{k}}, \quad v = p_2\sqrt{\frac{\tilde{k}}{mk^2}} \tag{4}$$

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and the following relations

$$k_0 = \xi k, \quad \tilde{k}_0 = \xi \tilde{k} \quad \text{where}\, \xi \geq 0, \tag{5}$$

the analysed ODEs are cast in the nondimensional form

$$\begin{pmatrix} \dot{x} \\ \dot{u} \\ \dot{y} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} u \\ x - x^3 + f_\xi(x,y) \\ v \\ y - y^3 - f_\xi(x,y) \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon_1 \Gamma' \cos(\omega' t) - \varepsilon_2 T_1'(u - w') \\ 0 \\ -\varepsilon_3 T_2'(v - w') \end{pmatrix}, \tag{6}$$

where

$$T_i'(p_i' - w') = T_{i0}' \operatorname{sgn}(p_i' - w') - B_{i1}'(p_i' - w') + B_{i2}'(p_i' - w')^3, \tag{7}$$

$$\Gamma' = \Gamma\sqrt{\frac{\tilde{k}}{k^3}},\, \omega' = \omega\sqrt{\frac{m}{k}},\, T_{i0}' = T_{i0}\sqrt{\frac{\tilde{k}}{k^3}}, \tag{8}$$

$$B_{i1}' = \frac{B_{i1}}{\sqrt{mk}},\, B_{i2}' = \frac{k^2 B_{i2}}{\sqrt{m^3 \tilde{k}^3}},\, w' = w\sqrt{\frac{\tilde{k}m}{k^2}}, \tag{9}$$

and

$$f_\xi(x,y) = \xi(x-y) - \xi(x-y)^3. \tag{10}$$

For $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0$ one gets the unperturbed system with the associated Hamiltonian

$$H = \frac{u^2}{2} + \frac{v^2}{2} + \frac{1}{4}(x^2-1)^2 + \frac{1}{4}(y^2-1)^2 + \frac{\beta}{4}\left((x-y)^2 - \frac{\alpha}{\beta}\right)^2. \tag{11}$$

Next, we consider a linearization along a homoclinic orbit $\gamma(t)$:

$$\gamma(t) = \begin{pmatrix} q(t) \\ \dot{q}(t) \\ -q(t) \\ -\dot{q}(t) \end{pmatrix}, \tag{12}$$

where:

$$q(t) = \sqrt{\frac{2(1+2\xi)}{1+8\xi}} \operatorname{sech}\left(t\sqrt{1+2\xi}\right). \tag{13}$$

The linearized system of the unperturbed equations (6) in vicinity of the homoclinic orbit $\gamma(t)$ reads[1]:

$$\begin{cases} \dot{\psi}_1 = \psi_2 \\ \dot{\psi}_2 = \left(1 + \xi - 3(1+4\xi) q^2(t)\right)\psi_1 + \xi\left(12q^2(t) - 1\right)\psi_3 \\ \dot{\psi}_3 = \psi_4 \\ \dot{\psi}_4 = \left(1 + \xi - 3(1+4\xi) q^2(t)\right)\psi_3 + \xi\left(12q^2(t) - 1\right)\psi_1 \end{cases} \tag{14}$$

[1]For more details see [3][4]

Solving these equations we obtain the fundamental solutions:

$$\{\psi^{(1)}, \psi^{(2)}, \psi^{(3)}, \psi^{(4)}\}. \tag{15}$$

Next, we seek a special solution such that $\psi^{(n)}(t) \xrightarrow{t \to \pm\infty} \infty$, so combining the equations (14) we get:

$$\ddot{\phi}_1 = (1+2\xi)\left(1 - 6\operatorname{sech}^2\left(t\sqrt{1+2\xi}\right)\right)\phi_1, \tag{16}$$

where $\phi_1 \equiv \psi_1 - \psi_3$. Since $\dot{q}(t)$ is a solution to the above equation and applying the following substitution $\dot{q}(t) \to r(t)\dot{q}(t)$, we obtain:

$$\ddot{r}\dot{q} + 2\dot{r}\ddot{q} = 0. \tag{17}$$

Integrating of (17) and owing to the obtained results, the solution reads

$$\phi_1(t) = \frac{3}{4}C_1 t\dot{q}(t) - \frac{1}{2}C_1 \operatorname{ctgh}(t)\dot{q}(t) + \frac{1}{8}C_1 \sinh(2t)\dot{q}(t) + C_2\dot{q}(t). \tag{18}$$

It is easy to see that the above solution possesses the desired asymptotics and it is denoted by $\psi^{(2)}(t)$.

