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ANALYSIS OF PERIODIC OSCILLATIONS OF PARTIAL DIFFERENTIAL EQUATIONS USING SMALL δ METHOD

I.V. Andrianov
Prydneprovye State Academy of Civil Engineering and Architecture, 24a Chernyshevskogo St., Dnepropetrovsk 320005, Ukraine

J. Awrejcewicz
Department of Automatics and Biomechanics, Technical University of Łódź, 1/15 Stefanowskiego St., 90-924 Łódź, Poland
e-mail: awrejcew@ck-sg.p.lodz.pl

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Abstract. Two non-linear problems are considered: a non-linear wave equation with Dirichlet boundary conditions and a linear wave equation with non-linear boundary conditions. A so called small δ method is applied. It has been shown, among others that a "problem of small denominators" is omitted using the introduced approach. A relation of the used technique to other exiting methods is discussed.

1. Introduction

A problem of construction of periodic solutions to non-linear partial differential equations has attached recently an attention of many researchers. Some of possible approaches in a frame of quasi-linear technics have been presented in various monographs. Among others we must mention the KAM (Kolmogorov-Arnol'd-Moser) theory [1-4], averaging method [5, 6] renormalization approach with an introduction of artificial small parameter [7, 8] or multi-scale approach [9].

2. One dimensional space problem

Let us consider the equation

$$u_{tt} = u_{xx} - (\omega^2 - 1)u^3 \tag{2.1}$$

with the Dirichlet boundary conditions

$$u(0,t) = u(\pi,t) = 0. \tag{2.2}$$

We take $\omega^2 = \text{const}, \quad 1 < \omega^2 < \infty$.

From a physical point of view the boundary problem (2.1), (2.2) describes a longitudinal bar vibrations in non-linear elastic medium.

We introduce a small parameter in the following way

$$u_{tt} = u_{xx} - (\omega^2 - 1)u^{1+2\delta}. \tag{2.3}$$

A solution to the boundary value problem (2.3), (2.2) has the form

$$u = u_0 + \delta u_1 + \delta^2 u + ... \qquad (2.4)$$

The time variable t is changed according to Poincaré-Lindstedt method

$$t = \tau/\omega, \qquad (2.5)$$

$$\omega^2 = 1 + \alpha_1\delta + \alpha_2\delta^2 + ... \qquad (2.6)$$

After introducing of (2.4) - (2.6) to the boundary value problem (2.3), (2.2) and splitting in regard to δ the following recurrent system of equations is obtained

$$u_{0\tau\tau} = u_{0xx} - (\omega^2 - 1)u_0, \qquad (2.7)$$

$$u_{1\tau\tau} + \alpha_1 u_{0\tau\tau} = u_{1xx} - (\omega^2 - 1)u_1 - (\omega^2 - 1)u_0 \ln(u_0{}^2), \qquad (2.8)$$

$$\begin{aligned} u_{2\tau\tau} + \alpha_2 u_{0\tau\tau} + \alpha_1 u_{1\tau\tau} &= u_{2xx} - (\omega^2 - 1)u_2 - \\ &- (\omega^2 - 1)\{u_1 \ln(u_0{}^2) + 2u_1 + 0.5u_0[\ln(u_0{}^2)]^2\}, \end{aligned} \qquad (2.9)$$

$$u_i(0,t) = u_i(\pi,t) = 0, \quad i = 0,1,2,... \qquad (2.10)$$

We are going to find a periodic solution to the boundary value problem (2.3), (2.2) taking into account the following initial conditions

$$u(x,0) = \sin x, \qquad (2.11)$$

$$u_t(x,0) = 0. \qquad (2.12)$$

In order to satisfy the initial conditions (2.11), (2.12) two different approaches might be applied. One of them is related to a series development of initial conditions in regard to δ. Thus, we get in each of the approximation (excluding zero order solution) the homogeneous boundary conditions [10]. In the second approach [11] a zero order solution is obtained with an accuracy of an arbitrary constant, which is then defined by a final solution. This approach represents a kind of renormalization method and seems to be more suitable for our purpose.

Let us take a zero order solution of (2.7), (2.10)-(2.12) in the form

$$u_0 = A \sin x \cos(\omega\tau),$$

where the constant A will be further defined.

