MELNIKOV METHOD AND CHAOS PREDICTION IN A SHAFT-BUSH SYSTEM INCLUDING TRIBOLOGICAL PROCESSES

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ABSTRACT: Periodic and chaotic dynamics of a bush being in a frictional contact with a rotating shaft is analysed analytically and numerically. An analytical chaos prediction using the Melnikov method is carried out, and then verified and further analysed numerically. Furthermore, more complex dynamics of the mentioned system using abrasive wear models and frictional heat generations are analysed numerically and some important conclusions are given.

I. INTRODUCTION

Chaotic vibrations in lumped systems with friction have been widely analysed (see, for instance [1-3]). In Refs. [3] the analytical conditions for chaos occurrence are formulated applying the Melnikov method [4] for harmonically driven systems. In this work, the bush – rotating shaft system (the shaft is kinematically (harmonically) excited) is analysed analytically using the Melnikov method. A prediction of a chaotic zone has been then verified numerically. Vibrations induced by friction and accompanied by thermal and wear processes are less investigated problems. Analysis of the onset of such vibrations as well as the behaviour of the contact characteristics (contact temperature, contact pressure and wear) may contribute to an explanation of the complex phenomena observed in such mechanical systems as pad brakes, grinding machines or in machine tools with a required very high tolerance.

Friction, wear, frictional heat generation and heat expansion belong to very complicated processes, which interact with each other, creating multidimensional and complex pairs of frictional objects. In a case of an un-stationary frictional process, changes of friction parameters are mutually dependent. We are going to investigate if frictional heat generation and wear may influence the periodic or chaotic motion which appears in a dynamical system modeling a bush-shaft unit with kinematic excitation.

The classical problem concerning vibrations of a pad attached to a rotating shaft linked with housing through springs corresponds to a schematic model of a braking pad or Pronny's clamp. It has been investigated in the works of Andronov *et al.*[5] A thermo-elastic contact between a rotating shaft and the non-movable and non-inertial pad has been investigated in references Pyryev and Hrylitskyy[6], and Pyryev[7]. Dynamical problems of the thermo-elastic contact with friction generated heat transfer have been analysed in references Olesiak and Pyryev [8, 9].

In this work the more complicated problem of a flat and axially symmetric thermo-elastic contact of the rotating shaft with the bush fixed by springs is addressed. The self-excited chaotic vibrations caused by friction, including wear, are analysed.

II. BASIC EQUATIONS

Elastic and heat transferring cylinder with a radius R_1 is inserted into bush. The internal radius of the bush attached on the cylinder is equal to R_1. The bush is linked with the housing by springs. We assume, that the bush is a perfect rigid body, and that radial springs have the stiffness coefficient k_1, whereas tangent springs are characterized by non-linear stiffness k_2 and k_3 (Duffing type stiffness) related to a unit length of the bush.

