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SMOOTH AND NON-SMOOTH REGULAR AND CHAOTIC DYNAMICS OF A STRONGLY NON-LINEAR TRIPLE PENDULUM

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This report briefly describes the large project of investigations of the flat triple physical pendulum with arbitrary situated barriers imposed on the position of the system. Some examples of dynamical behaviour of the special case of the triple pendulum (three coupled identical rods with horizontal frictionless barrier) are shown.

INTRODUCTION

A pendulum plays a very important role in mechanics since many interesting non-linear dynamical behaviour can be illustrated and analyzed using this simple system[1-4].

But a single degree-of-freedom models are only the first step to understand a real behaviour of either natural or engineering systems. Many physical objects are modeled by a few degrees of freedom. This is well known in mechanics, but even in physics an attempt to investigate coupled pendulums is recently observed. For instance, in references[1,5] both the theoretical and experimental (laboratory) investigations have been carried out on two and three coupled pendulums.

On the other hand, it is well known that impact and friction accompanies almost all real behaviour, leading to non-smooth dynamics. The non-smooth dynamical systems are analyzed in both pure[6] and applied sciences[7]. The non-classical bifurcations are analysed in systems with dry friction[8] and in systems with impacts[9-11].

Our work matches all the mentioned research directions. The scope of the project contains modeling of a flat triple physical pendulum with arbitrary situated barriers imposed on the position of the system (including impact and sliding motion modeling), numerical schemes for system simulation developing, stability investigation of the orbit analysis methods in the case of non-smooth system and its application in the investigated system in order to investigate non-smooth dynamics, as well as, classical and non-classical bifurcations[12-15]. Another goal of our research is focused on the applications of the investigated system (a piston-connecting rod-crankshaft system modeling)[15].

Here we only shortly present main governing equations and some examples of dynamical behaviour of the special case of the triple pendulum: three identical coupled rods with horizontal frictionless barrier.

MODEL OF THE INVESTIGATED SYSTEM AND METHODS OF ITS ANALYSIS

Three joined stiff physical pendulums coupled by viscous damping, moving on the plane are presented in Fig. 1. The system position is defined by three angles ψ_i $(i=1,2,3)$, and each of the bodies is harmonically excited by $\bar{f}_{e,i}$ $(i=1,2,3)$. It is assumed that the mass centers of the links lie on the lines including the joints and one of the principal central inertia axes of each link (z_{Ci}) is perpendicular to a movement pendulum plane. The set of possible configurations of the system is bounded by the arbitrarily situated stiff frictionless barriers.

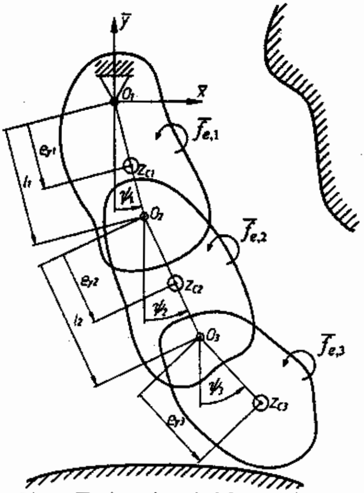

Fig. 1. The investigated triple pendulum.

The system is governed by the following set of differential equations together with the set of algebraic inequalities representing stiff obstacles in their non-dimensional form

$$\mathbf{M}(?)? + \mathbf{N}(?)?^2 + \mathbf{C}? + \mathbf{p}(?) = \mathbf{f}_e(?,?,t), \tag{1a}$$

$$h_i(?) \geq 0, \qquad i = 1,\dots,m \tag{1b}$$

where

$$? = \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{Bmatrix}, \quad ? = \begin{Bmatrix} \ddot{\psi}_1 \\ \ddot{\psi}_2 \\ \ddot{\psi}_3 \end{Bmatrix}, \; ?^2 = \begin{Bmatrix} \dot{\psi}_1^2 \\ \dot{\psi}_2^2 \\ \dot{\psi}_3^2 \end{Bmatrix}, \; ? = \begin{Bmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \\ \dot{\psi}_3 \end{Bmatrix}, \tag{2a}$$

