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# COMPLEX VIBRATIONS OF SPHERICAL AND CONICAL SHELLS WITH VARIABLE THICKNESS

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*Abstract.* In this work chaotic vibrations of deterministic and geometrically nonlinear elastic spherical and a conical axially symmetric shells with non-constant thickness subject to sinusoidal-type load are analyzed. Hybrid type variation equations are used to model the investigated problem. It is assumed that the shell material is isotropic and that the Young law holds. Both influence of inertial forces tangent to the mean shell's surface and inertial rotation of normal cross section are neglected. A transition from PDEs to ODEs with respect to time (the Cauchy problem) is realized through the Ritz procedure. Then the Cauchy problem is solved using the fourth order Runge-Kutta method.

New routes from harmonic to chaotic vibrations are detected and illustrated. Influence of various system parameters (shell deflection, amplitude and frequency of the excitation force, thickness variability, approximation modes number) on systems dynamics is analyzed, among others.

## 1. Introduction

Although many papers and books are devoted to analysis of spherical and shallow shells exhibiting thickness variability, mainly linear and stability problems are analyzed. In contrary, in this work we are focused on analysis of chaotic vibrations of the above mentioned shells. It should be emphasized that spherical shallow shells with constant thickness and rectangular plates subject to longitudinal sinusoidal – type loads are studied in references [1-7].

## 2. Problem formulation and the method of its solution

The shallow shell being a closed 3D object in $R^3$ with attached curvilinear coordinates c $\alpha, \beta, \gamma$ is studied. The action principle reads

$$-\delta(U_u + U_c) + \iint R\delta w ds = 0 , \tag{1}$$

where the first term represents the virtual work of elastic forces acting in the shell's material, whereas the second term expresses load, inertial and dissipative virtual works of the form

$$R = q - \frac{h\gamma}{g}(\ddot{w} + \varepsilon \dot{w}) \tag{2}$$

In result of computations the following variation equation is obtained

$$\begin{aligned}&\delta\iint\left\{\frac{D}{2}\left[(\Delta w)^2 - (1-\nu)L_2(w,w)\right] - \left[\Delta_k\varphi + L_2\left(\frac{1}{2}w + w_0, \varphi\right)\right]w - \frac{1}{2Eh}\left[(\Delta\varphi)^2 - (1+\nu)L_2(\varphi,\varphi)\right]\right\}ds - \\ &-\iint\left[q - \frac{h\gamma}{g}(\ddot{w} + \dot{w})\right]\delta w ds = 0.\end{aligned} \tag{3}$$

In order to solve equation (3), where the deflection function $w$ and the Airy's (stress) function $\varphi$ are the independent and being sought functions, a direct application of the Ritz method fails. A reason is that the left hand side of this equation is not the variation of a functional. Let us develop $w$ and $\varphi$ into two truncated series of linear functions satisfying main boundary conditions. In the case of axially deformed shallow rotational shell, the mentioned functions read

$$w = \sum_{i=1}^{n} x_i(t) w_i(\rho), \qquad \varphi = \sum_{i=1}^{n} y_i(t)\varphi_i(\rho), \tag{4}$$

where the coefficients $x_i(t)$ , $y_i(t)$ are being sought time dependent functions. Substituting (4) into (3), carrying out a variation procedure and comparing to zero the coefficients standing by $\delta x_i$ , $\delta y_i$ , the following ODEs with respect to $x_i(t)$ and $y_i(t)$ are obtained

$$\begin{aligned}&A_{ik}(\ddot{x}_k + \varepsilon \dot{x}_k) + B_{ik}x_k + C_{ip}y_p + D_{ikp}x_k y_p = Q_i q_0, \\ &C_{pi}x_i + E_{pj}y_j + \frac{1}{2}D_{pki}x_k x_i = 0,\end{aligned} \tag{5}$$

where $A_{ik}, B_{ik}, C_{ip}, D_{ikp}, E_{pj}$ denote matrix coefficients.

