6th CONFERENCE
on
DYNAMICAL SYSTEMS
THEORY AND APPLICATIONS
Łódź, December 10-12, 2001

AVERAGING ANALYSIS OF IN-PLANE PROBLEMS FOR FLEXIBLE ELEMENTS WITH ATTENUATIONS

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The main problem is particularly important from a practical application point of view [1]. To solve it various numerical [1], matrix algorithms [2] as well as nonsmooth functions approach [3] are used. In a case of large number of periodical nonhomogenities very promising seems to be averaging method, which has been clearly demonstrated using examples of bending of plates with attenuations [4-7]. In this work we consider an averaging of an in-plane problem of elasticity for plates with periodical attenuations. Our considerations are within a frame of simple scheme described in references [8-19].

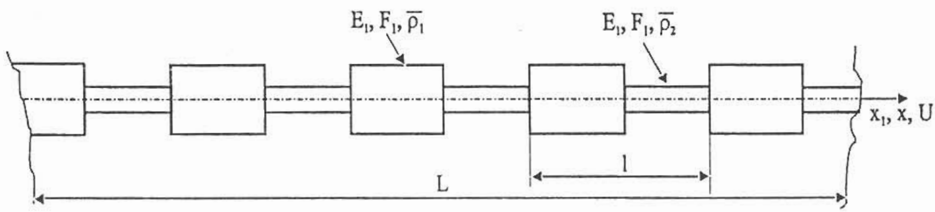

Figure 1.

1. We consider first longitudinal oscillations of a rod composed of linked sequence of elements with different characteristics (see Figure 1). Two neighbourhood rod's elements motions are governed by the following equations:

$$(EF)_i U_{ix_1x_1} - (\overline{\rho}F)_i U_{itt} = f_i(x_1,t); \;\; i=1,2, \tag{1.1}$$

where $f_i(x_1,t)$ define the forces acting on the rod's elements.

The following coupling condition between the neighborhood parts holds:

$$U_1 = U_2,\ \ T_1 = T_2 \qquad \text{on the contact} \tag{1.2}$$

where: $T_i = (EF)_i U_{ix_1},\ \ i = 1,2.$

We transform the relations (1.1), (1.2) to the following form:

$$U_{ixx} - \rho_i U_{itt} = \varphi_i(x,t),\ \ i = 1,2; \tag{1.3}$$

$$U_1 = U_2;\ \ U_{1x} = \varepsilon_1 U_2 \qquad \text{on the contact} \tag{1.4}$$

where: $\rho_i = L^2 \overline{\rho}_i / E_i$; $\varphi_1 = L^2 f_i(x,t)/E_i$; $x = x_1/L$ (see Figure 1); $\varepsilon_1 = (EF)_2/(EF)_1$.

Observe that the analysed object periodicity is characterized by parameter ε. As ε one can take a ratio of a length l of periodical part to length of a whole rod L (see Figure 1), or a ratio of l to a characteristic period of external load, and so on. We take $\varepsilon = \ell/L$ and we use two-scale method by introducing fast $\xi = x/\varepsilon$ and slow x variable, respectively. Therefore, we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x} + \varepsilon^{-1}\frac{\partial}{\partial \xi}. \tag{1.5}$$

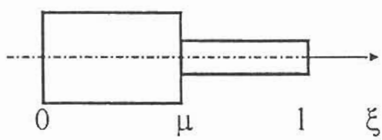

Figure 2.

A typical periodically appearing cell is shown in Figure 2. The being sought functions U_i are represented by the series:

$$U_i = U_0(x,t) + \varepsilon^{\alpha_i} U_i^{(1)}(x,\xi,t) + \varepsilon^{\alpha_i+1} U_i^{(2)}(x,\xi,t) + \ldots,\quad i = 1,2, \tag{1.6}$$

where: $U_i^{(k)}(x,\xi,t) = U_i^{(k)}(x,\xi+1,t)$; $i = 1,2$; $k = 1,2,\ldots$; the parameters α_i depend on a solution changes.

