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CHAOS GENERATED BY A DESTRUCTION OF FOUR HOMOCLINIC ORBITS

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Abstract. Chaotic dynamics of a rotated pendulum is predicted analytically using the Melnikov's technique. It is shown, among others, that the one-degree-of-freedom system can exhibit either two or four homoclinic orbits and one or two Melnikov criteria can be applied, respectively. A formula of system parameters has been derived in such way that all homoclinic orbits are destroyed.

1. Introduction

After occurrence of first reports of chaotic dynamics exhibited by simple oscillators, there is still an attempt to detect chaotic orbits in possibly simple dynamical systems [1, 2, 4, 5, 7]. However, chaotic behaviour presented in the mentioned references is mainly detected and analysed through various numerical approaches. In contrary, our attention is focused on an analytical estimation of chaotic thresholds in the system parameters space.

2. Melnikov's Method

The dynamics of a one-degree-of-freedom nonlinear oscillator can be governed by the equation

$$\ddot{x} + S(x) = \varepsilon G(x,\dot{x},t), \tag{1}$$

where: x is the displacement, $S(x)$ represents the nonlinear stiffness, $G(x,\dot{x},t)$ is the time T-periodical function depending on the displacement and velocity, and $\varepsilon \geq 0$ is the small parameter. Eq. (1) can be written in the form of two first order differential equations

$$\begin{aligned} \dot{v} &= -S(x) + \varepsilon G(x,v,t), \\ \dot{x} &= v, \end{aligned} \tag{2}$$

where v is the velocity. For $\varepsilon = 0$ we get an unperturbed system of the following form

$$\begin{aligned} \dot{v} &= -S(x), \\ \dot{x} &= v. \end{aligned} \tag{3}$$

Let us suppose that the unperturbed system governed by Eq. (3) has an equilibrium point p_0 at the origin of the phase plane. If the point p_0 is of the saddle type, then it is possible to find a homoclinic orbit q_0 associated to that point. In the case when $G(x,v,t)$ is T-periodic in time, the perturbed (2) can be written as an autonomous one

$$\begin{aligned} \dot{v} &= -S(x) + \varepsilon G(x,v,\eta), \\ \dot{x} &= v, \\ \dot{\eta} &= \omega, \end{aligned} \tag{4}$$

where the frequency $\omega = 2\pi/T$. The phase space of the system (4) for $\varepsilon = 0$ has the cycle structure caused by η and the hyperbolic orbit. Therefore, we can define a Poincaré map $P_\varepsilon : \Sigma^{\eta_0} \to \Sigma^{\eta_0}$, which transforms $\Sigma^{\eta_0} = \{(x,v,\eta)(\eta = \eta_0 \in [0,T)\}$ to itself and which has a saddle fixed point with a homoclinic orbit on Σ^{η_0}, corresponding to the phase portrait of the system (4). For $\varepsilon > 0$ the homoclinic orbit splits and it yields the stable $W^s(p_\varepsilon)$ and unstable $W^n(p_\varepsilon)$ manifolds of the hyperbolic point p_ε lying near p_0, of the form

$$\begin{aligned} W^s(p_\varepsilon) &= \left\{ (x,v) \in \Sigma^{\eta_0} \middle| \lim_{n \to +\infty} P_\varepsilon^n = p_\varepsilon \right\}, \\ W^n(p_\varepsilon) &= \left\{ (x,v) \in \Sigma^{\eta_0} \middle| \lim_{n \to +\infty} P_\varepsilon^{-n} = p_\varepsilon \right\}. \end{aligned}$$