In the above the following dimensional quantities are introduced: $t = \bar{t}\,\tau$, $\varepsilon = \bar{\varepsilon}/\tau$, where $\tau = \frac{a}{h_0}\sqrt{\frac{a^2\gamma}{Eg}}$, and the load parameter $q_0(t) = q(t)\,a^4 / Eh_0^4$, $\bar{w} = \frac{w}{h_0}$, $\bar{x}_i = \frac{x_i}{h_0}$, $\bar{\varphi} = \frac{\varphi}{Eh_0^3}$, $\bar{y}_i = \frac{y_i}{Eh_0^3}$, $\bar{h} = \frac{h(\rho)}{h_0}$. In equations (5) the bars over non-dimensional quantities are omitted.

Solving the second equation of system (5) with respect to $y_i$, one gets

$$y_i = \left[ E_{jp}^{-1} C_{ps} + \frac{1}{2}\left( E_{jp}^{-1} D_{pi} x_i \right)_s \right] x_s \,. \tag{6}$$

Multiplying the first equation of (5) by $A_{ik}^{-1}$, and denoting $\dot{x}_i = r_i$, the following Cauchy problem is defined

$$\begin{gathered} \dot{r}_i = -\bar{\varepsilon}\, r_i + \left[ A_{ik}^{-1} C_{kj} + \left( A_{ik}^{-1} D_{ks} x_s \right)_j \right] y_j - A_{ik}^{-1} B_{ks} x_s + q_0(\bar{t}) A_{ik}^{-1} Q_k \,, \\ x_i = r_i \,, \qquad i,k,s = 1,2,3,\ldots,n; \qquad p,j = 1,2,\ldots,m. \end{gathered} \tag{7}$$

Note that the applied transformation is allowed since matrices $A_{ik}^{-1}$ and $E_{jp}^{-1}$ exist because the coordinate functions are linearly independent. Equations (7) are solved using the fourth order Runge-Kutta method. The system (7) is integrated using the following initial conditions

$$x_i = 0, \ \dot{x}_i = 0 \ \text{ for } t = 0 \,.$$

The movably supported spherical and conical shells, treated as plates with initial deflection $w_0 = -k(1-\rho^2)$, $w_0 = -k(1-\bar{\rho})$, correspondingly, where $k = H/h_o$ denotes the deflection arrow, are studied. The being approximated functions for the given boundary conditions read

$$w_i(\rho) = (1-\rho^2)^{i+1}, \quad \varphi_j(\rho) = \rho^{2j} \,. \tag{8}$$

**3. Reliability of the obtained results**

The earlier introduced Ritz algorithm allows to solve a wide class of static as well as dynamical problems. A solution to static problems can be obtained through the 'set-up' method applied firstly by Fedos'ev [8]. Solving the Cauchy problem for $\varepsilon = \varepsilon_{cr}$ for different values of transversal the following maps are defined $\{q_i\} \to \{w_i, \varphi_i\}$, and hence the dependencies $q[w(0)]$ are constructed and the stress-strain design state is analyzed. The obtained results are compared with those obtained using the Ritz method for static higher order problems ($m = n = 3$) by Kantor [9]. The graph $q(w)$ obtained by Kantor fully coincide with those obtained through our approach. The digits in the parameter $q$ are associated with the deflection arrow $k$.

In order to compare various dynamical behaviors of the investigated systems, a concept of phase space is applied. The partial differential equations governing continuous systems (4) are substituted by infinite dimensional ODEs due to development of (3) into systems of ODEs. This substitution requires a brief comment. Namely, in practice instead of infinite dimensional systems, the truncated systems are used. One may expect that an increase of equations number yields a certain threshold number, and further increase of equations does not bring anything qualitatively and quantitavely new in the obtained results. A key role during this procedure plays the attractor dimension. However, even if an attractor dimension is bounded, an important role play effects associated through truncation of the system (8). Note that in a case of improper choice of the basis used for transformation into ODEs, an associated truncated system may exhibit unreal attractors. The described so far phenomenon occurred in two dimensional equations modeling heat fluid convection. The Lorenz system [10], representing three modes approximation, exhibits chaotic dynamics. However, increase of modes number decreases chaos dimension. For sufficiently large modes number chaos is not exhibited at all. In reference [11] it is shown, that for large Prandtl numbers $\delta$, the considered two- dimensional Boussinesque convection is characterized by critical values of the Rayleigh number Ra associated with one- and two-dimensional vibration processes, whereas further increase of Ra yields a one frequency periodic convection. Owing to the described so far examples, it is recommended to include a sufficient large number of modes either through the Bubnov-Galerkin or Ritz methods.