According to Gruendler's theory (see [4]) we compute only the following quantities:

$$K_{2i}(t,t_0) = \det\{\psi^{(1)}, \frac{\partial h(\gamma(t), t+t_0, 0)}{\partial \varepsilon_i}, \psi^{(3)}, \psi^{(4)}\}, \tag{19}$$

where $K_{2i}(t,t_0)$ represents the projection onto the direction $\psi^{(2)}(t)$ of the ε_i of the h evaluated along $\gamma(t)$. In our case the perturbation term has the following form:

$$h(x,t,\varepsilon) = \begin{pmatrix} 0 \\ \varepsilon_1 \Gamma' \cos(\omega' t) - \varepsilon_2 T_1'(u - w') \\ 0 \\ -\varepsilon_3 T_2'(v - w') \end{pmatrix}. \tag{20}$$

Making use of (20) and (19) we get:

$$K_{21}(t,t_0) = 2\Gamma'\dot{q}\cos(\omega'(t+t_0)), \tag{21}$$

$$K_{22}(t,t_0) = -2\dot{q}(t)T_1'(\dot{q} - w'), \tag{22}$$

$$K_{23}(t,t_0) = 2\dot{q}(t)T_2'(\dot{q} - w'). \tag{23}$$

The Melnikov-Gruendler function is defined as follows:

$$M(t_0) = -\sum_{j=1}^{4} M_{2j}(t_0) = -\sum_{j=1}^{4}\int_{-\infty}^{\infty} K_{2j}(t,t_0)\,dt. \tag{24}$$

In order to find the Melnikov-Gruendler function, first we have to find the funtions $M_{2j}(t_0)$, where:

$$M_{2j}(t_0) = \int_{-\infty}^{\infty} K_{2j}(t,t_0)\,dt. \tag{25}$$

Taking into account (21) we get:

$$M_{21}(t_0) = -2\sqrt{2}\Gamma'\pi\omega'\sqrt{\frac{1+2\xi}{1+8\xi}} \times \operatorname{sech}\left(\frac{\pi\omega'}{2\sqrt{1+2\xi}}\right)\sin(\omega' t_0) \quad (26)$$

$$\begin{aligned} M_{22}(t_0) &= 2\int_{-\infty}^{\infty} \dot{q} T_1'(\dot{q} - w')\,dt \\ &= 2T_{10}'\int_{-\infty}^{\infty} \dot{q}\,\operatorname{sgn}(\dot{q} - w')\,dt \\ &- 2B_{11}'\int_{-\infty}^{\infty} \dot{q}(\dot{q} - w')\,dt \\ &+ 2B_{12}'\int_{-\infty}^{\infty} \dot{q}(\dot{q} - w')^3\,dt \\ &= -\frac{8}{3}B_{11}'\frac{1+2\xi}{1+8\xi}\sqrt{1+2\xi} \\ &+ \frac{32}{35}B_{12}'\frac{(1+2\xi)^3}{(1+8\xi)^2}\sqrt{1+2\xi} \\ &+ 8B_{12}'w'^2\frac{1+2\xi}{1+8\xi}\sqrt{1+2\xi} \\ &+ 2T_{10}'\int_{-\infty}^{\infty} \dot{q}\,\operatorname{sgn}(\dot{q} - w')\,dt. \end{aligned} \quad (27)$$

In order to integrate the last term in the above expression we transform the intergal into the following form:

$$\int_{-\infty}^{\infty} \dot{q}(t)\,\operatorname{sgn}(\dot{q}(t) - w')\,dt = \frac{1+2\xi}{1+8\xi}\sqrt{1+2\xi} \times \int_{-\infty}^{\infty} \dot{\tilde{q}}(t)\,\operatorname{sgn}\left(\dot{\tilde{q}}(t) - \tilde{w'}\right)dt, \quad (28)$$

where $\dot{\tilde{q}}(t) = -\sqrt{2}\operatorname{sech}(t)\operatorname{tgh}(t)$ and $\tilde{w'} = w'\frac{\sqrt{1+8\xi}}{1+2\xi}$. Analysing the above integral we find two different cases with respect to the value of $\tilde{w'}$. Assume first that $\tilde{w'} > 1/\sqrt{2}$, then:

$$\int_{-\infty}^{\infty} \dot{\tilde{q}}(t)\,\operatorname{sgn}\left(\dot{\tilde{q}}(t) - \tilde{w'}\right)dt = \operatorname{sgn}(-\tilde{w'})\int_{\infty}^{\infty} \dot{\tilde{q}}(t)\,dt = 0. \quad (29)$$

