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STICK-SLIP CHAOS PREDICTION IN A TWO DEGREES-OF-FREEDOM MECHANICAL SYSTEM WITH FRICTION

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Abstract. We consider two coupled oscillators with negative Duffing type stiffness which are self (due to friction) and externally (harmonically) excited. The fundamental solutions of the homoclinic orbit are constructed. Then, the Melnikov-Gruendler approach is used to define the Melnikov's function including smooth and stick-slip chaotic behaviour.

1. Introduction

The presented work has at least a few motivations. First, there exist a vast research devoted to analysis of low and high dimensional system with friction. Some fundamental problems of non-smooth dynamical systems with friction are addressed for example, in references [1,2]. Beginning from the pioneering work of Melnikov [9], the Melnikov-like approaches spread into different branches of science. We briefly address the Melnikov-like techniques to predict the onset of chaos in systems governed by ODEs or maps. A splitting of separatrices for high-frequency perturbations of a planar Hamiltonian system using the Melnikov technique is also examined (see [6]). Mainly Gruendler's work [7] served for us as the basic reference to start with a construction of a homoclinic orbit in our $4D$ mechanical system perturbed by friction and harmonic excitation, and then to derive the associated Melnikov's function. It is worth noticing that an important opened problem of the Melnikov's approach relies on its extension into analysis of higher order dynamical systems. This problem seems to be unsolved since it is difficult to establish a priori a homoclinic orbit associated with a highly dimensional system considered.

It is needless to say that a prediction of chaos in an analytical way in non-smooth objects modelled as systems in R^4 plays a crucial role for both theoretical and applicable reasons. A key role of research carried out in this direction plays the paper by Awrejcewicz and Holicke [3], where a chaotic threshold for both smooth and stick-slip chaotic behaviour in one degree-of-freedom system with friction has been obtained using directly the Melnikov's technique. On the other hand, it was impossible to extend directly the original Melnikov's method devoted to analysis of an analytic system in R^2. Therefore, we have applied the Gruendler extension of the Melnikov's method to R^4, which is further referred as the Melnikov-Gruendler approach. However, in the cited Gruendler's work [7] again an emphasis of C^2 systems is given. In contrary, in our research we extend the results obtained earlier (see [3]) to R^4. Although we do not give a rigorous definitions and proofs of a C^n vector field on R^n, but we show the computations of related integrals yielding a being sought chaotic threshold defined by the approproate Melnikov's function.

2. The analysed system

The analysed mechanical object consists of two stiff bodies with the masses m coupled via linear and nonlinear springs in the way shown in Figure 1.

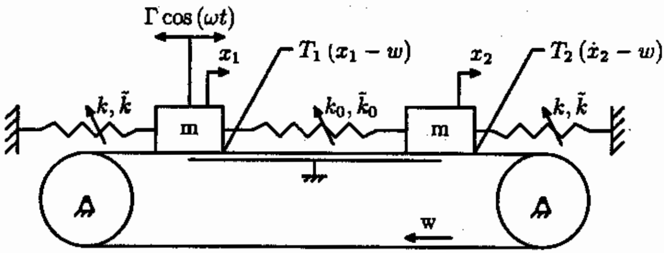

Fig. 1. Two degrees-of-freedom system with friction

Note that when the system is autonomous, i.e. $\Gamma = 0$, the self-excited oscillations appear, which are generated by frictional characteristics. The latter ones possess a decreasing part versus a relative velocity between both bodies and the tape moving with a constant velocity w. Although this problem belongs to classical ones and has been studied by vast number of researchers, an attempt to formulate threshold for chaos occurence in the analytical way failed. In what follows we show how to solve this problem using the Melnikov technique applied to our discontinuous system. It is also recommended to be familiar with the reference [3], where a similar like approach has been applied to predict chaos in a similar like system, but with one degree-of-freedom. Dynamics of our system is governed by the following ODEs:

$$\begin{cases} \dot{x}_1 &= p_1/m \\ \dot{p}_1 &= kx_1 - \bar{k}x_1^3 + k_0\left(x_1 - x_2\right) - \bar{k}_0\left(x_1 - x_2\right)^3 + \varepsilon_1 \Gamma \cos\left(\omega t\right) - \varepsilon_2 T_1\left(p_1/m - w\right) \\ \dot{x}_2 &= p_2/m \\ \dot{p}_2 &= kx_2 - \bar{k}x_2^3 - k_0\left(x_1 - x_2\right) + \bar{k}_0\left(x_1 - x_2\right)^3 - \varepsilon_3 T_2\left(p_2/m - w\right) \end{cases} \tag{1}$$

where $T_i\left(p_i/m - w\right) = T_{i0}\,\mathrm{sgn}\left(p_i/m - w\right) - B_{i1}\left(p_i/m - w\right) + B_{i2}\left(p_i/m - w\right)^3$ and w is the tape velocity, whereas $B_{11}, B_{12}, B_{21}, B_{22}, T_{10}, T_{20}$ are the friction coefficients. Introducing the following scaling

$$t \to t\sqrt{\frac{k}{m}}, \quad x = x_1\sqrt{\frac{\bar{k}}{k}}, \quad u = p_1\sqrt{\frac{\bar{k}}{mk^2}}, \quad y = x_2\sqrt{\frac{\bar{k}}{k}}, \quad v = p_2\sqrt{\frac{\bar{k}}{mk^2}} \tag{2}$$

and the following relations $k_0 = \xi k$, $\bar{k}_0 = \xi \bar{k}$ where $\xi \geq 0$. The analysed ODEs are cast in the nondimensional form

$$\begin{pmatrix} \dot{x} \\ \dot{u} \\ \dot{y} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} u \\ x - x^3 + f_\xi\left(x,y\right) \\ v \\ y - y^3 - f_\xi\left(x,y\right) \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon_1 \Gamma' \cos\left(\omega' t\right) - \varepsilon_2 T_1'\left(u - w'\right) \\ 0 \\ -\varepsilon_3 T_2'\left(v - w'\right) \end{pmatrix}, \tag{3}$$

where $f_\xi\left(x,y\right) = \xi\left(x - y\right) - \xi\left(x - y\right)^3$.

3. Melnikov-Gruendler's method

The method applied in the paper is due to Gruendler [?]. Although the theory is a generalization to a non-Hamiltonian case we apply it to a Hamiltonian one. Here we consider a mechanical system governed by the equation:

$$\dot{x}\left(t\right) = f\left(x\left(t\right)\right) + h\left(x\left(t\right), t, \epsilon\right), \tag{4}$$

where $f : R^4 \to R^4$ is a Hamiltonian vector field and $h : R^4 \times R \times B \subset R^4 \to R^4$ is periodic in t with frequency ω and satisfies $h\left(x\left(t\right), t, 0\right) = 0$. For $\epsilon = 0$ we obtain the unperturbed system. Let the unperturbed system possess a homoclinic orbit $\gamma\left(t\right)$ to a hyperbolic point at the origin.

The variational equation along $\gamma\left(t\right)$ is the following:

$$\dot{y}\left(t\right) = Df\left(\gamma\left(t\right)\right)y\left(t\right). \tag{5}$$

We seek a fundamental solution $\left\{\psi^{(1)}(t), \psi^{(2)}(t), \psi^{(3)}(t), \psi^{(4)}(t)\right\}$ to the equation (5) possessing some special properties. The properties are the following:

1. $\psi^{(4)}(t) = \dot{\gamma}(t)$[1]
2. The initial vectors $\psi^{(i)}(0)$ span a vector space
3. Each $\psi^{(i)}(t)$ has the exponential behaviour as $t \to \pm\infty$. Namely:
   $\psi^{(i)}(t) \sim t^{k_i}\, e^{\lambda_i t}\, v^{(i)} \qquad \text{as } t \to +\infty. \quad k_i \in N$
   $\psi^{(i)}(t) \sim t^{k_{\sigma(i)}}\, e^{\lambda_{\sigma(i)} t}\, \bar{v}^{(i)} \quad \text{as } t \to -\infty. \quad k_{\sigma(i)} \in N$
   where σ is a permutation on four symbols and $\left\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\right\}$ are the eigenvalues of $Df(0)$.
4. The signs of $\Re(\lambda_i)$ and $\Re\left(\lambda_{\sigma(i)}\right)$ in the exponential behaviour has to be such that:

$$\psi^{(1)}(t) = \begin{cases} \Re(\lambda_i) > 0 \\ \Re\left(\lambda_{\sigma(i)}\right) > 0 \end{cases} \qquad \psi^{(2)}(t) = \begin{cases} \Re(\lambda_i) > 0 \\ \Re\left(\lambda_{\sigma(i)}\right) < 0 \end{cases} \tag{6}$$

$$\psi^{(3)}(t) = \begin{cases} \Re(\lambda_i) < 0 \\ \Re\left(\lambda_{\sigma(i)}\right) < 0 \end{cases} \qquad \psi^{(4)}(t) = \begin{cases} \Re(\lambda_i) < 0 \\ \Re\left(\lambda_{\sigma(i)}\right) > 0 \end{cases} \tag{7}$$

Next we define an index set I by $i \in I$ if and only if $\psi^{(i)}(t) \xrightarrow{t \to \pm\infty} \infty$. Moreover we form the functions:

$$D(t) = \det\left\{\psi^{(1)}(t), \psi^{(2)}(t), \psi^{(3)}(t), \psi^{(4)}(t)\right\} e^{-\int_0^t \nabla f(\gamma(s))ds} \tag{8}$$

Since f is a Hamiltonian vector field we obtain $\nabla f = 0$. Thus the function $D(t)$ reduces to simpler form $D(t) = \det\left\{\psi^{(1)}(t), \psi^{(2)}(t), \psi^{(3)}(t), \psi^{(4)}(t)\right\}$. Let $K_{ij}(t, t_0)$[2] denote the result of replacing $\psi^{(i)}(t)$ in $D(t)$ by $\frac{\partial h(\gamma(t), t+t_0, 0)}{\partial \varepsilon_j}$. We define the function:

$$M_{ij}(t_0) = -\int_{\infty}^{\infty} K_{ij}(t, t_0)\, \mathrm{d}t, \quad i \in I \tag{9}$$

The function above measures the separation of stable and unstable manifolds. The Melnikov's function is defined as follows:

$$M(t_0) = \sum_{j=1}^{4} M_{ij}(t_0)\, \varepsilon_j, \quad i \in I \tag{10}$$

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[1]It is easy to show that $\dot{\gamma}(t)$ satisfies the equation (5)

[2]This function represents the projection onto the direction of $\psi^{(i)}(t)$ of the ε_j of the h evaluated along $\gamma(t)$.

4. Linearization along homoclinic orbit and fundamental solutions

Let us denote by $\gamma(t)$ the homoclinic orbit of the point $\{0,0,0,0\}$. It has (in our case) the following form

$$\gamma(t)=\begin{pmatrix} q(t)\\ \dot q(t)\\ -q(t)\\ -\dot q(t)\end{pmatrix},\quad \text{where}\quad q(t)=\sqrt{\frac{2(1+2\xi)}{1+8\xi}}\,\mathrm{sech}\left(t\sqrt{1+2\xi}\right). \tag{11}$$

The linearized system of the unperturbed equations (3) in vicinity of the homoclinic orbit $\gamma(t)$ reads

$$\dot\psi=F(t)\,\psi,\quad \text{where}\quad F(t)=\begin{pmatrix} 0 & 1 & 0 & 0\\ 1+\xi-3(1+4\xi)q^2(t) & 0 & -\xi+12\xi q^2(t) & 0\\ 0 & 0 & 0 & 1\\ -\xi+12\xi q^2(t) & 0 & 1+\xi-3(1+4\xi)q^2(t) & 0\end{pmatrix} \tag{12}$$