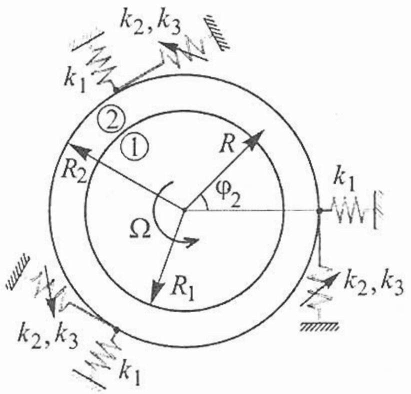

Fig.1 Analysed system

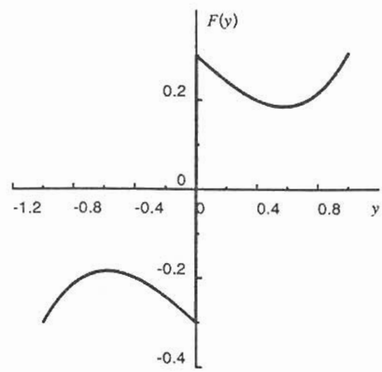

Fig.2 Friction versus velocity

The cylinder rotates with a such angular velocity $\Omega(t) = \Omega_* \omega_1(t)$, that the centrifugal forces may be neglected. We assume that the angular speed of the shaft rotation changes in accordance with $\omega_1 = \omega_* + \zeta_k \sin \omega' t$, where ζ_k is the dimensionless amplitude of the kinematic excitation. We assume that the bush ideally transforms heat and that initially the temperature is governed by the equation $T_{ot} + T_0 h_T(t)$ ($h_T(t) \to 2,\ t \to \infty$), and also that heat transfer is Newtonian between shaft and bush. The shaft starts to expand and a contact between the shaft and bush appears. We assume that between bush and shaft dry friction (related to unit length) appears defined by the function $F_t(V_w)$, where $V_w = \Omega R_1 - \dot{\varphi}_2 R_1$ is a relative velocity between the two given bodies. B_2 denotes the mass moment of inertia related to a length unit. We assume also that in accordance with the Amontos assumption the friction force F_{fr} is equal to the scalar product of the normal reaction force $N(t)$ and the friction coefficient, i.e. that $F_{fr} = f(V_w)N(t)$ is the friction force defining a resistance to the displacements of two bodies; $f(V_w)$ is the kinetic friction coefficient. The friction force F_{fr} generates heat on the contact surface for $R = R_1$, and wear U_z of bush occurs. $T_1(r,t)$ denotes the cylinder temperature which is equal to T_{ot} in the initial instant.

The problem is reduced to that of finding the displacement $\varphi_2(t)$ and the angular bush velocity $\dot{\varphi}_2(t)$, displacement $U(R,t)$ in the direction of the cylinder radius R, radial stress in the cylinder $\sigma_R(R,t)$, contact pressure $P(t) = N(t)/2\pi R_1 = -\sigma_R(R_1,t)$, cylinder temperature $T_1(R,t)$, and bush wear U_z in agreement with the Archard's assumption $\dot{U}_z(t) = K_z|V_w(\tau)|P(\tau)$.

Vibration of the bush being in a contact with thermo-elastic rotating shaft has the form:

$$\ddot{\varphi}(\tau) - \varphi(\tau) + b\varphi^3(\tau) = \varepsilon F(\omega_1 - \dot{\varphi})p(\tau), \; 0 < \tau < \infty, \tag{1}$$

$$\varphi(0) = \varphi^\circ, \; \dot{\varphi}(0) = \omega^\circ$$

where: $\omega_1 = \omega_* + \zeta_k \sin \omega\tau$. The non-dimensional contact pressure $p(\tau)$, shaft wear $u_z(\tau)$ and shaft temperature $\theta(r,\tau)$ are found from the equations:

$$p(\tau) = Bi\,\tilde{\omega}\int_0^\tau G_p(\tau-\xi)\dot{h}_T(\xi)\,\mathrm{d}\xi - k_z\int_0^\tau |\omega_1 - \dot{\varphi}|p(\xi)\,\mathrm{d}\xi + \gamma\tilde{\omega}\int_0^\tau \dot{G}_p(\tau-\xi)F(\omega_1-\dot{\varphi})p(\xi)(\omega_1-\dot{\varphi})\,\mathrm{d}\xi, \tag{2}$$

$$\theta(r,\tau) = Bi\tilde{\omega}\int_0^\tau G_\theta(r,\tau-\xi)\dot{h}_T(\xi)\,\mathrm{d}\xi + \gamma\tilde{\omega}\int_0^\tau \dot{G}_\theta(r,\tau-\xi)F(\dot{\phi}-\dot{\varphi})p(\xi)(\dot{\phi}-\dot{\varphi})\,\mathrm{d}\xi, \tag{3}$$

where:

$$\{G_p(\tau), G_\theta(1,\tau)\} = \frac{\{0.5, 1\}}{Bi\tilde{\omega}} - \sum_{m=1}^{\infty}\frac{\{2Bi, 2\mu_m^2\}}{\mu_m^2\tilde{\omega}(Bi^2+\mu_m^2)}\mathrm{e}^{-\mu_m^2\tilde{\omega}\tau}. \tag{4}$$

Note that μ_m are the roots of the characteristic equation ($m = 1,2,3,\ldots$)

$$Bi\,J_0(\mu) - \mu\,J_1(\mu) = 0 \tag{5}$$

where: $J_n(\mu)$ is the first order Bessel function with argument μ.