The projection $\bar{d}(t_0)$ [6, 3] on the normal to the homoclinic orbit $q_0(t-t_0) = (x_0(t-t_0),\ v_0(t-t_0))$ of the distance $d(t_0)$ between $W^s(p_\varepsilon)$ and $W^n(p_\varepsilon)$ can be obtained as follows

$$\bar{d}(t_0) = -\varepsilon \frac{M(t_0)}{(f(} + O(\varepsilon^2),$$

where: $|f| = v^2 + S^2(x)$ and

$$M(t_0) = \int_{-\infty}^{\infty} v_0(t) G[q_0(t), t + t_0] dt$$

is called the Melnikov function. For $M(t_0) = 0$ the stable $W^s(p_\varepsilon)$ and unstable $W^n(p_\varepsilon)$ manifolds of the hyperbolic point p_ε intersect, and when $M(t_0)$ has simple zeros (additionally $\partial M(t_0)/\partial t_0 \neq 0$), then by the Smale-Birkhoff homoclinic theorem, the power of the set of the intersection points equals to the power of the set of the integer numbers ($\aleph_0$), and chaotic motions can appear.

Some systems can exhibit more than one homoclinic orbit. In this case there are more than one Melnikov functions. If we combine system parameters in such way that two or more Melnikov functions disappear, then a point in a parameters space, where two or more homoclinic orbits are destroyed, will be detected.

3. Analysed System

A forced rotated pendulum with small damping is shown in Fig. 1.

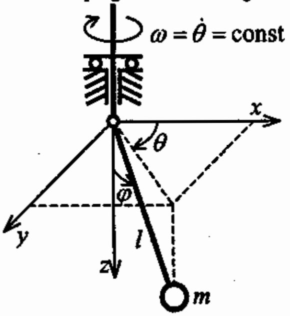

Fig. 1. Analysed system.

The analysed system is governed by the following second order differential equation

$$ml^2\ddot{\varphi} - ml^2\omega^2 \sin\varphi\cos\varphi + mgl\sin\varphi = \varepsilon\left(A\cos\omega_1 t - D\dot{\varphi}\right),$$

and hence

$$\ddot{\varphi} - \tfrac{1}{2}\omega^2 \sin 2\varphi + \omega_0^2 \sin\varphi = \varepsilon\left(\gamma\cos\omega_1 t - \delta\dot{\varphi}\right), \tag{5}$$

where: $\omega_0 = \sqrt{g/l}$, $\gamma = A/\left(ml^2\right)$, $\delta = D/\left(ml^2\right)$, ε - small parameter.

Eq. (5) is transformed into the following form

$$\dot{\chi} = \tfrac{1}{2}\omega^2 \sin 2\varphi - \omega_0^2 \sin\varphi + \varepsilon\left(\gamma\cos\omega_1 t - \delta\chi\right),$$
$$\dot{\varphi} = \chi.$$

For $\varepsilon = 0$ one gets the unperturbed system with the associated Hamiltonian

$$H\left(\varphi,\dot{\varphi}\right) = \frac{\dot{\varphi}^2}{2} + \frac{1}{4}\omega^2\cos 2\varphi - \omega_0^2\cos\varphi .$$

There are two main critical points of the system: $\varphi_{01} = 0$ and $\varphi_{02} = \pm\pi$. For $|\omega| > \omega_0 > 0$ two other ones occur

$$\varphi_{03,4} = \pm\arccos\left(\frac{\omega_0^2}{\omega^2}\right), \qquad \left|\varphi_{03,4}\right| < \pi/2 .$$

Observe that the point φ_{02} is always a hyperbolic saddle. The φ_{01} one changes its type depending of the parameters value. For $|\omega| < \omega_0$ it is a centre, whereas for $|\omega| > \omega_0$ we have a hyperbolic saddle critical point. The additional points $\varphi_{03,4}$ are centres, when they exist.

