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STABILITY OF MULTI-LAYERED SANDWICH CYLINDERS UNDER TORSION

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Wide application of multi - layered construction in aeronautics, rocket techniques and transportation caused interesting publications on theory of stability and vibrations of such constructions. This paper is intended to present stability problem of a circular cylindrical shell subjected to torsion. Assumptions of a deformation process of a shell are presented in the Figure 1.

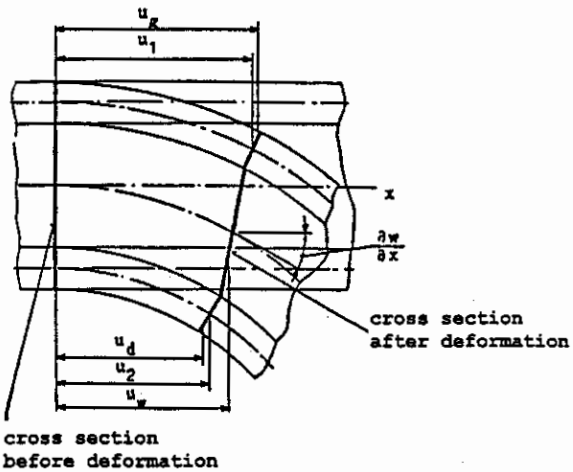

Figure 1: Deflection of the shell cross-section

According to the assumed model displacements of elements of a shell are defined by the following formulas:

a) for the upper layer:

$$u_g = u_1 - \left(z + h + \frac{t}{2}\right)\frac{\partial w}{\partial x},$$
$$v_g = v_1 - \left(z + h + \frac{t}{2}\right)\frac{\partial w}{\partial y}, \tag{1}$$
$$w_g = w_1,$$

b) for the lower layer:

$$u_d = u_2 - \left(z - h - \frac{t}{2}\right)\frac{\partial w}{\partial x},$$
$$v_d = v_2 - \left(z - h - \frac{t}{2}\right)\frac{\partial w}{\partial y}, \tag{2}$$
$$w_d = w_2,$$

c) for the inner filling:

$$u_w = \frac{1}{2}(u_1 + u_2) - \frac{z}{2h}\left(u_1 - u_2 - t\frac{\partial w}{\partial x}\right),$$
$$v_w = \frac{1}{2}(v_1 + v_2) - \frac{z}{2h}\left(v_1 - v_2 - t\frac{\partial w}{\partial y}\right), \tag{3}$$
$$w_w = w_3.$$

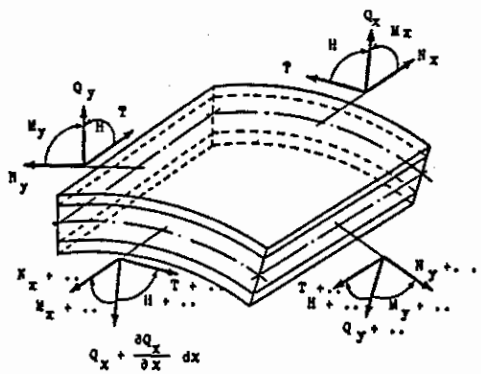

Figure 2: Loads of a shell element

Applying know formulas from the theory of elasticity we can describe internal

forces and momentums of a shell in a following way:

$$
\begin{aligned}
N_x &= 2B\left(\frac{\partial u_\alpha}{\partial x}+\nu\frac{\partial v_\alpha}{\partial y}-\nu\frac{w}{R}\right),\\
N_y &= 2B\left(\frac{\partial v_\alpha}{\partial y}+\nu\frac{\partial u_\alpha}{\partial x}-\frac{w}{R}\right),\\
T &= B(1-\nu)\left(\frac{\partial u_\alpha}{\partial y}+\frac{\partial v_\alpha}{\partial x}\right),\\
Q_x &= -2G_w\left[u_\beta-\left(h+\frac{t}{2}\right)\frac{\partial w}{\partial x}\right]-2D(1-\nu)\frac{\partial^3 w}{\partial x\partial y^2}+\\
&\quad -2D\left(\frac{\partial^3 w}{\partial x^3}+\nu\frac{\partial^3 x}{\partial x\partial y^2}\right)-\frac{B}{2}(1-\nu)t\left(\frac{\partial^2 u_\beta}{\partial y^2}+\frac{\partial^2 v_\beta}{\partial x\partial y}\right)+\\
&\quad -Bt\left(\frac{\partial^2 u_\beta}{\partial x^2}+\nu\frac{\partial^3 v_\beta}{\partial x\partial y}\right),\\
Q_y &= -2G_w\left[v_\beta-\left(h+\frac{t}{2}\right)\frac{\partial w}{\partial y}\right]-2D(1-\nu)\frac{\partial^3 w}{\partial x^2\partial y}+\\
&\quad -2D\left(\frac{\partial^3 w}{\partial y^3}+\nu\frac{\partial^3 w}{\partial x^2\partial y}\right)-\frac{B}{2}(1-\nu)t\left(\frac{\partial^2 u_\beta}{\partial x\partial y}+\frac{\partial^2 v_\beta}{\partial x^2}\right)+\\
&\quad -Bt\left(\frac{\partial^2 v_\beta}{\partial y^2}+\nu\frac{\partial^2 u_\beta}{\partial x\partial y}\right),\\
M_x &= -2D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)-2B\left(h+\frac{t}{2}\right)\left(\frac{\partial u_\beta}{\partial x}+\nu\frac{\partial v_\beta}{\partial y}\right),\\
M_y &= -2D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)-2B\left(h+\frac{t}{2}\right)\left(\frac{\partial v_\beta}{\partial y}+\nu\frac{\partial u_\beta}{\partial x}\right),\\
H &= -2D(1-\nu)\frac{\partial^2 w}{\partial x\partial y}-B(1-\nu)\left(h+\frac{t}{2}\right)\left(\frac{\partial u_\beta}{\partial y}+\frac{\partial V_\beta}{\partial x}\right).
\end{aligned}
\tag{4}
$$