$$\mathbf{M}(?) = \begin{bmatrix} 1 & \nu_{12}\cos(\psi_1-\psi_2) & \nu_{13}\cos(\psi_1-\psi_3) \\ \nu_{12}\cos(\psi_1-\psi_2) & \beta_2 & \nu_{23}\cos(\psi_2-\psi_3) \\ \nu_{13}\cos(\psi_1-\psi_3) & \nu_{23}\cos(\psi_2-\psi_3) & \beta_3 \end{bmatrix},$$

$$\mathbf{N}(?) = \begin{bmatrix} 0 & \nu_{12}\sin(\psi_1-\psi_2) & \nu_{13}\sin(\psi_1-\psi_3) \\ -\nu_{12}\sin(\psi_1-\psi_2) & 0 & \nu_{23}\sin(\psi_2-\psi_3) \\ -\nu_{13}\sin(\psi_1-\psi_3) & -\nu_{23}\sin(\psi_2-\psi_3) & 0 \end{bmatrix}, \tag{2b}$$

$$\mathbf{C} = \begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}, \qquad \mathbf{p}(?) = \begin{Bmatrix} \sin\psi_1 \\ \mu_2\sin\psi_2 \\ \mu_3\sin\psi_3 \end{Bmatrix}, \qquad \mathbf{f}_e(?,?,t) = \begin{Bmatrix} f_{e,1}(?,?,t) \\ f_{e,2}(?,?,t) \\ f_{e,3}(?,?,t) \end{Bmatrix}.$$

Moreover, the state of activity of each obstacle is the part of the state of the system. Each obstacle can be inactive, with instantaneous contact with some links or with continuous contact with them. In the second case an impact takes place. In the third case the system just slides along the obstacle. The impact is modeled by the use of generalized Newton's impact law based on the restitution coefficient rule[7]. The sliding state is modeled by the introducing of the normal forces from the obstacle to equations (1a).

The system response is obtained numerically by the Runge-Kutta integration of the differential equations between each two successive discontinuity points (where the activity of the obstacles changes: the impact takes place or the time interval of sliding begins or ends). These points are detected by halving integration step until obtaining assumed precision.

After the simulation of the system, the next step was the stability analysis of the solution in the investigated model, which in fact is piece-wise smooth (PWS) one. The classical methods and algorithms basing on the linear perturbation equations are used with the modifications taking into account the perturbations jump in the discontinuity points[16]. The numerical software for Lyapunov exponents calculation and periodic orbit stability analysis (seeking for periodic orbits and their bifurcations analysis) was developed.

For more details on modeling, relations between real and non-dimensional parameters, numerical algorithms, etc., see works[12-15].

NUMERICAL EXAMPLES

For numerical examples the special case of the triple pendulum is chosen: three identical coupled rods, for which we have: $\beta_2 = 4/7$, $\beta_3 = 1/7$, $\mu_2 = 3/5$, $\mu_3 = 1/5$, $\nu_{12} = 9/14$, $\nu_{13} = 3/14$, $\nu_{23} = 3/14$. The first rod is harmonically forced: $\mathbf{f}_e(?\,,?\,,t) = \mathbf{f}_e(t) = [q_1 \cos \omega_1 t, 0, 0]^T$ and the horizontal barrier is imposed on the position of the system: $h_1(?\,) = \eta - \cos\psi_1 \geq 0$, $h_2(?\,) = \eta - (\cos\psi_1 + \cos\psi_2) \geq 0$, $h_3(?\,) = \eta - (\cos\psi_1 + \cos\psi_2 + \cos\psi_3) \geq 0$, where η is the non-dimensional parameter determining the barrier position and the restitution coefficient e is associated with the barrier. Initial conditions $\mathbf{x}(0) = \mathbf{x}_0$ for examples are given by the use of the state vector $\mathbf{x} = [\psi_1, \psi_2, \psi_3, \dot{\psi}_1, \dot{\psi}_2, \dot{\psi}_3, \omega_1 t]^T$. The variables x_{O4} and y_{O4} are the non-dimensional coordinates of the third rod's end (O_4) position.