A homoclinic orbit of the critical point φ_{02} satisfies the equation

$$\frac{d\varphi}{dt} = \pm\sqrt{2\omega_0^2\left(\cos\varphi + 1\right) + \omega^2\left(1 - \cos^2\varphi\right)}. \tag{6}$$

Integrating Eq. (6) one gets

$$q_{02} = \begin{cases} \varphi_{02}(t) = \operatorname{sgn} t \cdot \arccos\left(\dfrac{4\left(\omega^2 + \omega_0^2\right)}{2\omega^2 + \omega_0^2\left(1 + \cosh\left(2\sqrt{\omega^2 + \omega_0^2}\,t\right)\right)} - 1\right) \\ \dot{\varphi}_{02}(t) = \pm\dfrac{4\omega_0\left(\omega^2 + \omega_0^2\right)\cosh\left(\sqrt{\omega^2 + \omega_0^2}\,t\right)}{2\omega^2 + \omega_0^2\left(1 + \cosh\left(2\sqrt{\omega^2 + \omega_0^2}\,t\right)\right)} \end{cases} \tag{7}$$

The other homoclinic orbit associated to the critical point φ_{02} takes the form

$$\frac{d\varphi}{dt} = \pm\sqrt{2\omega_0^2\left(\cos\varphi - 1\right) + \omega^2\left(1 - \cos^2\varphi\right)}, \qquad |\omega| > \omega_0. \tag{8}$$

Integrating Eq. (8) one gets

$$q_{01} = \begin{cases} \varphi_{01}(t) = \pm\arccos\left(1 - \dfrac{8\left(\omega^2 - \omega_0^2\right)z}{\left(\omega_0^2 - z\right)^2 + 4\omega^3 z}\right) \\ \dot{\varphi}_{01}(t) = \pm\dfrac{4\left(\omega_0^2 - z\right)\left(\omega^2 - \omega_0^2\right)\sqrt{z}}{\left(\omega_0^2 - z\right)^2 + 4\omega^2 z} \end{cases}, \qquad |\omega| > \omega_0. \tag{9}$$

where: $z = \omega_0^2 e^{2\sqrt{\omega^2 - \omega_0^2}\,t}$.

Summing up, there are exist two or four homoclinic orbits. A number of homoclinic solutions depends on the system parameters. For $|\omega| < \omega_0$ the analysed system has two ones described by Eq. (7), and for $|\omega| > \omega_0$ there are four homoclinic orbits - see Eq. (9). The homoclinic orbits are shown in Fig. 2.

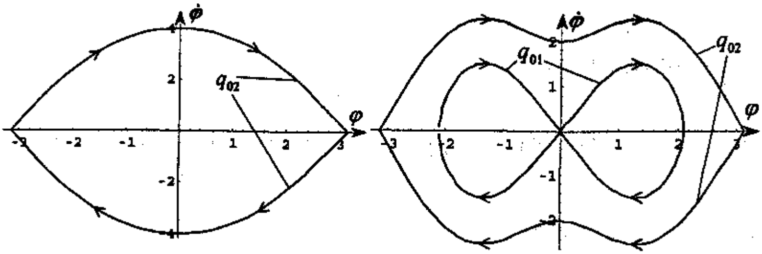

Fig. 2. Homoclinic orbits for $\omega_0 = 2$, $\omega = 1$ (to the left) and for $\omega_0 = 1$, $\omega = 2$ (to the right).

4. Melnikov's functions

The Melnikov's function along homoclinic orbit q_{02} yields

$$M_2(t_0) = \int_{-\infty}^{\infty} \dot\varphi_{02}(t)\left(\gamma\cos\omega_1(t+t_0) - \delta\dot\varphi_{02}(t)\right)dt \,. \tag{10}$$

Substituting (7) into (10) one gets

$$M_{s2}(t_0) = I_1 - I_2 , \tag{11}$$

where:

$$I_1 = 4\gamma\omega_0\left(\omega^2+\omega_0^2\right)\int_{-\infty}^{\infty}\frac{\cosh\left(\sqrt{\omega^2+\omega_0^2}\,t\right)\cos\omega_1(t+t_0)}{2\omega^2+\omega_0^2\left(1+\cosh\left(2\sqrt{\omega^2+\omega_0^2}\,t\right)\right)}dt ,$$

$$I_2 = 16\delta\omega_0^2\left(\omega^2+\omega_0^2\right)^2\int_{-\infty}^{\infty}\frac{\cosh^2\left(\sqrt{\omega^2+\omega_0^2}\,t\right)}{\left(2\omega^2+\omega_0^2\left(1+\cosh\left(2\sqrt{\omega^2+\omega_0^2}\,t\right)\right)\right)^2}dt \,.$$