Notations used in the above equations are described in the Figure 2.

The equilibrium equations have the following form:

$$\begin{aligned}
\frac{\partial^2 u_\alpha}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2 u_\alpha}{\partial y^2} + \frac{1+\nu}{2}\frac{\partial^2 v_\alpha}{\partial x \partial y} - \frac{\nu}{R}\frac{\partial w}{\partial x} &= 0,\\
\frac{\partial^2 v_\alpha}{\partial y^2} + \frac{1-\nu}{2}\frac{\partial^2 v_\alpha}{\partial x^2} + \frac{1+\nu}{2}\frac{\partial^2 u_\alpha}{\partial x \partial y} - \frac{1}{R}\frac{\partial w}{\partial y} &= 0,\\
\frac{Bh}{G_w}\left(\frac{\partial^2 u_\beta}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2 u_\beta}{\partial y^2} + \frac{1+\nu}{2}\frac{\partial^2 v_\beta}{\partial x \partial y}\right) - u_\beta + \left(h+\frac{t}{2}\right)\frac{\partial w}{\partial x} &= 0,\\
\frac{Bh}{G_w}\left(\frac{\partial^2 v_\beta}{\partial y^2} + \frac{1-\nu}{2}\frac{\partial^2 v_\beta}{\partial x^2} + \frac{1+\nu}{2}\frac{\partial^2 u_\beta}{\partial x \partial y}\right) - v_\beta + \left(h+\frac{t}{2}\right)\frac{\partial w}{\partial y} &= 0,\\
\frac{2B}{R}\left(\frac{\partial v_\alpha}{\partial y} + \nu\frac{u_\alpha}{x} - \frac{w}{R}\right) - 2B\left(h+\frac{t}{2}\right)\nabla^2\left(\frac{\partial u_\beta}{\partial x} + \frac{\partial v_\beta}{\partial y}\right) +&\\
-2D\nabla^4 w + N_x\frac{\partial^2 w}{\partial x^2} + N_y\frac{\partial^2 w}{\partial y^2} + 2T\frac{\partial^2 w}{\partial x \partial y} &= 0.
\end{aligned} \tag{5}$$

These equations can be converted to one partial differential equation with respect to the transversal deflection w:

$$\begin{aligned}
&\frac{2B}{R^2}(1-\nu^2)\left(1-\frac{Bh}{G_w}\nabla^2\right)\frac{\partial^4 w}{\partial x^4} - \nabla^4\left(N_x\frac{\partial^2 w}{\partial x^2} + N_y\frac{\partial^2 w}{\partial y^2} + \right.\\
&\left. + 2T\frac{\partial^2 w}{\partial x \partial y} - 2D\nabla^4 w\right)\left(1-\frac{Bh}{G_w}\nabla^2\right) + 2B\left(h+\frac{t}{2}\right)^2\nabla^8 w = 0.
\end{aligned} \tag{6}$$

