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Bifurcations of thin plates transversally and sinusoidally excited

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ABSTRACT: Parametric vibrations of a flexible plate with infinite length are analysed. The plate is considered as a system with infinite degrees-of-freedom. The method of calculations of the Lyapunov exponents spectrum of the considered system is proposed. In order to verify results the spectrum of Lyapunov exponents is also computated using known Bennetin's method. In addition, bifurcations and chaotic dynamics of the analysed system are reported.

1 INTRODUCTION

Although parametric vibrations of flexible plates with infinite length are considered in large amount of publications, in majority of them the mentioned continuous system is analyzed as a one–degree–of–freedom–system. In this work we consider complex vibrations and bifurcations of plates as a system with infinite degrees–of–freedom. The Bubnov–Galerkin method with high approximations and finite difference methods with approximation of $O(h^4)$ are applied to trace regular and chaotic motion of the mentioned dynamical system.

2 FUNDAMENTAL RELATIONS

We consider large deflections of anisotropic plate with infinite length assuming that one of the plate dimensions is essentially larger than the other one. To solve the problem, a unit width belt with the length b is separated from the rectangular plate. The governing differential equation has the following non–dimensional form

$$\frac{\partial^2 w}{\partial t^2}+\varepsilon\frac{\partial w}{\partial t}=-\frac{1}{6}\lambda\frac{\partial^4 w}{\partial x^4}+\lambda\left\{\int_0^1\left(\frac{\partial w}{\partial x}\right)^2dx\right\}\frac{\partial^2 w}{\partial x^2}+P_x(t)\frac{\partial^2 w}{\partial x^2}. \tag{1}$$

The equation (1) is transformed to non–dimensional form using the following relations

$$x=\bar{x}b,\quad w=h\bar{w},\quad P_x=\frac{Eh^3}{b^2}\bar{P}_x,\quad \lambda=\frac{1}{12(1-\nu^2)},$$

$$q=\frac{Eh^4}{b^4}\bar{q},\quad \varepsilon=h\bar{\varepsilon},\quad t=t_0\bar{t},\quad \varepsilon=1,\quad 0\le x\le 1,$$

$$t\in(0,\infty),\quad \nu=0{,}3,$$

where: b denotes the plate dimension, h is plate thickness, $w(x,t)$ is the deflection function, and $P_x(t)$ and $q(t)$ are the longitudinal and transversal loads, respectively.

The following boundary conditions are associated with the equation (1):

$$w=w''_{xx}=0\quad for\quad x=0;1, \tag{2}$$

$$w=w'_x=0\quad for\quad x=0;1. \tag{3}$$

The initial conditions read, respectively:

$$w(x)\big|_{t=0}=0.01\sin\pi x,\quad \dot{w}(x)\big|_{t=0}=0, \tag{4}$$

$$w(x)\big|_{t=0}=0.01(1-\cos 2\pi x),\quad \dot{w}(x)\big|_{t=0}=0. \tag{5}$$

A solution to the formulated Cauchy problem is found using two approaches; i.e. the Bubnov–Galerkin method and the method of finite differences with approximation of $O(h^4)$. Recall, that the Bubnov–Galerkin method reduces the initial–boundary problem to that of solution to a system of ordinary differential equations. Assume a solution in the form

$$w(x,t)=\sum_{i=0}^{N}A_i(t)w_i(x), \tag{6}$$

where $w_i(x)$ should satisfy one of the given boundary conditions (2), (3). Applying the Bubnov–Galerkin procedure to the equation (1) the following system of ordinary differential equations with respect to $A_i(t)$ is obtained:

$$\begin{aligned}&\sum_{i=0}^{N}\left(\ddot{A}_i+\varepsilon\dot{A}_i\right)a_{ik}=-\frac{1}{6}\lambda\sum_{i=0}^{N}A_i b_{ik}+\\&\lambda L(A_i(t))\sum_{i=0}^{N}A_i c_{ik}+P_y(t)\sum_{i=0}^{N}A_i c_{ik}+Q_k(t),\end{aligned} \tag{7}$$

where: $k=0,1,\ldots$.

Above the following notation is used:

$$a_{ik}=\int_0^1 w_i(x)w_k(x)dx,\quad b_{ik}=\int_0^1 w_i^{IV}(x)w_k(x)dx,$$

$$c_{ik}=\int_0^1 w_i^{''}(x)w_k(x),\quad Q_k=\int_0^1 q(t,x)dx, \tag{8}$$

$$L(A_i(t))=\int_0^1\left(\sum_{i=0}^{N}A_i(t)w_i^{'}\right)^2dx.$$

It is easy to reduce the second order differential equations to first order ones and then to solve them using, for example, Runge–Kutta method.