In what follows an influence of modes number in the Ritz procedure on example of spherical and conical shells vibrations with non-constant thickness and movably supported is studied. The input loads is uniformly distributed on the shell surface of the sinusoidal form

$$q = q_0 \sin \omega_p t \ . \qquad (9)$$

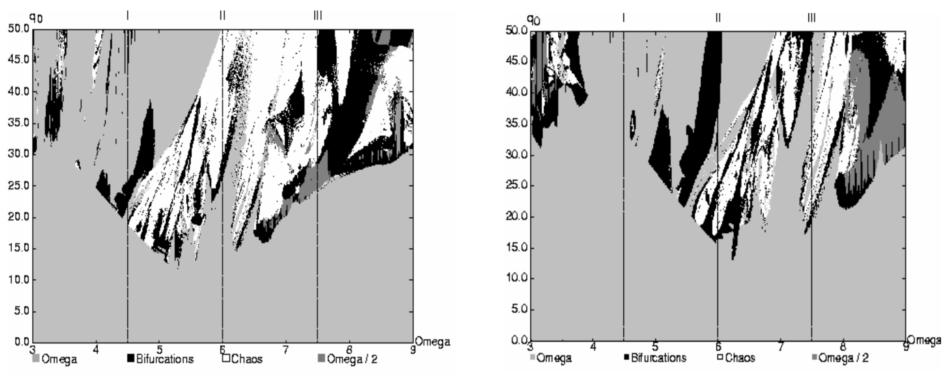
Our numerical simulations show that a computational convergence essentially depends on the deflection arrow $k$ and on the shells geometry. In the case of a constant thickness for $k \le 2$, and for a sphere, as well as for a cone beginning from $n \ge 2$, the convergence occurs. In the case of the sphere for $2 < k \le 3$ the obtained series is convergent beginning from $n \ge 2$, whereas for $k > 3$ the convergence begins from $n \ge 4$. In the case of the cone for $2 < k \le 3$, the convergence occurs for $n \ge 4$, whereas for $k > 3$ it begins for $n \ge 5$.

In this report only results associated with $n = 6$ are illustrated and discussed.

**4. Spherical shells with constant and non-constant thickness**

Let us analyze spherical shallow shells with constant and non-constant thickness: $h = h_0(1 + c\rho)$, movably supported on their edges, where a shell's meridian form is initiated by the deflection $w_0 = -k(1 - \rho^2)$. Recall that the load is defined through (9), and the initial conditions are zero.

A picture of transition to chaos differs essentially owing to increase of the arrow deflection *k*. Analysis of the dependence $\{q_0, \omega_p\}$ versus *k* shows, that the plate (*k*=0) $\{q_0, \omega_p\}$ is characterized only by harmonic vibrations with constraints on deflection $w(0) \le 5$ and $q_0 \le 100$. Zones of chaos and bifurcations begin to increase with increase of *k*. For *k* = 1 there exist only two zones with a margin inclusion of chaotic zones, whereas increase of $k \ge 1.5$ yields one more bifurcation zone.