In equations (1)-(5) the following non-dimensional quantities are introduced:

$$r = \frac{R}{R_1}, \; \tau = \frac{t}{t_*}, \; \varphi = \frac{\varphi_2}{\Omega_* t_*}, \; p = \frac{P}{P_*}, \; u_z = \frac{U_z}{U_*}, \; \theta = \frac{T_1 - T_{ot}}{T_0}, \; \varphi^\circ = \frac{\varphi_2^\circ}{\Omega_* t_*}, \; \omega^\circ = \frac{\omega_2^\circ}{\Omega_*},$$

$$\omega = \omega' t_*, \; b = \frac{k_3 R_2^4 t_*^4 \Omega_*^2}{B_2}, \; \varepsilon = \frac{P_* t_* 2\pi R_1^2}{B_2 \Omega_*}, \; Bi = \frac{\alpha_T R_1}{\lambda_1}, \; k_z = \frac{\Omega_* P_* t_* K_z R_1}{U_*}, \; \gamma = \frac{2E_1\alpha_1 R_1^2 \Omega_*}{\lambda_1(1-2\nu)},$$

$$\tilde{\omega} = \frac{t_* a_1}{R_1^2}, \; h_T(\tau) = h_T(t_*\tau), \; F(\omega_1 - \dot{\varphi}) = f(R_1\Omega_*(\omega_1 - \dot{\varphi})),$$

and the following notation has been used:

$$t_* = \sqrt{\frac{B_2}{k_* R_2^2}}, \; P_* = \frac{2E_1\alpha_1 T_0}{1-2\nu}, \; k_* = k_1(l_0/l_1 - 1)(1 + l_1/R_2) - k_2,$$

where: l_0 is the un-stretched spring length, l_1 is the length of the compressed spring for $\varphi_2 = 0$,

E_1 is the elasticity modulus, v is the Poisson coefficient, α_1 is the shaft expansion coefficient, α_T is heat taking up coefficient, a_1 is the temperature compensation coefficient, λ_1 is the heat transfer coefficient, K_z is the wear coefficient. Friction dependence on velocity is approximated by the function

$$F(y) = F_0 \operatorname{Sgn}(y) - \alpha y + \beta y^3 , \quad \operatorname{Sgn}(y) = \begin{cases} \{y/|y|\} & \text{for } y \neq 0, \\ [-1, 1] & \text{for } y = 0. \end{cases} \tag{6}$$

III. MELNIKOV METHOD

Dynamics of system (1) for $x = \varphi$, $y = \dot{\varphi}$ and without the tribological processes ($\gamma = 0, k_z = 0$) is governed by the following equations

$$\begin{aligned} \dot{x} &= p_0(x, y) + \varepsilon p_1(x, y, \omega\tau, \varepsilon), \\ \dot{y} &= q_0(x, y) + \varepsilon q_1(x, y, \omega\tau, \varepsilon), \end{aligned} \tag{7}$$

where:

$$\begin{aligned} &p_0(x, y) = y, &&p_1(x, y, \omega\tau, \varepsilon) = 0, \\ &q_0(x, y) = x - bx^3, &&q_1(x, y, \omega\tau, \varepsilon) = F(\omega_* + \zeta_k \sin\omega\tau - y). \end{aligned} \tag{8}$$