Expression I_1 yields

$$I_1 = I_{11} + I_{12} , \tag{12}$$

where:

$$I_{11} = -4\gamma\omega_0\left(\omega^2+\omega_0^2\right)\sin\omega_1 t_0\int_{-\infty}^{\infty}\frac{\cosh\left(\sqrt{\omega^2+\omega_0^2}\,t\right)\sin\omega_1 t}{2\omega^2+\omega_0^2\left(1+\cosh\left(2\sqrt{\omega^2+\omega_0^2}\,t\right)\right)}dt ,$$

$$I_{12} = 4\gamma\omega_0\left(\omega^2+\omega_0^2\right)\cos\omega_1 t_0\int_{-\infty}^{\infty}\frac{\cosh\left(\sqrt{\omega^2+\omega_0^2}\,t\right)\cos\omega_1 t}{2\omega^2+\omega_0^2\left(1+\cosh\left(2\sqrt{\omega^2+\omega_0^2}\,t\right)\right)}dt \,.$$

Note that the integrand in I_{11} is an odd function, hence

$$I_{11} = 0 \,.$$

Using (12) one obtains

$$I_1 = \gamma A_{12} \cos \omega_1 t_0 , \tag{13}$$

where the constant A_{12} reads

$$A_{12} = 8\omega_0 \left(\omega^2 + \omega_0^2\right) \int_0^\infty \frac{\cosh\left(\sqrt{\omega^2 + \omega_0^2 t}\right)\cos\omega_1 t}{2\omega^2 + \omega_0^2\left(1 + \cosh\left(2\sqrt{\omega^2 + \omega_0^2 t}\right)\right)} dt .$$

Integrating expression I_2 one gets

$$I_2 = 2\delta \frac{\omega_0^2}{\omega}\left(\operatorname{arctgh}\left(\frac{\omega}{\sqrt{\omega^2 + \omega_0^2}} \operatorname{tgh}\left(\sqrt{\omega^2 + \omega_0^2 t}\right)\right) + \frac{\omega\sqrt{\omega^2 + \omega_0^2}\sinh\left(2\sqrt{\omega^2 + \omega_0^2 t}\right)}{2\omega^2 + \omega_0^2\left(1 + \cosh\left(2\sqrt{\omega^2 + \omega_0^2 t}\right)\right)}\right)\Bigg|_{-\infty}^{\infty}$$

and

$$I_2 = 4\delta\left(\frac{\omega_0^2}{\omega}\operatorname{arctgh}\left(\frac{\omega}{\sqrt{\omega^2 + \omega_0^2}}\right) + \sqrt{\omega^2 + \omega_0^2}\right). \tag{14}$$

Substituting (13) and (14) into (11) we obtain

$$M_2(t_0) = -\gamma A_{12} \sin\omega_1 t_0 - 4\delta\left(\frac{\omega_0^2}{\omega}\operatorname{arctgh}\left(\frac{\omega}{\sqrt{\omega^2 + \omega_0^2}}\right) + \sqrt{\omega^2 + \omega_0^2}\right).$$

Then, the first Melnikov criterion is given by the formula

$$\left|\gamma A_{12}\right| > 4\delta\left(\frac{\omega_0^2}{\omega}\operatorname{arctgh}\left(\frac{\omega}{\sqrt{\omega^2 + \omega_0^2}}\right) + \sqrt{\omega^2 + \omega_0^2}\right). \tag{15}$$

The next Melnikov's function along homoclinic orbit q_{01} yields

$$M_1(t_0) = \int_{-\infty}^{\infty} \dot\varphi_{01}(t)\left(\gamma\cos\omega_1(t + t_0) - \delta\dot\varphi_{01}(t)\right)dt . \tag{16}$$