This equation describes the generalized case of a sandwich shell subjected to external loads N_x, N_y and T. Assuming that internal layer is much thicker than outer and applying only torsional load we can simplify [1], [2], [3], [4] the equations (5) and (6) to the following one:

$$\begin{aligned}
\frac{\partial^2 u_\alpha}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2 u_\alpha}{\partial y^2} + \frac{1+\nu}{2}\frac{\partial^2 v_\alpha}{\partial x \partial y} - \frac{\nu}{R}\frac{\partial w}{\partial x} &= 0,\\
\frac{\partial^2 v_\alpha}{\partial y^2} + \frac{1-\nu}{2}\frac{\partial^2 v_\alpha}{\partial x^2} + \frac{1+\nu}{2}\frac{\partial^2 u_\alpha}{\partial x \partial y} - \frac{1}{R}\frac{\partial w}{\partial y} &= 0,\\
\frac{Bh}{G_w}\left(\frac{\partial^2 u_\beta}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2 u_\beta}{\partial y^2} + \frac{1+\nu}{2}\frac{\partial^2 v_\beta}{\partial x \partial y}\right) - u_\beta + \left(h+\frac{t}{2}\right)\frac{\partial w}{\partial x} &= 0,\\
\frac{Bh}{G_w}\left(\frac{\partial^2 v_\beta}{\partial y^2} + \frac{1-\nu}{2}\frac{\partial^2 v_\beta}{\partial x^2} + \frac{1+\nu}{2}\frac{\partial^2 u_\beta}{\partial x \partial y}\right) - v_\beta + \left(h+\frac{t}{2}\right)\frac{\partial w}{\partial y} &= 0,\\
\frac{2B}{R}\left(\frac{\partial v_\alpha}{\partial y} + \nu\frac{u_\alpha}{x} - \frac{w}{R}\right) - 2B\left(h+\frac{t}{2}\right)\nabla^2\left(\frac{\partial u_\beta}{\partial x} + \frac{\partial v_\beta}{\partial y}\right) +&\\
+2T\frac{\partial^2 w}{\partial x \partial y} &= 0
\end{aligned} \tag{7}$$

$$\frac{2B}{R^2}(1-\nu^2)\left(1-\frac{Bh}{G_w}\nabla^2\right)\frac{\partial^4 w}{\partial x^4}-\nabla^4\left(1-\frac{Bh}{G_w}\nabla^2\right)2T\frac{\partial^2 x}{\partial x\partial y}+ \\ +2B\left(h+\frac{t}{2}\right)^2\nabla^8 w=0. \tag{8}$$

The following boundary conditions can be applied: for simple support of the edge

$$\begin{aligned}
&1.\ w=0, \qquad 2.\ u_\alpha=0,\\
&3.\ v_\alpha=0, \qquad 4.\ v_\beta=0,\\
&5.\ -2D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)-2B\left(h+\frac{t}{2}\right)\left(\frac{\partial u_\beta}{\partial x}+\nu\frac{\partial v_\beta}{\partial y}\right)=0,\\
&6.\ -2D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)=0
\end{aligned} \tag{9}$$

and for the clamped edges

$$\begin{aligned}
&1.\ w=0, \qquad 3.\ v_\alpha=0, \qquad 5.\ \frac{\partial u_\beta}{\partial x}=0,\\
&2.\ u_\alpha=0, \qquad 4.\ v_\beta=0, \qquad 6.\ \frac{\partial^2 w}{\partial x^2}=0.
\end{aligned} \tag{9a}$$

The solution to differential equations can be presented in the following form:

$$\begin{aligned}
u_\alpha&=\sum_{j=1}^{10}J_j\beta_{Aj}\sin\left(\frac{\lambda_j}{R}x-\frac{n}{R}y\right),\\
v_\beta&=\sum_{j=1}^{10}J_j\beta_{Cj}\sin\left(\frac{\lambda_j}{R}x-\frac{n}{R}y\right),\\
u_\alpha&=\sum_{j=1}^{10}J_j\beta_{Fj}\sin\left(\frac{\lambda_j}{R}x-\frac{n}{R}y\right),\\
v_\beta&=\sum_{j=1}^{10}J_j\beta_{Kj}\sin\left(\frac{\lambda_j}{R}x-\frac{n}{R}y\right),\\
w&=\sum_{j=1}^{10}J_j\sin\left(\frac{\lambda_j}{R}x-\frac{n}{R}y\right).
\end{aligned} \tag{10}$$

Detailed description of solution of the equations (7), (8) is presented in [5]. Numerical computations were performed according to the detailed description presented in [5]. During the computations external load was increased from 0 to the value for which stability loss occurred. The computations were performed with different value of the parameter n - which describes form of the stability loss. It equals to a number of waves on a circumference of a cylinder. The lowest value of a load corresponds to a stability loss.