Observe that for a sufficiently small ε the system (7) has one homoclinic orbit of the form

$$x_0(\tau) = \sqrt{\frac{2}{b}} \frac{1}{\cosh(\tau)}, \quad y_0(\tau) = -\sqrt{\frac{2}{b}} \frac{\sinh(\tau)}{\cosh^2(\tau)}, \tag{9}$$

and the Mielnikov function is defined by the formula [3]

$$M(\tau_0) = \int_{-\infty}^{+\infty} (q_0 p_1 - q_1 p_0)\Big|_{\substack{x=x_0(\tau-\tau_0)\\ y=y_0(\tau-\tau_0)}} \mathrm{d}\tau = -\int_{-\infty}^{+\infty} q_1 p_0\Big|_{\substack{x=x_0(\tau-\tau_0)\\ y=y_0(\tau-\tau_0)}} \mathrm{d}\tau, \tag{10}$$

where: $x_0(\tau), y_0(\tau)$ is the solution of unperturbed system of equations ($\varepsilon = 0$), which corresponds to the homoclinic orbit, and τ_0 is the parameter characterizing a position of the point belonging to this orbit. According to the Melnikov theory, if the function $M(\tau_0)$ has simple zero root, then for sufficiently small ε the motion governed by the equations (7) will be chaotic. The Melnikov function has the form

$$\begin{aligned} M(\tau_0) &= I(\tau_0) + 2C + 2\zeta\sqrt{A^2 + B^2}\sin(\omega\tau_0 + \varphi) + \\ &6\beta\zeta_k^2\left(I_{220}\cos^2\omega\tau_0 + I_{202}\sin^2\omega\tau_0 - 2\omega_* I_{111}\sin\omega\tau_0\cos\omega\tau_0\right) + \\ &2\beta\zeta_k^3\left(-I_{130}\cos^3\omega\tau_0 - 3I_{112}\sin^2\omega\tau_0\cos\omega\tau_0\right) \end{aligned} \tag{11}$$

where:

$$A = (\alpha - 3\beta\omega_*^2)I_{110} - 3\beta I_{310}, \quad B = 6\beta\omega_* I_{201},$$

$$C = \beta I_{400} - (\alpha - 3\beta\omega_*^2)I_{200}, \quad \varphi = \arctan(A/B),$$

$$I_{njk} = \int_0^\infty [y_0(t)]^n [\sin(\omega t)]^j [\cos(\omega t)]^k \, \mathrm{d}t. \tag{12}$$

After integration of (12) we obtain

$$I_{200}=\frac{2}{3b},\ I_{400}=\frac{8}{35b^2},\ I_{201}=\frac{\pi\omega(2-\omega^2)}{6b\sinh(\pi\omega/2)},\ I_{110}=-\frac{1}{\sqrt{2b}\cosh(\pi\omega/2)},$$

$$I_{310}=\frac{\omega(11+10\omega^2-\omega^4)}{120b\sqrt{2b}}\left\{\psi\left(\frac{1-i\omega}{4}\right)-\psi\left(\frac{3-i\omega}{4}\right)+\psi\left(\frac{1+i\omega}{4}\right)-\psi\left(\frac{3+i\omega}{4}\right)\right\},$$

$$I_{130}=-\frac{3\pi\omega}{8\sqrt{2b}}\left\{\cot\left(\frac{\pi(1-i\omega)}{4}\right)+\cot\left(\frac{3\pi(1-i\omega)}{4}\right)-\cot\left(\frac{\pi(3-i\omega)}{4}\right)-\cot\left(\frac{\pi(1-3i\omega)}{4}\right)\right\},$$

$$I_{112}=\frac{\pi\omega\cosh(\pi\omega/2)}{\sqrt{2b}\left(1-2\cosh(\pi\omega)\right)},\ I_{111}=-\frac{\pi\omega}{\sqrt{2b}\cosh(\pi\omega)},\ I_{220}=\frac{\pi\omega(2\omega^2-1)+\sinh(\pi\omega)}{3b\sinh(\pi\omega)},$$

$$I_{202}=\frac{\pi\omega(1-2\omega^2)+\sinh(\pi\omega)}{3b\sinh(\pi\omega)},\ \psi(z)=\frac{\Gamma'(z)}{\Gamma(z)},$$

where $\psi(z)$ denotes the derivative of the natural logarithm of the function $\Gamma(z)$.