(a) (b)

Fig. 1. Charts of regular and chaotic motions and their bifurcations in the parameters $\{q_0, \omega_p\}$ plane for spherical shells for $k = 5$ with constant (a) and non-constant (b) thickness

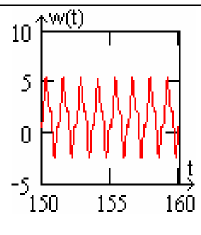
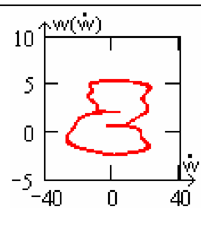
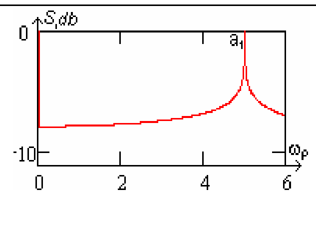
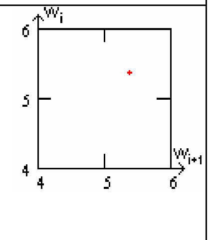
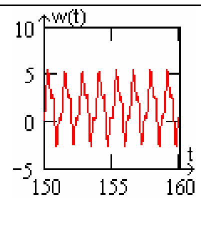
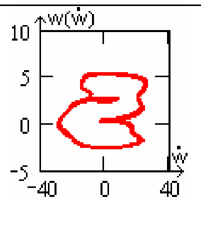
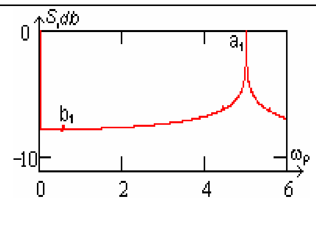
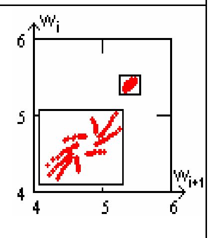
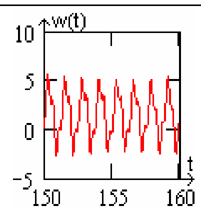
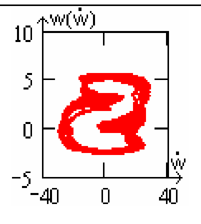
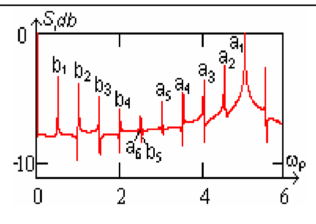
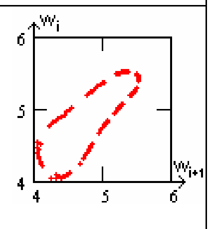
In Fig. 1 charts associated with control parameters $\{q_0, \omega_p\}$ for constant (left) and non-constant (right) thicknesses are reported for $c = 0.1$. Change of the shell thickness essentially influences the system state. Analysis of the obtained results shown in Fig. 1 allows for conclusion that shells with non-constant thickness have smaller domain of chaotic vibrations. In words, both regular and chaotic vibration of a shell can be controlled through a change of shell cross section.

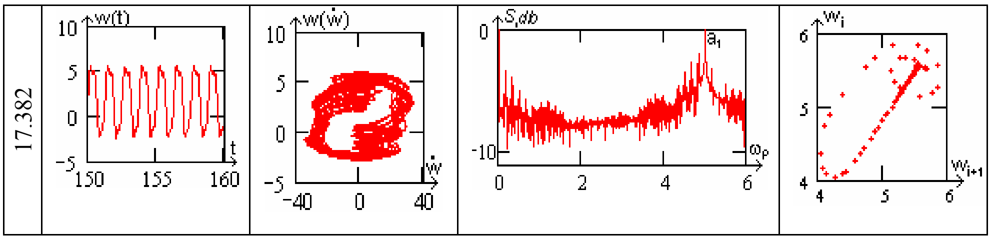
A scenario from regular to chaotic state of a shell with non-constant thickness is similar to that of constant thickness. In both cases a sharp deflection increase is observed of a doubled shell's thickness value during a transition from one to another state. The observed process can be here interpreted as a dynamical stability loss of spherical shells subject to sinusoidal load.

Although a turbulence phenomenon is known since a hundred of years, its mathematical model has been proposed by Landau [12]. It is worth noticing that no one of the existing hypotheses used so far to analyze deterministic vibrations of spherical shells with arbitrary boundary conditions and with an arbitrary deflection arrow can be satisfactorily applied to govern a transition to chaos. Note that the applied control parameters play an essential role during a transition of the considered mechanical system into chaos while changing an amplitude and frequency of external input.