3 FINITE DIFFERENCE METHOD

To reduce the partial differential equations (1) to ordinary differential equations the finite difference method with $O(h^4)$ approximation is applied to the spatial coordinate x. In the grid space

$$G_k=\left\{0\le x_i\le 1,\ x_i=ih,\ h=\frac{1}{N},\ i=1,\ldots,N\right\},$$

the partial derivative are substituted by the difference approximations

$$\begin{aligned}&\left(\frac{\partial(\cdot)}{\partial x}\right)_i\approx\frac{1}{12h}\left[(\cdot)_{i-2}-8(\cdot)_{i-1}+8(\cdot)_{i+1}-(\cdot)_{i+2}\right]=\\&=\Lambda_x(\cdot)+0\left(h^4\right)\end{aligned} \tag{9}$$

$$\begin{aligned}&\left(\frac{\partial^2(\cdot)}{\partial x^2}\right)_i\approx\\&\approx\frac{1}{h^2}\left[(\cdot)_{i+2}+16(\cdot)_{i+1}-30(\cdot)_i+16(\cdot)_{i-1}+(\cdot)_{i-2}\right]=\\&=\Lambda_{x^2}(\cdot)+0\left(h^4\right)\end{aligned} \tag{10}$$

$$\begin{aligned}&\left(\frac{\partial^4(\cdot)}{\partial x^4}\right)_i\approx\frac{1}{h^4}\left[(\cdot)_{i-2}-4(\cdot)_{i-1}-6(\cdot)_i-4(\cdot)_{i+1}+(\cdot)_{i+2}\right]=\\&=\Lambda_{x^4}(\cdot)+0\left(h^4\right)\end{aligned} \tag{11}$$

and the equation (1) is transformed to the following set of equations

$$\begin{aligned}&\frac{d^2w_i}{dt^2}+\varepsilon\frac{dw_i}{dt}=-\frac{1}{6}\lambda\Lambda_{x^4}(w_i)+\\&\lambda\left\{\int_0^1(\Lambda_x(w_i))^2dx\right\}\Lambda_{x^2}(w_i)+\\&P_x(t)\Lambda_{x^2}(w_i)+q(t,ih).\end{aligned} \tag{12}$$

Note that in the equations (12) integration along the plate length is carried out using the Simpson's formula. Applying the approximation $O(h^4)$ we have two series of the out-contour points. The following observation holds for the boundary conditions (2), (3):

- for the balls supported edge $(\cdot)_{-i}=-(\cdot)_i$,
- for the clomped edge $(\cdot)_{-i}=-(\cdot)_i$.

Introducing the change of variables

$$\frac{dw_i}{dt}=w_i^{'}, \tag{13}$$

the equation (12) is reduced to the following first order differential equations with respect to the deflections w_i and velocities $w_i^{'}$:

$$\begin{aligned}&\frac{dw_i^{'}}{dt}+\varepsilon\frac{dw_i^{'}}{dt}=-\frac{1}{6}\lambda\Lambda_{x^4}(w_i)+\\&\lambda\left\{\int_0^1(\Lambda_x(w_i))^2dx\right\}\Lambda_{x^2}(w_i)+\\&P_x(t)\Lambda_{x^2}(w_i)+q(t,ih).\end{aligned} \tag{14}$$

The obtained ODEs (13) and (14) are solved using Runge–Kutta method of the second order.

4 APPLICATION OF THE BUBNOV-GALERKIN METHOD

A solution (6) is sought in the form

$$w_i(x)=\sin(2i+1)\pi x. \tag{15}$$

In the case of parametric vibrations $P_x(t)=P_x\sin(\omega_p t)$ and for $q(t,x)=0$, the system (7) is transformed to the following one–degree–of freedom system:

$$\ddot{A}_0(t)+\varepsilon\dot{A}_0(t)+\left(\frac{\pi^4}{6}\lambda-\pi^2P_y(t)\right)A_0(t)+ \tag{16}$$

$$+\frac{\pi^4}{2}\lambda A_0^3(t) = 0$$

The obtained second order ordinary differential equation, , is reduced to the system of two first order ordinary differential equations, which are then solved using Runge–Kutta method of the second order.

5 LYAPUNOV CHARACTERISTIC EXPONENTS

We briefly recall the know algorithm of Lyapunov characteristic exponents computation. Theory of the Lyapunov exponents has been applied by Oseledec 1968. A connection between the Laypunov exponents and Kolmogorov entropy has been considered by Bennetin et al., 1978, 1979 and rigorously formulated by Pesin 1977. In the literature devoted to investigation of chaotic vibrations of various dynamical systems the method proposed by Bennetin et al. is widely used.