Next we obtain the following equations

$$\begin{cases}\ddot\psi_1=\left(1+\xi-3(1+4\xi)q^2(t)\right)\psi_1+\xi\left(12q^2(t)-1\right)\psi_3\\ \ddot\psi_3=\left(1+\xi-3(1+4\xi)q^2(t)\right)\psi_3+\xi\left(12q^2(t)-1\right)\psi_1\end{cases} \tag{13}$$

A combination of the equations (17) yields

$$\ddot\phi_1=(1+2\xi)\left(1-6\,\mathrm{sech}^2\left(t\sqrt{1+2\xi}\right)\right)\phi_1,\quad \phi_1\equiv\psi_1-\psi_3 \tag{14}$$

It is easy to see that $\psi^{(4)}(t)=\dot\gamma(t)$ satisfies the above equation. In order to find another solution, the following substitution is applied: $\dot q(t)\to r(t)\,\dot q(t)$. Since $\dot q(t)$ is a solution to (19) one gets

$$\ddot r\dot q+2\dot r\ddot q=0 \tag{15}$$

Integrating of (20) and owing to the obtained results, the solution reads

$$\phi_1(t)=r(t)\,\dot q(t)=\left(\frac{3}{4}C_1t-\frac{1}{2}C_1\,\mathrm{ctgh}(t)+\frac{1}{8}C_1\sinh(2t)+C_2\right)\dot q(t) \tag{16}$$

The above solution possesses the following asymptotics $\phi_1(t)\xrightarrow{t\to\pm\infty}\mathrm{e}^{\pm t\sqrt{1+2\xi}}$. Next, summing up equations (17) we obtain

$$\ddot\phi_2=h(t,\xi)\,\phi_2,\quad h(t,\xi)=1-6\frac{1+2\xi}{1+8\xi}\,\mathrm{sech}^2\left(t\sqrt{1+2\xi}\right),\quad \phi_2=\psi_1+\psi_3 \tag{17}$$

Suppose that $y_1(t)$ is a solution of the above equation then $y_2(t) = y_1(-t)$ is also the solution because $h(t,\xi)$ is an even function with respect to t. Furthermore, it can be shown that $y_1(t)\dot{y}_2(t) - \dot{y}_1(t)y_2(t) = \Omega(\xi)$, where $\Omega(\xi)$ is a time-independent function.

The fundamental solutions of the above equations are the following:

$$\psi^{(1)}(t) = \begin{pmatrix} y_1 \\ \dot{y}_1 \\ y_1 \\ \dot{y}_1 \end{pmatrix}, \quad \psi^{(2)}(t) = \begin{pmatrix} r\dot{q} \\ \dot{r}\dot{q} + r\ddot{q} \\ -r\dot{q} \\ -\dot{r}\ddot{q} - r\ddot{q} \end{pmatrix}, \quad \psi^{(3)}(t) = \begin{pmatrix} y_2 \\ \dot{y}_2 \\ y_2 \\ \dot{y}_2 \end{pmatrix}, \quad \psi^{(4)}(t) = \begin{pmatrix} \dot{q} \\ \ddot{q} \\ -\dot{q} \\ -\ddot{q} \end{pmatrix} \tag{18}$$

5. Computing the Melnikov-Gruendler function

In our case, a perturbation term associated with (3) reads

$$h(t,\varepsilon) = \left(0, \varepsilon_1 \Gamma' \cos(\omega' t) - \varepsilon_2 T_1' (u - w'), 0, -\varepsilon_3 T_2' (v - w')\right)^T. \tag{19}$$