In Fig. 2 the periodic solution of period $9T$ (where T is the period of external forcing) with impacts is presented. Fig. 3 shows the periodic solution with part of the trajectory lying on the surface of the barrier (sliding state) and with the broken symmetry. Another example is shown in Fig. 4, where the post-transient solution is situated on the 2-D tori with two incommensurate frequencies (quasi-periodic solution with Poincaré section in the form of continuous line). Fig. 5 presents the bifurcational diagram exhibiting grazing bifurcation. If the vertical position of the barrier η is changed, the periodic orbit without impacts (Fig. 6) touches the barrier tangentially for the critical value of η, and with further changes of the bifurcational parameter the chaotic attractor appears suddenly (Fig 7). Lyapunov exponents spectra for the presented attractors are given in Table 1.

CONCLUDING REMARKS

This paper briefly reports the larger project of investigations of the flat triple physical pendulum with arbitrary situated barriers imposed on the position of the system. Some numerical examples for certain special case of the triple pendulum are shown. On the one hand, the investigated system is very rich source of non-linear dynamics including non-smooth dynamics, but on the other hand it can also serve as a model for many real physical and technical objects.

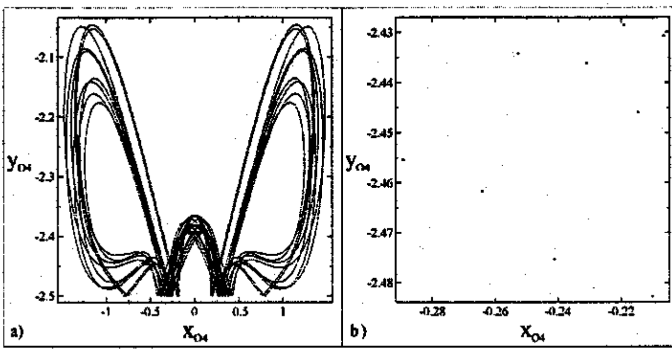

Fig. 2 Periodic solution for parameters: $c_1 = c_2 = c_3 = 0.1$, $q_1 = 0.7485$, $\omega_1 = 1$, $\eta = 2.5$, $e = 1$, and initial conditions $\mathbf{x}_0^T = [1,-1,1,0,0,0,0]$. Trajectory (a) and Poincaré section (b).

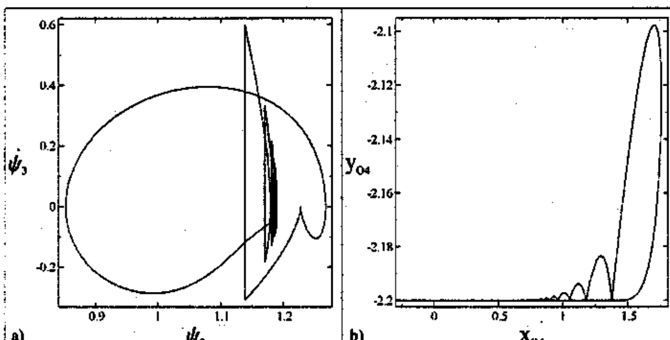

Fig. 3 Periodic orbit containing sliding state for parameters: $c_1 = c_2 = c_3 = 0.2$, $q_1 = 0.75$, $\omega_1 = 1$, $\eta = 2.2$, $e = 0.8$, and initial conditions $\mathbf{x}_0^T = [1,1,1,0,0,0,0]$.