Although in general we apply the Bennetin et al., 1978, 1979 method, but for low dimensional systems one may solve the problem in an analytical way.

For instance, for our system (16) the associated Jacobian has the form

$$I(A_0, t) = \begin{pmatrix} 0 & 1 \\ -\frac{\pi^2}{12(1-\upsilon^2)} - 3\frac{\pi^2}{4(1-\upsilon^2)}A_0(t)^2 + P_x(t) - \varepsilon \end{pmatrix}. \quad (17)$$

Assuming λ_1, λ_2 to be eigenvalues of the Jacobian, one may solve the system of differential equations: $\frac{dA}{dt} = I \times A$ with the initial conditions $A(0) = \begin{pmatrix} A_0 & (0) \\ A_0^{\cdot} & (0) \end{pmatrix}$ to get

$$A(t) = \begin{pmatrix} \frac{1}{\lambda_1 - \lambda_2}\{(-\lambda_2 A_0(0) + A_0^{\cdot}(0))e^{\lambda_1 t} - (-\lambda_{11} A_0(0) + A_0^{\cdot}(0))e^{\lambda_2 t}\} \\ \frac{1}{\lambda_1 - \lambda_2}\{\lambda_1(-\lambda_2 A_0(0) + A_0^{\cdot}(0))e^{\lambda_1 t} - \lambda_2(-\lambda_{11} A_0(0) + A_0^{\cdot}(0))e^{\lambda_2 t}\} \end{pmatrix}. \quad (18)$$

Observe, that in general λ_1 and λ_2 are complex values and hence the latter formula has different representation for complex and purely real λ. Therefore, the mentioned drawbacks of the Lyapunov exponents computation are omitted here, since instead of numerical integrations we apply the analytic formula (18).

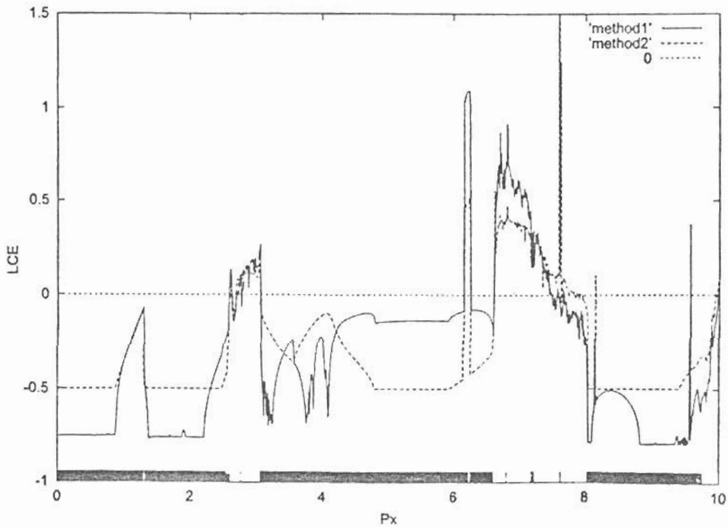

Figure 1. Lyapunov characteristic exponent estimated using the Bennettin et al. 1978, 1979 method (method 1) and our method (method 2) versus P_x

This proposed approach essentially decreases the computational time and increases accuracy of the results. A comparison of two described methods shows that they in practice are equivalent. A difference in results is less than 1%.

The analytical form (18) has also a physical interpretation. Observe that the length of the vectors x_i exponentially increases (decreases) with respect to the real parts of λ_1 and λ_2.

Therefore, the Lyapunov exponents computation can be estimated by a direct averaging of real parts of the Jacobian matrix eigenvalues along the investigated phase curve. The proposed method is more suitable to estimate Lyapunov exponents contrary to the classical one, which is only approximated numerical realization of the original idea.

However, although the qualitative behavior of two discussed methods is similar, but quantitative differences are rather expected. As an example a comparison of computations of maximal Lyapunov exponent versus excitation amplitude P_x for fixed ω_p equal to eigenfrequency of the considered system is shown in Figure 1. The obtained results coincide well with the Fourier analysis (FFT) of these vibrations. In words, the harmonic vibrations correspond to negative values of the maximal Lyapunov exponent, chaotic vibrations correspond to its positive value, and in the case of a bifurcation the LCE is close to 0.