Therefore, one gets[3]

$$\frac{\partial h(\gamma(t), t+t_0, 0)}{\partial \varepsilon_1} = \begin{pmatrix} 0 \\ \Gamma' \cos(\omega'(t+t_0)) \\ 0 \\ 0 \end{pmatrix}, \frac{\partial h(\gamma(t), t+t_0, 0)}{\partial \varepsilon_2} = \begin{pmatrix} 0 \\ -T_1'(\dot{q}(t) - w') \\ 0 \\ 0 \end{pmatrix}, \tag{20}$$

$$\frac{\partial h(\gamma(t), t+t_0, 0)}{\partial \varepsilon_3} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -T_2'(\dot{q}(t) - w') \end{pmatrix}, \frac{\partial h(\gamma(t), t+t_0, 0)}{\partial \varepsilon_4} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{21}$$

Observe that only $K_{2j}(t,t_0)$ should be found, since $\psi^{(2)}(t) \xrightarrow{t\to\pm\infty} \infty$. First K_{21} is found

$$K_{21}(t,t_0) = \det \begin{pmatrix} y_1 & 0 & y_2 & \dot{q} \\ \dot{y}_1 & \Gamma' \cos(\omega'(t+t_0)) & \dot{y}_2 & \ddot{q} \\ y_1 & 0 & y_2 & -\dot{q} \\ \dot{y}_1 & 0 & \dot{y}_2 & -\ddot{q} \end{pmatrix} \tag{22}$$
$$= 2\Gamma' \dot{q} \cos\left(\omega'(t+t_0)\right)(y_1\dot{y}_2 - \dot{y}_1 y_2) = 2\Omega(\xi)\Gamma' \dot{q} \cos\left(\omega'(t+t_0)\right)$$

Second, K_{22} and K_{23} are found

$$K_{22}(t,t_0) = -2\Omega(\xi)\dot{q}T_1'\left(\dot{q} - w'\right), \quad K_{23}(t,t_0) = -2\Omega(\xi)\dot{q}T_2'\left(\dot{q} - w'\right) \tag{23}$$

[3]For more details see section 3

According to (13) we obtain

$$M_{21}(t_0) = -2\sqrt{2}\Gamma'\Omega(\xi)\frac{\pi\omega'}{\sqrt{1+8\xi}}\operatorname{sech}\left(\frac{\pi\omega'}{2\sqrt{1+2\xi}}\right)\sin(\omega' t_0) \tag{24}$$

$$\begin{aligned} M_{22}(t_0) &= 2\Omega(\xi)\int_{-\infty}^{\infty} T_1'(\dot{q}-w')\,dt = 2\Omega(\xi)T_{10}'\int_{-\infty}^{\infty}\dot{q}\operatorname{sgn}(\dot{q}-w')\,dt \\ &- 2\Omega(\xi)B_{11}'\int_{-\infty}^{\infty}\dot{q}(\dot{q}-w')\,dt + 2\Omega(\xi)B_{12}'\int_{-\infty}^{\infty}\dot{q}(\dot{q}-w')^3\,dt \\ &= -\frac{8}{3}\Omega(\xi)B_{11}'\frac{1+2\xi}{1+8\xi}\sqrt{1+2\xi} + \frac{32}{35}\Omega(\xi)B_{12}'\frac{(1+2\xi)^3}{(1+8\xi)^2}\sqrt{1+2\xi} \\ &+ 8\Omega(\xi)B_{12}'w'^2\frac{1+2\xi}{1+8\xi}\sqrt{1+2\xi} + 2\Omega(\xi)T_{10}'\int_{-\infty}^{\infty}\dot{q}\operatorname{sgn}(\dot{q}-w')\,dt \end{aligned} \tag{25}$$