The first order solution has the form

$$\begin{aligned} u_{1\tau\tau} - u_{1xx} + (\omega^2 - 1)u_1 = L \equiv A \sin x \cos(\omega\tau)\{\alpha_1\omega^2 + \ln A - \\ - (\omega^2 - 1)[\ln(\sin^2 x) + \ln(\cos^2(\omega\tau))]\}. \end{aligned} \qquad (2.13)$$

The constant α_1 is found using a condition of avoiding a secular term:

$$\int_0^{\pi/2}\int_0^{\pi/(2\omega)} L \sin x \cos(\omega\tau) dx d\tau = 0$$

which yields

$$\alpha_1 = -\frac{\ln A}{\omega^2} - \frac{\omega^2 - 1}{2\omega^2}(2\ln 2 - 1)(1 + \omega).$$

Now the functions sin x ln ($\sin^2$ x) and cos ($\omega\tau$) ln ($\cos^2$ $\omega\tau$) are developed into the Fourier series, and L reads

$$L = -A(\omega^2 - 1)\left[\sin x \sum_{j=2}^{\infty} T_j \cos(j\omega\tau) + \cos(\omega\tau)\sum_{k=2}^{\infty} X_k \sin kx\right], \qquad (2.14)$$

where:

$$T_j = -\frac{4}{j^2-1}\ , \quad j = 3, 5, 7... \qquad X_k = -\frac{4}{k^2-1}\ , \quad k = 3, 5, 7... \tag{2.15}$$

The particular solution of (2.13) can be presented in the form

$$u^{(1)} = u^{(11)} + u^{(12)}, \tag{2.16}$$

where:

$$u^{(11)} = A(\omega^2-1)\sin x \sum_{j=2}^{\infty} T_j^{(1)} \cos(j\omega\tau),$$

$$u^{(12)} = -A(\omega^2-1)\sin(\omega\tau) \sum_{j=2}^{\infty} X_k^{(1)} \sin kx, \tag{2.17}$$

$$T_j^{(1)} = \frac{T_j}{\omega^2(j^2-1)}; \quad X_k^{(1)} = \frac{1}{k^2-1}.$$

The function $u^{(1)}$ satisfies the initial condition (2.12). The function $u^{(11)}$ gives a condition to define the constant A in this approximation

$$u_0 + \delta u^{(11)} = 1$$

Therefore

$$1 = A + A\delta(\omega^2-1)\sum_{j=2}^{\infty} T_j^{(1)}. \tag{2.18}$$

In order to compensate residual function in the initial condition (2.11) introduced by the function $u^{(12)}$ it is necessary to modify a general solution of a homogeneous equation of the first order approximation

$$u_{\tau\tau}^{(2)} - u_{xx}^{(2)} + (\omega^2-1)u^{(2)} = 0.$$

In result one obtains

$$u^{(2)} = A(\omega^2-1)\sum_{k=2}^{\infty} X_k^{(1)} \sin(kx)\cos(\omega_k^{(0)}\tau), \tag{2.19}$$

where: $\omega_k^{(0)} = \sqrt{k^2+\omega^2-1}$.

Let us consider a second order approximation equation of the form

$$\begin{aligned} &u_{2\tau\tau} - u_{2xx} + (\omega^2-1)u_2 = \\ &= \alpha_2\omega^2 \sin x\cos(\omega\tau) + (\omega^2-1)(L_1+L_2+L_3), \end{aligned} \tag{2.20}$$

where:

$$L_1 = (2+\ln A)u_1^{(1)} - \alpha_1 u_{\tau\tau}^{(1)},$$

$$L_2 = 0.5u_0[\ln(u_0)^2]^2 + u^{(11)}\ln(\cos^2(\omega\tau)) + u^{(12)}\ln(\sin x), \tag{2.21}$$

$$L_3 = [2+\ln(u_0{}^2)]u^{(2)} - \alpha_1 u_{\tau\tau}^{(2)} + u^{(11)}\ln(\sin^2 x) + u^{(12)}\ln(\cos^2(\omega\tau)).$$

The terms occurring in the right-hand side of the function L_1 do not include resonance harmonics. In contrary, in the function L_2 there occur only resonance harmonic sin x cos($\omega\tau$). In the function L_3, in spite of the already mentioned harmonic, there appears also the resonance one of the form sin(kx) cos ω_kx, k = 2, 3,... .

A proper choice of the constant α_2 allows for omitting sin x cos($\omega\tau$). In order to avoid problems concerning other resonance harmonics the following procedure can be applied.