Substituting (9) into (16) we obtain

$$M_{s1}(t_0) = I_3 - I_4 , \tag{17}$$

where:

$$I_3 = 4\gamma\left(\omega^2 - \omega_0^2\right)\int_{-\infty}^{\infty}\frac{\left(\omega_0^2 - z\right)\sqrt{z}}{\left(\omega_0^2 - z\right)^2 + 4\omega^2 z}\cos\omega_1(t + t_0)dt,$$

$$I_4 = 16\delta\left(\omega^2 - \omega_0^2\right)^2\int_{-\infty}^{\infty}\frac{\left(\omega_0^2 - z\right)^2 z}{\left(\left(\omega_0^2 - z\right)^2 + 4\omega^2 z\right)^2}dt, \qquad |\omega| > \omega_0 ,$$

and $z = \omega_0^2 e^{2\sqrt{\omega^2 - \omega_0^2}\, t}$. Expression I_3 yields

$$I_3 = I_{31} + I_{32} , \tag{18}$$

where:

$$I_{31}=\gamma\cos\omega_1 t_0\int_{-\infty}^{\infty}\frac{4\left(\omega_0^2-z\right)\left(\omega^2-\omega_0^2\right)\sqrt{z}}{\left(\omega_0^2-z\right)^2+4\omega^2 z}\cos\omega_1 t dt,$$

$$I_{32}=-\gamma\sin\omega_1 t_0\int_{-\infty}^{\infty}\frac{4\left(\omega_0^2-z\right)\left(\omega^2-\omega_0^2\right)\sqrt{z}}{\left(\omega_0^2-z\right)^2+4\omega^2 z}\sin\omega_1 t dt.$$

Observe that the integrand of I_{31} is an odd function, and therefore

$$I_{31}=0.$$

Using (18) one obtains

$$I_3=-\gamma A_{32}\sin\omega_1 t_0, \tag{19}$$

where the constant A_{32} reads

$$A_{32}=\int_{-\infty}^{\infty}\frac{4\left(\omega_0^2-z\right)\left(\omega^2-\omega_0^2\right)\sqrt{z}}{\left(\omega_0^2-z\right)^2+4\omega^2 z}\sin\omega_1 t dt.$$

Introducing the variable change $z=\omega_0^2 e^{2\sqrt{\omega^2-\omega_0^2}t}$, the expression I_4 takes the form

$$I_4=8\delta\left(\omega^2-\omega_0^2\right)^{\frac{1}{4}}\int_{-\infty}^{\infty}\frac{\left(\omega_0^2-z\right)^2 dz}{\left(\left(\omega_0^2-z\right)^2+4\omega^2 z\right)^2}.$$

Its integration yields

$$I_4=-\frac{2\delta}{\sqrt[4]{\omega^2-\omega_0^2}}\left(\frac{2\left(\omega_0^4+z\left(2\omega^2-\omega_0^2\right)\right)}{\left(\omega_0^2-z\right)^2+4\omega^2 z}+\frac{\omega_0}{\sqrt{\omega^2-\omega_0^2}}\operatorname{arcctg}\left(\frac{2\omega^2-\omega_0^2+z}{2\omega_0\sqrt{\omega^2-\omega_0^2}}\right)\right)\Bigg|_{-\infty}^{\infty},$$

and

$$I_4=-2\pi\delta\omega_0\left(\omega^2-\omega_0^2\right)^{-\frac{1}{4}}. \tag{20}$$

Substituting (19) and (20) into (17), one gets

$$M_1\left(t_0\right)=\gamma A_{32}\cos\omega_1 t_0+2\pi\delta\omega_0\left(\omega^2-\omega_0^2\right)^{-\frac{1}{4}}.$$

Then, the second Melnikov criterion is given by the formula

$$\left|\gamma A_{32}\right|>2\pi\delta\omega_0\left(\omega^2-\omega_0^2\right)^{-\frac{1}{4}}. \tag{21}$$

The Eq. (15) and Eq. (21) are shown in Fig. 3 in the (γ,ω) plane for $\omega_0=2$, $\omega_1=1$, $\delta=0{,}2$. For $\omega>2$, there exist two curves corresponding to both Melnikov functions $M_1(t_0)$ and $M_2(t_0)$ and they cross each other at the point $(\bar{\gamma},\bar{\omega})$.