6 APPLICATION OF BUBNOV-GALERKIN AND FINITE DIFFERENCE METHODS-NUMERICAL RESULTS

The system of equations (1), (2), (3) and (4), (5) is solved using both Bubnov–Galerkin and finite difference methods. The investigations are focused on construction of plate vibration chart in the parameter plane (P_x, ω_p) for each of the boundary conditions. It has been observed that vibrations are defined in full by the vibrations of the center of the plate $(x = 0.5)$, since other plate points move in a synchronized way. The intervals $0.5 < \omega_0 < 1.5$ are applied while constructing the chart, where ω_0 is the eigenfrequency of the plate for given boundary conditions.

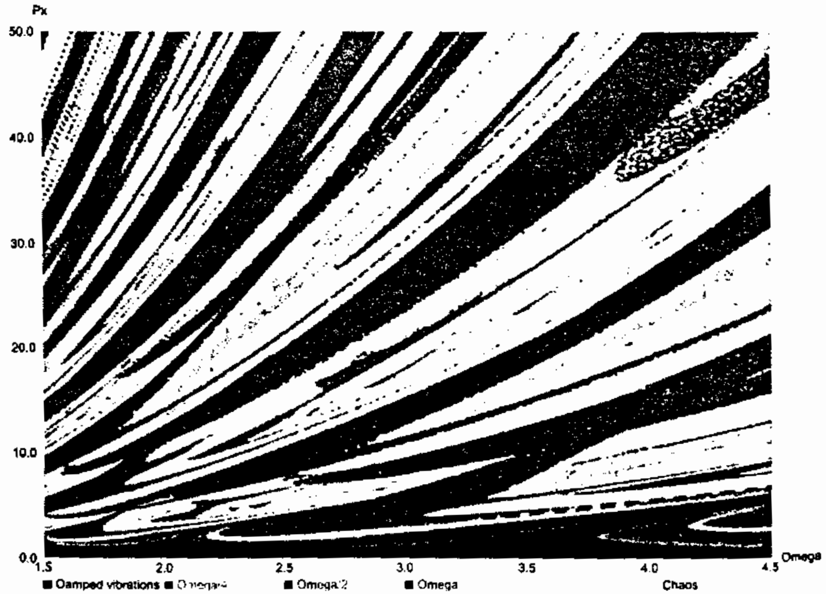

Figure 2. Regular and chaotic dynamics in the plane (P_x, ω_p)

The interval of P_x changes is chosen in a such way that a maximal deflection does not achieve 7 times of plate thickness.

For the boundary conditions (3) the eigenfrequency $\omega_0=3$, and the interval of "chart" construction is equal to [1.5, 4.5]. Interval of P_x changes is [0,50]. The "chart" corresponding to the applied boundary conditions is reported in Figure 2. Observe that results obtained using either Bubnov–Galerkin or finite difference methods are identical, which proves their reality (true values).

In both Bubnov–Galerkin and finite difference methods the used time step $dt = 0.0078125$. In addition, in the latter method $n = 8\ (dx = 0.125)$. Those parameters are found from the Runge's rule and they guarantee a convergence of the harmonic vibrations.

Let us now analyze the chart reported in Figure 2. We trace dynamics by observing the cross sections along the P_x axis, which correspond to the change of external amplitude excitation (the frequency of excitation is constant). Applying such way of analysis one may separate the plane into three parts: [1.5, 2.2]–low frequencies; [2.2, 3.8]–average frequencies; [3.4, 4.5]–high frequencies.

For small values of amplitudes of the exciting force the introduced energy to the system is to small to realize the undamped stationary vibrations. The lowest part of the chart corresponds to damped vibrations. The first stiff bifurcation is represented by the top boundary of this part. In words, a slight increase of the amplitude of external excitation yields a qualitative change of vibrations.

A transition from damped to harmonic vibrations takes place. In addition, two different cases may be distinguished after the 'stiff' bifurcation: the frequency of vibrations is either equal to the external excitation frequency or it is twicely smaller. This frequency serves as criterion for partition of the plane into different frequency zones.

For zones of low and high frequencies the vibrations with $\omega_p/2$ appear, whereas for zone of averaged frequencies the vibrations with ω_p occur. The obtained numerical results suggest that the harmonic plate vibrations are possible only for three following frequencies: $\omega_p/4$, $\omega_p/2$, and ω_p.

We briefly follow the scenario of bifurcations for average frequencies zone. After the stiff bifurcation the harmonic vibrations with the frequency ω_p appear. The next increase of P_x leads to the bifurcation series and to transition to chaotic vibrations. Then, again harmonic vibrations appear, but with the frequency ω_p and the related transition is not associated with a series of bifureations. A similar like scheme of frequencies switching is typical also for two other parts of the parameter plane with the only one difference: instead of the initial frequency ω_p now $\omega_p/2$ appears.