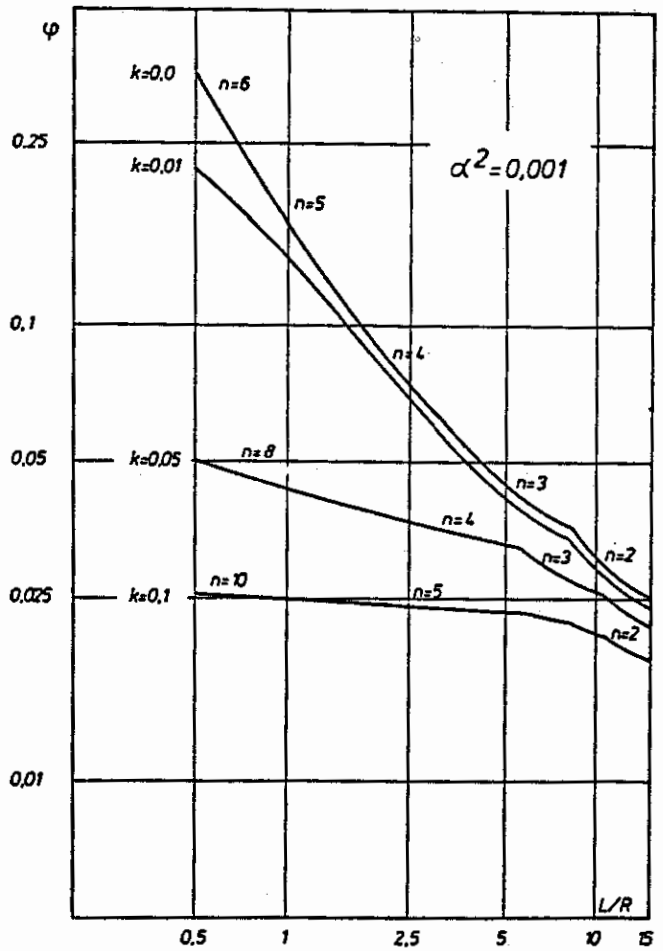

Figure 3: Critical load of a shell for $\alpha^2 = 0,001$

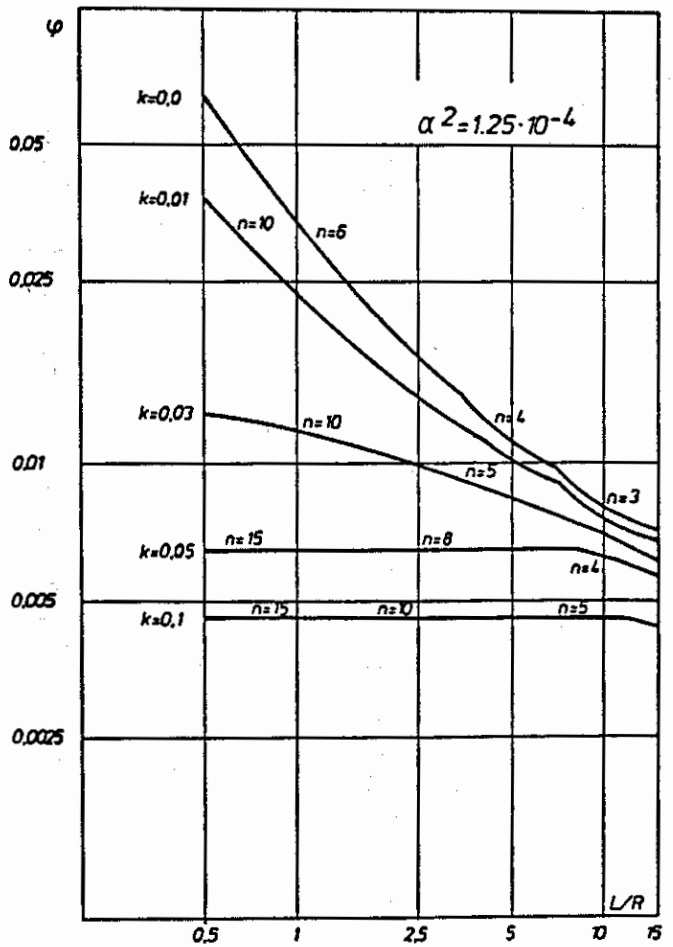

Figure 4: Critical load of a shell for $\alpha^2 = 0,000125$

Graphical results of numerical computations for the selected cases are presented in the Figures 3 and 4.

Results of computations for a simple support differ from results obtained by other authors in 13.5% [6] and 0.14% [7].

Results of computations are presented in the graphs obtained for the different values of a thickness coefficient $\alpha = \frac{h+t/2}{R}$ and filling stiffness coefficient $k = \frac{Eth}{(1-\nu^2)R^2}$. Critical load is described by a coefficient $\varphi = \frac{(1-\nu^2)M_s}{2\pi R^2 Et}$. In the presented paper the solution was obtained assuming linear combination of the eigenvalues of a system of differential equations. This was the first approach to solve the problem of dynamic stability of a sandwich shell subjected to a harmonic load. Results of further results will be also published.

References

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