In equation (11) the term $I(\tau_0)$ is defined by the formula

$$I(\tau_0)=-F_0\int_{-\infty}^{+\infty}y_0(t)\mathrm{Sgn}(\omega_r)\,\mathrm{d}t=2F_0\sqrt{\frac{2}{b}}\sum_m\frac{\mathrm{sgn}(\omega_r'(t_m))}{\cosh t_m},\tag{13}$$

where t_m are the roots of the equation

$$\omega_r(t_m)=\omega_*+\zeta_k\sin\left(\omega(t_m+\tau_0)\right)-y_0(t_m)=0,\tag{14}$$

and $\omega_r'(t)=\zeta_k\omega\cos\left(\omega(t+\tau_0)\right)-x_0(t)+bx_0^3(t)$.

If the Melnikov function (11) changes sign, then one may expect an occurrence of chaos. The Melnikov function is further analysed in more detail. Observe that for large values of the parameter b and small values ζ_k we obtain $I(\tau_0)\equiv 0$. In addition, the remaining terms in (11) are small and do not change the Melnikov function sign. Beginning from a certain threshold value ζ_{ch} the integral $I(\tau_0)\neq 0$ and it starts to play a dominant role in $M(\tau_0)$. Therefore, the function $M(\tau_0)$ starts to change its sign, when the function $I(\tau_0)$ is not equal to zero, which means that the function $\omega_r(t)$ changes its sign. This observation yields the following formula for ζ estimation:

$$\zeta_{ch}=\omega_*-1/\sqrt{2b}.\tag{15}$$

To check a reliability of ad hoc formulated estimation (15) the numerical calculations have been carried out and the following parameters have been fixed: $b=9$, $\varepsilon=0.1$, $\omega=2$, $\omega_*=0.4$, $F_0=0.3$, $\alpha=0.3$, $\beta=0.3$. The function $M(\tau_0)$ changes its sign for $\zeta_{ch}\approx 0.168$, whereas the value obtained using formula (15) is equal to $\zeta_{ch}=0.164$.

IV. NUMERICAL ANALYSIS

The numerical analysis of the equation (1) without tribological processes ($\gamma=0,\ k_z=0$) has been carried out. Poincaré sections and phase trajectories are shown in Figure 3(a). For

$\zeta_k = 0.165$ points 1 correspond to period – 6 motion, whereas chaos is reported for $\zeta_k = 0.2$ (points 2). It is clearly seen that a chaotic distribution of the Poincaré points is located in the vicinity of the homoclinic orbit (9). The points located on the surface (x, y) are separated in time step equal to the period of kinematic excitation $2\pi/\omega$.

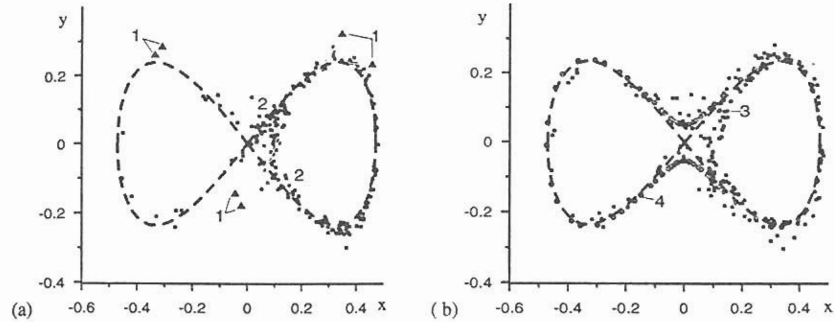

Fig.3 Poincaré section of the bush ($1 - \zeta_k = 0.165$, $\gamma = 0$, $k_z = 0$; $2 - \zeta_k = 0.2$, $\gamma = 0$, $k_z = 0$; $3 - \zeta_k = 0.2$, $\gamma = 20$, $k_z = 0$; $4 - \zeta_k = 0.2$, $\gamma = 20$, $k_z = 0.1$; dashed curves corresponds to the homoclinic orbit).

V. CONCLUSIONS

The frictional heat generation does not change qualitatively the detected chaotic motion for the considered shaft velocity (see points 3 in Figure 3(b)). Both contact pressure and temperature oscillate also chaotically. However, the threshold of chaos occurrence ζ_{ch} decreases. Wear, after some time, leads to achievement of zero value contact pressure. The bush movement is represented by the points 4 (see Figure 3(b)).

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