In addition, in the investigated "chart" the areas with harmonic vibrations with the frequency $\omega_p/4$, and the areas where bifurcations are observed, are separated.

For areas with bifurcations two special cases are addressed: (i) a transition from harmonic to chaotic vibrations, and (ii) an investigation areas being internal ones for zones of harmonic vibrations.

The singular cases occurring on the borders between frequency zones are investigated separately. The jump–like switching between frequencies is observed, i. e. either a lack of bifurcation series or transition to chaos are detected. The discussed frequency zones are: [2.1, 2.2] and [3.8, 4.2].

In the mentioned frequency intervals one may observe stiff bifurcation twicely. After the first stiff bifurcation stable harmonic vibrations appear. Next, in the short interval of P_x the vibrations again are damped. Then again stiff bifurcation occurs, and the harmonic vibrations with different frequency appear.

Following the introduced study of the chard one may also detect self–similarity of its parts, particularly in the part corresponding to small amplitudes of external force. The shapes of areas are repeated in all three zones, and higher frequencies correspond to larger area dimension. As it has been already mentioned, neighboring zones are distinguished through switching of frequency order for harmonic vibrations. Therefore, similar areas have only qualitatively the same vibrations, but they may differ with respect to quantizied exponents.

Since on the same intervals of parameter changes a route to chaos is associated with bifurcation series, the problem of comparison with Feigenbaum scenario is addressed.

The first 8 bifurcations are numerically traced using the Bubnov–Galerkin method. As an example, the data obtained for the eigenfrequency $\omega_p = 3$ required for the Feigenbaum constant computation are reported in Table 1.

Table 1

	1	2	3	4	5	6	7	8
P_x	2.573266	2.603611	2.313630	2.616679	2.616790	2.616790	2.616814	2.61819
C_i		3.028	3.975	4.770	4.759	4.708	4.683	

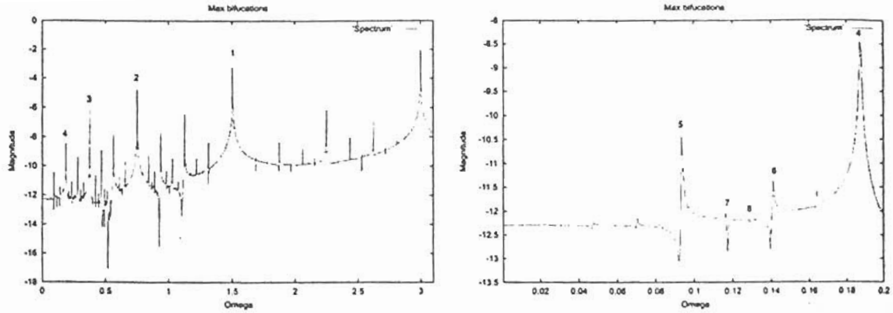

Figure 3. Frequency spectrum with marked eight bifurcations

The obtained results coincide very well with the theoretically obtained value

$$C_\infty = \lim_{n\to\infty} \frac{P_x^n - P_x^{n-1}}{P_x^{n+1} - P_x^n} \approx 4.6692....$$

In Figure 3 the frequency spectrum for $\omega_p = 3$, $P_x = 2.616819$ with eight bifurcations is reported.

A route to chaos is strictly associated with the Feigenbaum scenario.

7 CONCLUSIONS

Both Bubnov-Galerkin and finite difference methods are used to analyse regular and chaotic dynamics of a flexible plate with infinite length. The analytical method for computation of Lyapunov characteristics exponent is proposed.

The series of period doubling bifurcations in the plane (P_x, ω_p) are easily traced in the power spectrum (Figure 3). Eight bifurcations are numerically detected.

The process of chaotization is characterized by a broad band noise occurred in the spectrum support. The peaks corresponding to fundamental frequencies and their harmonics remain clearly distuiguished.

The sequence of period doubling bifurcations approaches the strange chaotic attractor.

After the Andronov–Hopf bifurcation the stable limit cycle is born, and the previously stable equilibrium (singular point) becomes unstable of the saddle–focus type with one dimensional stable w^s and two dimensional unstable w^n manifolds. First, the unstable manifold w^n of the equilibrium is the limit cycle. Then two smallest multiplicators of the limiting cycle become complex – conjugated. The manifold w^n starts to wind up on the limit cycle and the configuration similar to a funnel appears. All attractors from a certain space are pulled into this funnel, and the trajectories move in a chaotic manner.

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