Note that while investigating of chaotic vibrations the typical classical diagrams are not constructed here, and a bifurcation scale is constructed depended on $\{q_0, \omega_p\}$ for a fixed value of $\omega_p$. This approach is motivated by an earlier observation that in the chart $\{q_0, \omega_p\}$ various transitions to chaos are possible. The new scenario from regular to chaotic dynamics occurred in spherical and conical shells for both constant and non-constant thickness are reported (Table 1), where also a signal $w(t)$, phase portrait, power spectrum and Poincaré maps are included. Increase of the parameter $q_0$ yields a period doubling bifurcation, i.e. a second frequency occurs which is linearly independent from the first one.

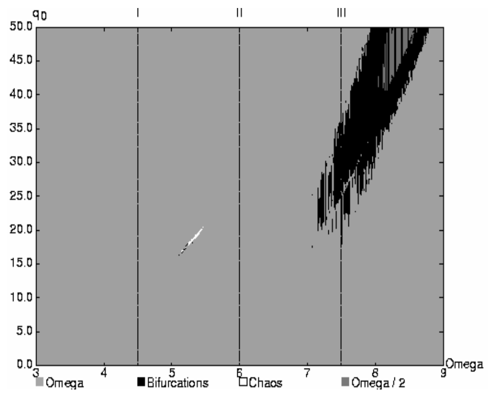
Table 1

| $q_0$ | $w$(t) | Phase portrait | Power spectrum | Poincaré map |
|---|---|---|---|---|
| 15. |  |  |  |  |
| 15.92 |  |  |  |  |
| 16.3 |  |  |  |  |

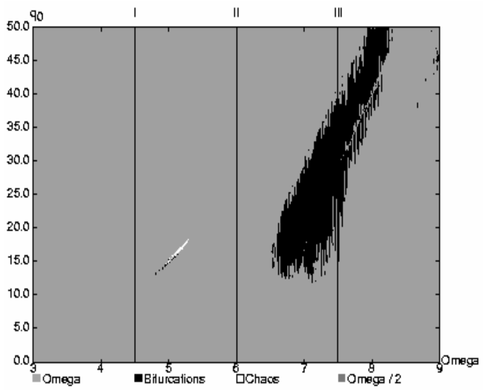



## 5. Investigation of chaotic vibrations

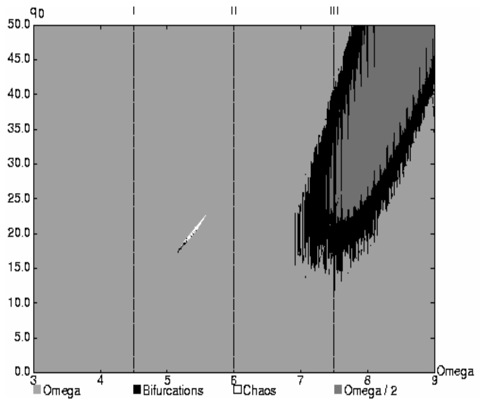
Let us compare analysis of spherical and conical shells applying movable clamping and zero values of initial conditions. Analysis of the dependence $\{q_0, \omega_p\}$ versus deflection arrow *k* shows, that an influence of shell geometry on the vibration character increases with increase of *k*. For example, for the plate (*k*=0) $\{q_0, \omega_p\}$ only harmonic vibrations occur for $w(0) \le 5$ and $q_0 \le 100$. For *k*=1 in both a cone and a sphere, a small chaotic zone appears between frequencies $\omega_p = 5$ and $\omega_p = 6$. For *k* = 1.5 in both charts bifurcation zones appear. For *k* = 2 zones of multiple frequencies are added. For $k \ge 3$, an influence of the shell's geometry yields sufficient differences in the charts for the sphere and the cone. In Figure 2 the charts of the control parameters $\{q_0, \omega_p\}$ of the cone (left) and the sphere for $k = 1.5; 2; 5$, where the geometry influence is clearly expressed, are shown.