Observe that in the consider system three key parameters ε, ε_1 and μ appear. First of them is small in comparison to two other and consequently it can serve as a basis for an order estimation of other parameters. The averaged relations can be only obtained for fully defined relations between parameters. We introduce parameters β_1, β_2 and β_3 together with the parameters α_1, α_2 using the formulas:

$$\varepsilon_1 \sim \varepsilon^{\beta_1}, (1-\varepsilon_1) \sim \varepsilon^{\beta_2}, \mu\varepsilon^{\beta_3}, (1-\mu) \sim \varepsilon^{\beta_4}. \tag{1.7}$$

A choice of the parameters of asymptotic integrations $\alpha_i, \beta_k (i = 1,2,; k = 1 \div 4)$ is carried out using a routine but, in general, a tedious procedure [20-23]. In result we obtain two following fundamental cases:

a) $\alpha_1 = \alpha_2 = 2,\ \beta_1 = 0,\ \beta_2 = 1,\ \beta_3 = \beta_4 = 0$. The choice of the parameters corresponds to rods with approximately similar length and similar characteristics.

Dynamics of a cell $0 \le \xi \le 1$ is governed in the first approximation by the equation:

$$\frac{\partial^2 U_i}{\partial \xi^2} = A_i(U_o), \quad i = 1,2; \tag{1.8}$$

$$\text{for } \xi = \mu \quad U_1^{(1)} = U_2^{(1)}; \tag{1.9}$$

$$U_{1\xi}^{(1)} = \varepsilon_1 U_{2\xi}^{(1)} - (1-\varepsilon_1) U_{ox}; \tag{1.10}$$

$$U_1^{(1)}\Big|_{\xi=0} = U_2^{(1)}\Big|_{\xi=1}; \tag{1.11}$$

$$U_1^{(1)}\Big|_{\xi=0} = \varepsilon_1 U_{2\xi}^{(1)}\Big|_{\xi=0} - (1-\varepsilon_1) U_{ox}. \tag{1.12}$$

where: $A_i(U_0) = \varphi_i - U_{oxx} + \rho_1 U_{ott}$.

The conditions (1.11) and (1.12) are yielded by the periodicity condition (1.7).

Integrating (1.8) gives:

$$U_i^{(1)} = C_i^{(1)}(x,t) + C_i^{(2)}(x,t)\xi + 0.5 A_i \xi^2, i = 1,2. \tag{1.13}$$

A solvability condition in relation to constants $C_i^{(k)}$ is obtained after a substitution of solution (1.13) to relations (1.9)-(1.12), and it is governed by the following averaged equation:

$$A_1 \mu + \varepsilon_1 (1-\mu) A_2 = 0. \tag{1.14}$$

Observe that, from a physical point of view this equation corresponds to averaging of parameters due to the Voight approach.

The relations (1.9)-(1.12) yield only the difference $C_1^{(1)} - C_2^{(1)}$, and we take arbitrary $C_2^{(1)} = 0$. Other parameters are defined as follows:

$$\begin{aligned} C_1^{(1)} &= C_2^{(2)} + 0.5 A_2; \\ C_1^{(2)} &= \varepsilon_1 (C_2^{(2)} + A_2) - (1-\varepsilon_1) U_{ox}; \\ C_2^{(2)} &= \frac{A_1 \mu^2 + A_2(1 + 2\varepsilon_1 \mu - \mu^2) - (1-\varepsilon_1)\mu U_{ox}}{2[\mu(1-\varepsilon_1) - 1]}. \end{aligned} \tag{1.15}$$

Now we briefly discuss a question of boundary conditions satisfaction. Assume, for example, that we have U = 0 for x = 0; 1. Then the following interpretation can be given: the considered system is represented by an averaged rod, whereas a real structure holds in rod's ends neighborhood (Figure 3). This problem can be solved, for example, using matrices approach [24].

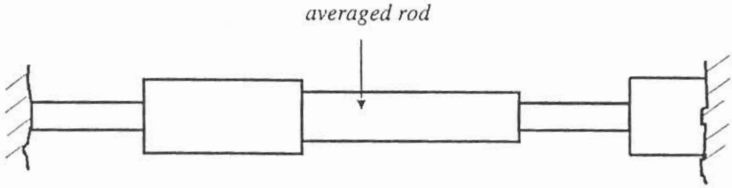

Figure 2.

b) $\alpha_1 = 3, \alpha_2 = 1, \beta_1 = 1. \beta_2 = 0, \beta_3 = 0, \beta_4 = 1$. This case corresponds to one of the most important for practice, i. e. when weak short additives separate long stiff elements [1].