The second case we obtain for $\tilde{w'} < 1/\sqrt{2}$:

$$\int_{-\infty}^{\infty} \dot{\tilde{q}}\,\operatorname{sgn}\left(\dot{\tilde{q}} - \tilde{w'}\right)dt = -\int_{-\infty}^{t_1} \dot{\tilde{q}}dt + \int_{t_1}^{t_2} \dot{\tilde{q}}dt - \int_{t_2}^{\infty} \dot{\tilde{q}}dt = 2\sqrt{2}\left(\operatorname{sech}(t_2) - \operatorname{sech}(t_1)\right) \quad (30)$$

where:

$$t_1 = \ln\left(\frac{1}{\tilde{w'}}\sqrt{1+\sqrt{1-2\tilde{w'}^2}}\right) + \ln\left(1 - \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{1-2\tilde{w'}^2}}\right),$$

$$t_2 = \ln\left(\frac{1}{\tilde{w'}}\sqrt{1-\sqrt{1-2\tilde{w'}^2}}\right) + \ln\left(1 - \sqrt{\frac{1}{2} - \frac{1}{2}\sqrt{1-2\tilde{w'}^2}}\right).$$

Next, we substitute the obtained result to (27) and we find:

$$\begin{aligned} M_{22}(t_0) &= -\frac{8}{3}B_{11}'\frac{1+2\xi}{1+8\xi}\sqrt{1+2\xi} \\ &+ \frac{32}{35}B_{12}'\frac{(1+2\xi)^3}{(1+8\xi)^2}\sqrt{1+2\xi} \\ &+ 8B_{12}'w'^2\frac{1+2\xi}{1+8\xi}\sqrt{1+2\xi} \\ &+ 4\sqrt{2}T_{10}'\frac{(1+2\xi)^{3/2}}{1+8\xi}\theta\left(\frac{1}{\sqrt{2}} - \tilde{w'}\right) \\ &\times(\operatorname{sech} t_2 - \operatorname{sech} t_1), \end{aligned} \quad (31)$$

where $\theta(x)$ is Heaviside's function. In the similar way we obtain $M_{23}(t_0)$. Making use of the obtained results and (24), we get the Melnikov-Gruendler function:

$$\begin{aligned} M(t_0) &= -2\sqrt{2}\Gamma'\pi\omega'\sqrt{\frac{1+2\xi}{1+8\xi}}\operatorname{sech}\left(\frac{\pi\omega'}{2\sqrt{1+2\xi}}\right) \\ &\times\sin(\omega' t_0) - \frac{8(1+2\xi)^{3/2}}{3(1+8\xi)}(B_{11}' - B_{21}') \\ &+ 8\frac{(1+2\xi)^{3/2}}{1+8\xi}(B_{12}' - B_{22}')\left(w'^2 + \frac{4(1+2\xi)^2}{35(1+8\xi)}\right) \\ &+ 4\sqrt{2}(T_{10}' - T_{20}')\frac{(1+2\xi)^{3/2}}{1+8\xi} \\ &\times(\operatorname{sech}(t_2) - \operatorname{sech}(t_1)). \end{aligned} \quad (32)$$

Note that for $\xi = 0$ we obtain Melnikov's function for one degree-of-freedom which is in accordance with the result obtained in [1].

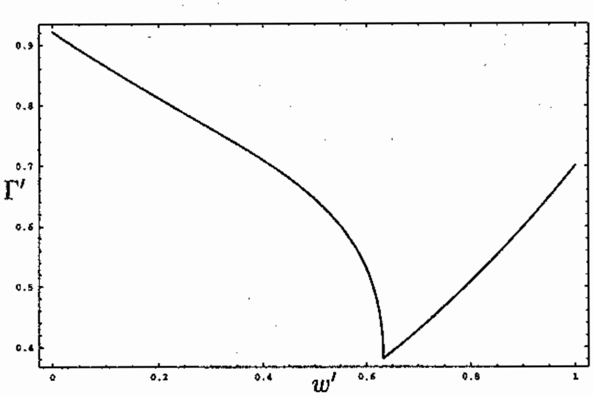

Figure 2: The threshold curve.

It is clear that having analytical form of the Melnikov-Gruendler function various control parameter can be taken to show regular and/or chaotic behavior. Let us take, following the paper [1], two of them i.e. $\{\Gamma', w'\}$. The obtained curve (see Figure 2) defines a chaotic threshold. Namely, above the mentioned curve chaos is expected, whereas below a regular behavior is expected. Observe a cusp in Figure 2 that corresponds to a switch between smooth and stick-slip dynamics. The switch takes place exactly for the tape velocity $w' = 1/\sqrt{2}$. In order to plot the threshold curve we have taken the following values $\xi = 0.1, T'_{10} = 0.45, T'_{20} = 0.15, B'_{11} = 0.25, B'_{21} = 0.15, B'_{12} = 0.35, B'_{22} = 0.2$. To sum up in this paper an important problem related to stick-slip chaos prediction is solved. It possesses a challenging impact on analysis of all mechanical systems with friction since many of them can be modelled by two degrees-of-freedom objects.

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