The following change of variables is introduced

$$\omega_k = \omega_k^{(0)} + \beta_k^{(1)}\delta + \beta_k^{(2)}\delta^2 + \ldots \qquad (2.22)$$

and thus

$$\begin{aligned} u^{(2)} &= A(\omega^2 - 1)\sum_{k=2}^{\infty} X_k \sin(kx)\cos(\omega_k^{(0)}\tau) + \\ &+ A(\omega^2 - 1)\delta \sum_{k=2}^{\infty} X_k \beta_k^{(1)} \sin(kx)\cos(\omega_k^{(0)}\tau). \end{aligned} \qquad (2.23)$$

Now, by a proper choice of the constant $\beta_k^{(1)}$ one can avoid all the resonance terms appearing in the right-hand side of equation (2.20).

A generalisation of the presented above method into a higher dimensional case can be extended rather easily.

3. Problem with nonlinear boundary conditions

We consider the following equation governing the behaviour of waves

$$u_{tt} = u_{\alpha\alpha} \qquad (3.1)$$

with the following boundary conditions:

$$u(0,t) = 0, \quad u(1,t) + u_t(1,t) + \varepsilon u^3(1,t) = 0. \qquad (3.2)$$

A similar problem for $|\varepsilon| << 1$ can be efficiently solved by means of the perturbation technique.

3.1. Zero approximation

By taking $\varepsilon = 1$, we are going to solve the fundamental problem (3.1) - (3.2). From a physical point of view, the considered problem governs, for instance, vibration of a string or longitudinal vibrations of a rod with non-linear boundary conditions. After splitting the initial boundary problem with regard to powers of the "small parameter δ" the following recurrent system of linear boundary value problems is obtained:

$$\begin{aligned} &u_{0tt} = u_{0xx}, \\ &u_i(0,t) = 0, \quad i = 0,1,2,\ldots \\ &u_{itt} = u_{ixx} - \sum_{p=0}^{i} \alpha_{i-p} u_{ptt}, \quad i = 1,2,3,\ldots, \quad \alpha_0 = 0, \end{aligned} \qquad (3.3)$$

$$\begin{aligned} &\text{for} \quad x = 1 \quad u_{0x} + 2u_0 = 0, \\ &u_{1x} + 2u_1 = -u_0 \ln(u_0^2), \\ &u_{2x} + 2u_2 = -u_1 \ln(u_0^2) - 2u_1 - 0.5u_0[\ln(u_0^2)]^2 \equiv M_1. \end{aligned} \qquad (3.4)$$

A solution to the boundary value problem (3.3) can be presented in the form

$$u_0 = A \sin \omega_0 x \sin \omega_0 t,$$

where the frequency ω_0 is found from the following transcendental equation $\omega_0 = -2\,\mathrm{tg}\,\omega_0$.

3.2 First order approximation

The first order approximations have the following form

$$u_{1xx} - u_{1tt} = \alpha_1 A\omega_0^2 \sin(\omega_0 x)\sin(\omega_0 t), \quad (3.5)$$

$$u_1(0,t) = 0. \quad (3.6)$$

For x = 1 we have

$$u_{1x} + 2u_1 = A_1 \sin(\omega_0 t)[\ln A_1^2 + \ln(\sin^2(\omega_0 t))] \equiv M_2, \quad (3.7)$$

where: $A_1 = -A\sin\omega_0$.

A particular solution to equation (3.5), satisfying the boundary condition (3.6), has the form:

$$u_1^{(1)} = -\frac{1}{2}A\omega_0^2 x\cos(\omega_0 x)\sin(\omega_0 t). \quad (3.8)$$

The resonance term $A_1 R_1 \sin(\omega_0 t)$ is obtained from the right-hand side of the boundary condition (3.7), where:

$$R_1 = \ln A_1^2 + 0.5 - \ln 2.$$

The constant α_1 is obtained as a result of avoiding secular terms, and it reads:

$$\alpha_1 = R_1/(6+\omega_0^2).$$

Next, the right-hand side of the boundary condition (3.7) is represented by the Fourier series

$$M_2 = A_1 \sum_{k=2}^{\infty} T_k \sin(k\omega_0 t),$$

where: $T_k = \frac{\omega_0}{\pi}\int_0^{\frac{2\pi}{\omega_0}} \sin(\omega_0 t)[\ln A_1^2 + \ln(\sin^2(\omega_0 t))]\sin(k\omega_0 t)dt$.