A solution to the cell problem yields the following averaged equation:

$$A_1(U_0) = 0\,, \qquad (1.16)$$

and the 'fast' improvement:

$$U_2^{(1)} = U_{ox}(1-\xi)\,. \qquad (1.17)$$

2. Now we consider in-plane problem of theory of elasticity. It can be formulated within full equations of elasticity theory, as it was done in reference [1]. However, we use a different approach.

The simplified relations of in-plane theory of elasticity are used to solve a sequence of harmonical problems instead of bi-harmonical one [8-19].

The following simple physical scheme can be used (Figure 4). The load P(x) causes a plate deformation in OY direction, and therefore the fundamental deformation is related to V (one can neglect other one, i. e. U = 0). Similarly, for Q(y) one can take V = 0.

The similar like computational schemes have been widely applied by engineers in airplanes and rockets constructions [8, 9]. Furthermore, after some modifications it has been also successfully applied to theory of composites [10-13]. A similar like simplifications have been also applied in references [14, 15].

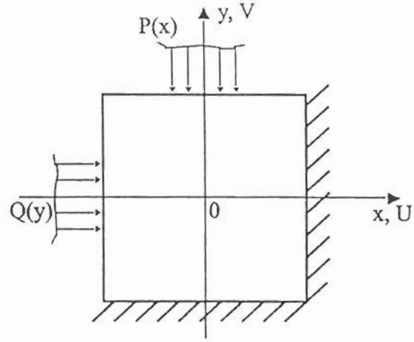

Figure 4.

The drawbacks of this partly empirical engineering approach are clear: possibility to increase the exactness of the results or difficulties to formulate a boundary value problem. It has been shown firstly in reference [16] that one can use either a ratio of stiffness in two main directions of strongly anisotropic materials or a ratio of rotational stiffness and one of a stretching-compressing stiffness as a small parameter. Using those approaches many important behaviours have been explained including that of singular character of asymptotics, as well as the initial biharmonic equation can be reduced to two Laplace equations coupled via boundary conditions, which allows to use theory of potential. In addition, a splitting of boundary conditions as well higher order approximations are constructed [17]. Besides, it has been shown that for an isotropic case (mostly unsuitable for asymptotics) an error introduced by first approximation is small [17]. Some mathematical aspects of the introduced asymptotics are analysed in references [18, 19]. The above discussion force us to consider an averaging procedure within a frame of Laplace equation.

The analysed initial problem is shown in Figure 5. The full equations governing in-plane orthotropic elasticity have the form:

$$\begin{aligned} &B_1^{(i)}U_{ix_1x_1} + G^{(i)}U_{iy_1y_1} + (B_1^{(i)}\nu^{(i)} + G^{(i)})V_{ix_1y_1} - \rho_i U_{itt} = f_i(x_1, y_1, t); \\ &B_2^{(i)}V_{iy_1y_1} + G^{(i)}V_{ix_1x_1} + (B_2^{(i)}\nu^{(i)} + G^{(i)})U_{ix_1y_1} - \rho_i V_{itt} = F_i(x_1, y_1, t), i = 1,2. \end{aligned} \tag{2.1}$$

with the attached contact continuation relations:

$$\begin{aligned} &U_1 = U_2; \;\; V_1 = V_2; \\ &T_x^{(1)} = T_x^{(2)}; \;\; T_{xy}^{(1)} = T_{xy}^{(2)}; \end{aligned} \tag{2.2}$$

where:

$$\begin{aligned} &T_x^{(i)} = B_1^{(i)}(U_{ix_1} + \nu_1^{(i)}V_{iy_1}); \\ &T_{xy}^{(i)} = G^{(i)}(U_{iy_1} + V_{ix_1}), \;\; i = 1,2. \end{aligned}$$