Let us consider the last integral

$$\int_{-\infty}^{\infty}\dot{q}\operatorname{sgn}(\dot{q}-w')\,dt = \frac{1+2\xi}{1+8\xi}\sqrt{1+2\xi}\int_{-\infty}^{\infty}\dot{\tilde{q}}\operatorname{sgn}(\dot{\tilde{q}}-\tilde{w}')\,dt, \tag{26}$$

where $\dot{\tilde{q}} = -\sqrt{2}\operatorname{sech}(t)\operatorname{tgh}(t)$ and $\tilde{w}' = w'\frac{\sqrt{1+8\xi}}{1+2\xi}$. Assume first that $\tilde{w}' > 1/\sqrt{2}$, then

$$\int_{-\infty}^{\infty}\dot{\tilde{q}}\operatorname{sgn}(\dot{\tilde{q}}-\tilde{w}')\,dt = 0 \tag{27}$$

Assume now that $\tilde{w}' < 1/\sqrt{2}$, then

$$\int_{-\infty}^{\infty}\dot{\tilde{q}}\operatorname{sgn}(\dot{\tilde{q}}-\tilde{w}')\,dt = -\int_{-\infty}^{t_1}\dot{\tilde{q}}dt + \int_{t_1}^{t_2}\dot{\tilde{q}}dt - \int_{t_2}^{\infty}\dot{\tilde{q}}dt = 2\sqrt{2}\left(\operatorname{sech}(t_2)-\operatorname{sech}(t_1)\right) \tag{28}$$

where $\operatorname{sech}(t_1) = \sqrt{\frac{1}{2}-\frac{1}{2}\sqrt{1-2\tilde{w}'^2}}$, $\operatorname{sech}(t_2) = \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-2\tilde{w}'^2}}$. Substituting the obtained result we find

$$\begin{aligned} \frac{1+8\xi}{4(1+2\xi)^{3/2}}M_{22}(t_0) &= -\frac{2}{3}\Omega(\xi)B_{11}' + 2\Omega(\xi)B_{12}'\left(w'^2 + \frac{4(1+2\xi)^2}{35(1+8\xi)}\right) \\ &+ \Omega(\xi)T_{10}'\sqrt{2}\theta\left(\frac{1}{\sqrt{2}}-\tilde{w}'\right)\left(\operatorname{sech}(t_2)-\operatorname{sech}(t_1)\right) \end{aligned} \tag{29}$$

where $\theta(x)$ is Heaviside's function. In the similar way we obtain function $M_{23}(t_0)$. Finally, we find Melnikov-Gruendler function

$$\begin{aligned} M(t_0) &= -\sqrt{2}\Gamma'\pi\omega'\operatorname{sech}\left(\frac{\pi\omega'}{2\sqrt{1+2\xi}}\right)\sin(\omega' t_0) - \frac{4(1+2\xi)^{3/2}}{3\sqrt{1+8\xi}}\left(B_{11}'+B_{21}'\right) \\ &+ 4\frac{(1+2\xi)^{3/2}}{\sqrt{1+8\xi}}\left(B_{12}'+B_{22}'\right)\left(w'^2 + \frac{4(1+2\xi)^2}{35(1+8\xi)}\right) \\ &+ 2\sqrt{2}\left(T_{10}'+T_{20}'\right)\frac{(1+2\xi)^{3/2}}{\sqrt{1+8\xi}}\theta\left(\frac{1}{\sqrt{2}}-\tilde{w}'\right)\left(\operatorname{sech}(t_2)-\operatorname{sech}(t_1)\right) \end{aligned} \tag{30}$$

6. Concluding remarks

In this paper an important problem related to stick-slip chaos prediction is successfully solved. It possesses a challenging impact on analysis of all mechanical systems with friction, since many of them can he modelled by two degrees-of-freedom objects [4].

Motivated mainly by two papers [3,7], the homoclinic orbit is defined analytically, and then the Melnikov-Grunedler method is applied. The Melnikov's integrals are computed for both qualitatively different cases i.e. for regular and discontinous onset of chaos. It is worth noticing that for $\xi = 0$ Melnikov-Gruendler function reduces to the one-dimensional case [3].

7. References

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