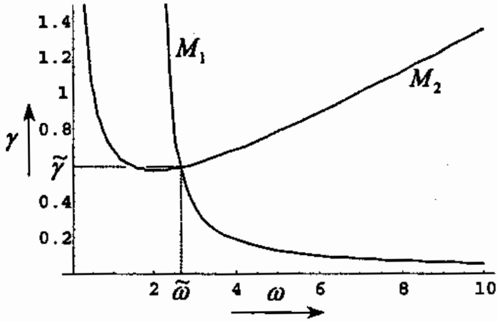

Fig. 3. Chaotic thresholds in the (γ,ω) plane ($\omega_0=2$, $\omega_1=1$, $\delta=0{,}2$).

Eq. (15) and Eq. (21) are shown in Fig. 4 in the (γ,ω_0) plane for $\omega=4$, $\omega_1=1$, $\delta=0{,}2$. For $\omega_0<4$, there exist two curves corresponding to both Melnikov functions $M_1(t_0)$ and $M_2(t_0)$, and they cross each other at the point $(\tilde{\gamma},\tilde{\omega}_0)$.

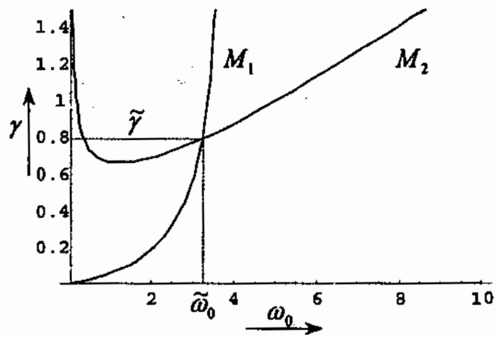

Fig. 4. Chaotic thresholds in the (γ,ω_0) plane ($\omega=4$, $\omega_1=1$, $\delta=0{,}2$).

Transforming Eq. (15) and Eq. (21) one gets

$$\frac{4\delta}{A_{12}}\left(\frac{\tilde{\omega}_0^2}{\tilde{\omega}}\operatorname{arctgh}\left(\frac{\tilde{\omega}}{\sqrt{\tilde{\omega}^2+\tilde{\omega}_0^2}}\right)+\sqrt{\tilde{\omega}^2+\tilde{\omega}_0^2}\right)=\frac{2}{A_{32}}\pi\delta\tilde{\omega}_0\left(\tilde{\omega}^2-\tilde{\omega}_0^2\right)^{-\frac{1}{4}}=\tilde{\gamma}\,.$$

The condition of crossing curves corresponding to Melnikov functions $M_1(t_0)$ and $M_2(t_0)$ is given by the formula

$$\frac{\tilde{\omega}_0^2}{\tilde{\omega}}\operatorname{arctgh}\left(\frac{\tilde{\omega}}{\sqrt{\tilde{\omega}^2+\tilde{\omega}_0^2}}\right)+\sqrt{\tilde{\omega}^2+\tilde{\omega}_0^2}-\frac{A_{12}}{2A_{32}}\pi\tilde{\omega}_0\left(\tilde{\omega}^2-\tilde{\omega}_0^2\right)^{-\frac{1}{4}}=0\,.$$

5. Conclusions

The classical Melnikov's approach is applied to predict chaotic behaviour of the rotated pendulum with small damping. It is illustrated that the system can posses either two $(|\omega|<\omega_0)$ or four

$(|\omega| > \omega_0)$ homoclinic orbits. Two Melnikov's criteria of chaos are computed and associated chaotic thresholds in the (γ, ω) and (γ, ω_0) parameter planes are reported.

6. References

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