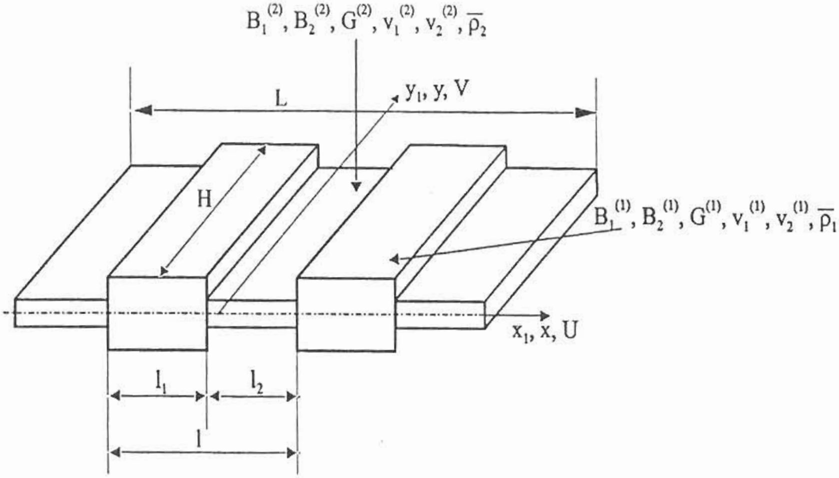

Figure 5.

We introduce the following 'small' parameters $\chi_i = G^{(i)} \big/ B_1^{(i)}$ and we assume $B_1^{(i)} \sim B_2^{(i)}$; $\nu_k^{(i)} \sim \chi_1$; $i = 1,2$; $k = 1,2$.

After asymptotical splitting in relation to χ_i the following Laplace equations and continuation conditions are obtained:

$$B_1^{(i)} U_{ix_1x_1} + G^{(i)} U_{iy_1y_1} - \overline{\rho}_i U_{itt} = f_i(x_1, y_1, t); \tag{2.3}$$

$$U_1 = U_2;\ B_1^{(1)} U_{1x_1} = B_1^{(2)} U_{2x_1} \quad \text{on a contact,} \tag{2.4}$$

$$B_2^{(i)} V_{iy_1y_1} + G^{(i)} V_{ix_1x_1} - \overline{\rho}_i V_{itt} = F_i(x_1, y_1, t); \tag{2.5}$$

$$V_1 = V_2;\ G^{(1)} V_{1x_1} = G^{(2)} V_{2x_1} \quad \text{on a contact.} \tag{2.6}$$

The averaging procedure of the problem (2.4), (2.5) can be carried out using Voight approach and leads to the result

$$B_2 V_{0y_1y_1} + G V_{0x_1x_1} - \rho V_{0tt} = F_i(x_1, y_1, t), \tag{2.7}$$

where:

$$B_2 = \frac{B_2^{(1)}\ell_1 + B_2^{(2)}\ell_2}{\ell};\ G = \frac{G^{(1)}\ell_1 + G^{(2)}\ell_2}{\ell};\ \rho = \frac{\overline{\rho}_1\ell_1 + \overline{\rho}_2\ell_2}{\ell};\ F = \frac{F_1\ell_1 + F_2\ell_2}{\ell};$$

In order to analyse a boundary value problem (2.3), (2.4) we use nondimensional quantities

$$U_{ixx} + \chi_i U_{iyy} - \rho_i U_{itt} = \varphi_i(x,y,t),$$

$$U_1 = U_2; U_{1x} = \varepsilon_1 U_{2x} \quad \text{on a contact.} \qquad (2.8)$$

where: $\rho_i = \overline{\rho}_i L^2 / B_1^{(i)}$; $\varphi_i = f_i L^2 / B_1^{(i)}$; $x = x_1 / L$; $y = y_1 / L$; $\varepsilon_1 = B_1^{(2)} / B_1^{(1)}$.

Observe that the problem governed by (2.8) is identical to that of the earlier discussed rod. All of the earlier results hold if

$$A_i(U_0) = \varphi_i - U_{0xx} - \chi_i U_{0yy} + \rho_i U_{0tt}.$$

For elements with similar stiffness and length the relations (1.13)-(1.15) hold, whereas for short and weak junctions the relations (1.16), (1.17) are valid.

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