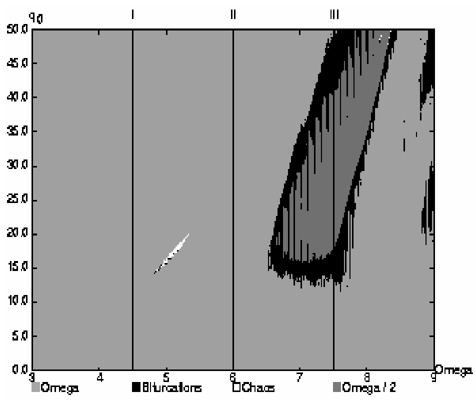

a) cone, $k = 1.5$



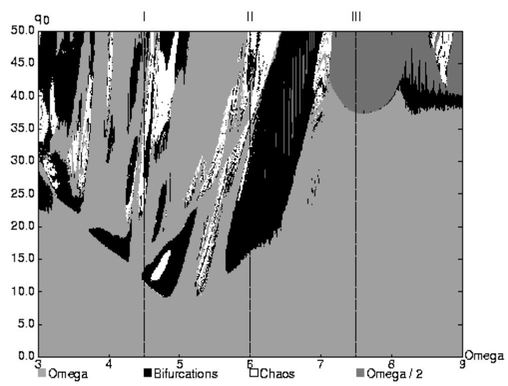

b) sphere, $k = 1.5$



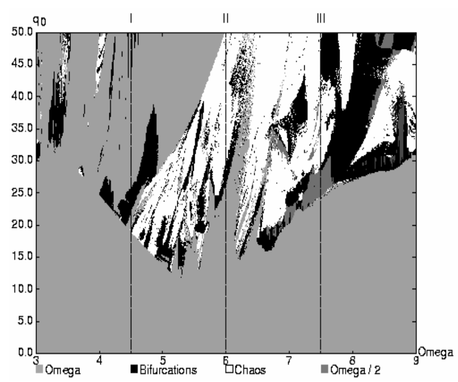

c) cone, $k = 2$




d) sphere, $k = 2$




e) cone, $k = 5$




f) sphere, $k = 5$

Fig.2. Charts of the control parameters $\{q_0, \omega_p\}$ of conical and spherical shells

Besides of the mentioned route to chaos, the parameters plane $\{q_0, \omega_p\}$ of conical and spherical shells with the arrow deflection $k \geq 4$, there are subspaces where a route to chaos follows the Feigenbaum scenario [13]. However, in the case of spherical shells the Feigenbaum constant can not be estimated, since only three bifurcations have been observed. Note that in the case of conical and spherical shells with $k \leq 3$ the Feigenbaum scenario is not detected.

For the conical shell with the deflection arrow $k = 5$ the following series is obtained

$$d_n = \frac{q_{0,n} - q_{0,n-1}}{q_{0,n+1} - q_{0,n}} = 4.66830065 .$$

Theoretical value, obtained for the function $f = \left(1 - cx^2\right)$ [13, Table1.25] reads: $d = 4.66916224$ . A difference between theoretical and numerical experiments for conical shells is of amount of 0.018%. The values of the series $q_{0,n}$ and the series $d$ are reported in Table 2.

Table 2

| n | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $Q_{0,n}$ | 9.605846 | 11.098 | 11.77755 | 11.9204 | 11.951 |
| $d_n$ | | 2.19579722 | 4.75708785 | 4.66830065 | |

## 6. Conclusions

In this work the method to study chaotic vibrations of shallow spherical and conical shells with respect to the series of parameters like a deflection arrow, amplitude and frequency of excitation, shell thickness and its parameters is proposed. The various charts of the shells vibrations depended on the control parameters $\{q_0, \omega_p\}$ are reported. First, an influence of modes number on the vibration character of spherical shells in bifurcation and chaotic zones is studied. A threshold occurrence, when either chaos or so called 'multi-modes turbulence' appears, is illustrated and discussed. Various dynamical phenomena of spherical shells versus deflection arrow are reported.

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