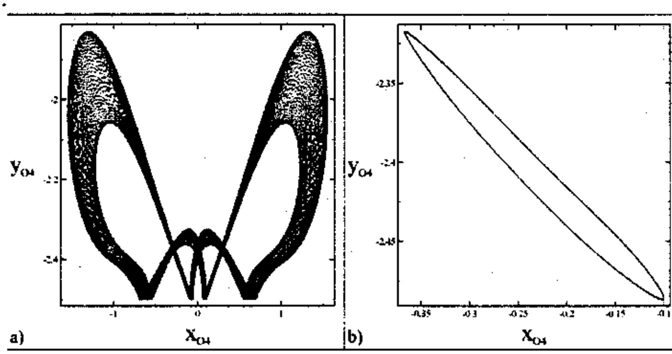

Fig. 4 Quasiperiodic solution for parameters: $c_1 = c_2 = c_3 = 0.1$, $q_1 = 0.7885$, $\omega_1 = 1.005$, $\eta = 2.5$, $e = 1$, and initial conditions $\mathbf{x}_0^T = [1,1,1,0,0,0,0]$. Trajectory (a) and Poincaré section (b).

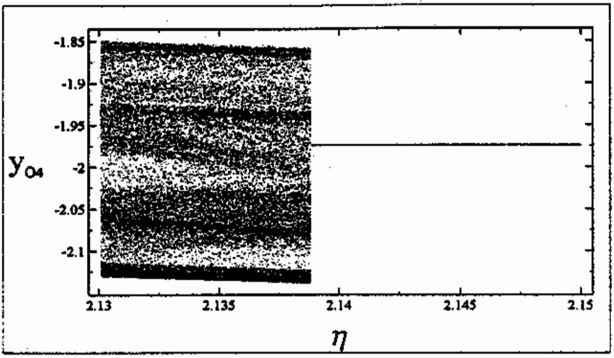

Fig. 5 Bifurcational diagram of Poincaré sections for $\eta \in (2.13, 2.15)$ and other parameters: $c_1 = c_2 = c_3 = 0.1$, $q_1 = 0.7885$, $\omega_1 = 1$ and $e = 1$. The diagram was performed with the start for $\eta = 2.15$, with initial conditions $\mathbf{x}_0^T = [0.52893, -1.31279, -2.53054, 1.22409, 0.97141, 0.20193, 0]$.

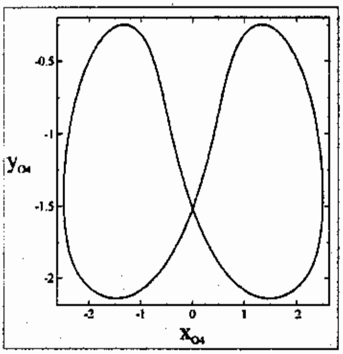

Fig. 6 Periodic orbit without impacts corresponding to the Fig. 5 for $\eta = 2.14$.

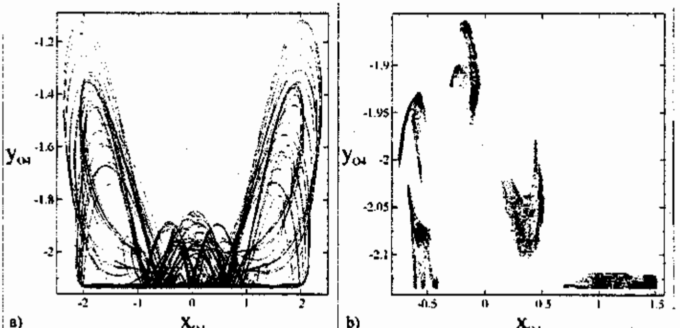

Fig. 7 Chaotic attractor corresponding to the Fig. 5 for $\eta = 2.135$. Trajectory (a) and Poincaré section (b).

Fig. no.	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	Attractor
2	0	-0.020	-0.020	-0.546	-0.998	-1.472	-2.085	limit cycle
4	0	0.000	-0.041	-0.043	-1.325	-1.702	-1.772	quasi-periodic
6	0	-0.11	-0.11	-0.26	-0.26	-1.71	-2.29	limit cycle
7	*0.12*	0	-0.16	-0.32	-0.87	-1.44	-1.85	chaotic

Table 1 Lyapunov exponents for the presented attractors.

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