To conclude, the following results are obtained:

$$T_k = 1 + \ln\frac{A^2\sin^2\omega_0}{4} \quad (\text{for} \quad k=1),$$

$$T_k = -\frac{4}{k^2-1} \quad (\text{for} \quad k=3,5,7,....), \quad (3.9)$$

$$T_k = 0 \quad (\text{for} \quad k=2,4,6,....).$$

When the term of $k = 1$ is neglected, then

$$M_2 = -A_1 \sum_{k=3,5,7,...}^{\infty} \frac{4}{k^2-2}\sin(k\omega_0 t), \; \omega_0 = 2.28893. \quad (3.10)$$

A solution to the boundary value problem (3.5) - (3.7) has the form $u_1 = u^{(1)} + u^{(2)}$,

where $u^{(1)}$ - solution to the homogeneous equation (3.5), and

$$u^{(2)} = A_1 \sum_{k=2}^{\infty} T_k^{(1)} \sin(\omega_0 kx)\sin(\omega_0 kt),$$
$$T_k^{(1)} = T_k [k\omega_0 \cos(k\omega_0) + 2\sin(k\omega_0)]. \quad (3.11)$$

In a similar way a second order approximation can be obtained.

4. Conclusion

As it has been shown, an application of "small δ method" can lead to omitting a problem with small denominators. A question arises: what it means when a problem of small denominators occurs. It indicates rather a wrong choice of a small parameter or an extremely complicated dependence of a sought function on a small parameter. The efforts focused on solution to the small denominator problem led to the fundamental results (KAM-theory). However, a final complete solution to the mentioned problem is not formulated yet. Therefore, a natural idea of searching another small parameter appears. The above given examples show that parameter δ occurring in the exponents of the series can serve as an alternative choice to avoid the mentioned drawbacks.

References

[1] WAYNE C. E. *Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory.* Communications of Mathematical Physics, 1990, vol. 127, pp. 479-528.

[2] CRAIG W., WAYNE C. E. *Newton's method and periodic solutions of nonlinear wave equations.* Communications of Pure and Applied Mathematics, 1993, vol. 46, pp. 1409-1498.

[3] WAYNE C. E. *Periodic solutions of nonlinear partial differential equations.* Notices of the American Mathematical Society, 1997, No 7, pp. 895-902.

[4] BOURGAIN J. *Construction of periodic solutions of nonlinear wave equations in higher dimension.* Geometry and Functional Analysis, 1995, vol. 5, pp. 105-140.

[5] KELLER J. B., KOGELMAN S. *Asymptotic solutions of initial value problems for non-linear partial differential equations.* SIAM J. Appl. Maths., 1970, vol. 18, pp. 748-758.

[6] MITROPOLSKY Yu., KHOMA G., GROMAK M. *Asymptotic Methods for Investigating Equations of Hyperbolic Type.* Kluwer Academic Publishers, N.Y., 1997.

[7] ANDRIANOV I. V., DANISHEVSKY V. V. *Asymptotic investigation of the nonlinear dynamic boundary value problem for rods.* Technische Mechanik, 1995, vol. 15, No 1, pp. 53-55.

[8] AWREJCEWICZ J., ANDRIANOV I. V., MANEVITCH L. I. *Asymptotic Approaches in Nonlinear Dynamics: New Trends and Applications*, Springer Verlag, Heidelberg, 1998.

[10] NAYFEH A. H. *Perturbation Methods.* Wiley Intersc., N.Y.,1973.

[11] MC RAE S. M., VRSKAY E. R. *Perturbation theory and the clasical limit of quantum mechanics.* Journal of Mathematical Physics, 1997, vol. No 6, pp. 2899-2921.

[12] LIDSKIJ B. V., SCHULMAN E. J. *Periodic solutions of the equation* $\ddot{u} + u_{xx} + u^3 = 0$. Functional Analysis and Applications, 1988, vol. 22, pp. 332-333.

[13] RABINOWITCH P. *Free vibrations for a semilinear wave equation.* Communications of Pure and Applied Mathematics, 1977, vol. 30, pp. 31-68.

[14] BREZIS H. *Periodic solutions of nonlinear vibrating strings and duality principles.* Bulletin of American Mathematical Society (N. S), 1983, vol. 8, pp. 409-426.