5th CONFERENCE
on
DYNAMICAL SYSTEMS
THEORY AND APPLICATIONS
Łódź, December 6-8, 1999

SMALL δ METHOD APPLIED FOR CONSTRUCTION OF PERIODIC SOLUTIONS TO NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

I.V. Andrianov, J. Awrejcewicz

Summary. A non-linear wave equation with Dirichlet boundary conditions is analysed. A so called small δ method is applied. It has been shown, among others that a "problem of small denominators" is omitted using the introduced approach.

1. Introduction

A problem of construction of periodic solutions to non-linear partial differential equations has attached recently an attention of many researchers [1-21]. Some of possible approaches in a frame of quasi-linear technics have been presented in various monographs. Among others we must mention the KAM (Kolmogorov-Arnol'd-Moser) - theory [15-18], averaging method [9, 11], renormalization approach with an introduction of artificial small parameter [19, 20], multi-scale approach [10].

The problems concerning the existence of periodic solutions have been considered in references [1, 2, 9, 12-18]. In reference [13] the variational technique is used, which will be briefly sketched. Consider the following problem

$$u_{tt} = u_{xx} + f(u), \qquad 0 < x < \pi , \tag{1}$$

$$u(0,t) = u(\pi,t) = 0 . \tag{2}$$

Under certain (fairly weak) conditions of the nonlinear term f(u), the existence of the periodic solution

$$u(x,t) = u(x,t+T), \quad T = \gamma\pi \tag{3}$$

is proved, where γ is the rational number. The last requirement refers to the approach used in the reference [13]. A review of the papers using the variational techniques is presented in reference [14].

An application of the KAM - theory allowed to examine a case of irrational γ occurred in (3) [12, 15-18]. In particular, in references [15, 16] the existence of time periodical solutions of the boundary value problems (1), (2) with $f(u) = \nu u + u^3$ and with fixed γ in (3) has been shown. A generalization of the technique presented in [15, 16] has been proposed in investigations [17, 18]. The variational approach and KAM - theory are complementary to one another and allow to cover a whole interval of frequencies (in the case of proof of existence of solutions). A construction of periodic solutions in time of non-linear partial differential equations is outlined in references [3-11, 19, 20].

The fundamental problem occurring during a formal construction of periodic solutions in all earlier discussed cases is related to that of small denominators. The mentioned problem can be briefly presented in the following manner. Suppose that a time periodic solution is constructed to the following boundary value problem

$$u_{tt} = u_{xx} + \varepsilon u^3, \qquad \varepsilon << 1, \tag{4}$$

$$u(0,t) = u(\pi,t) = 0. \tag{5}$$

Using a first approximation one obtains a linear wave equation. In order to get higher approximations one needs to insert a linear wave operator. The last operation leads to an occurrence of small denominators.

Therefore, a natural need for finding another parameter of asymptotic series to avoid the mentioned drawbacks is required. In the papers [22, 23] the so called "small δ method" is proposed, which relies on introduction of artificial "small parameter" δ in a power of non-linear terms.

According to that approach equation (4) should be rewritten to the form

$$u_{tt} = u_{xx} + \varepsilon u^{1+2\delta},$$

and its solution can be sought in the series form of small parameter δ.

An application of the outlined procedure to a series of nonlinear equations showed its high effectivity.

2. One dimensional space problem

Let us consider the equation

$$u_{tt} = u_{xx} - (\omega^2 - 1)u^3 \tag{6}$$

with the Dirichlet boundary conditions

$$u(0,t) = u(\pi,t) = 0 . \tag{7}$$

We take $\omega^2 = const, \quad 1 < \omega^2 < \infty$.

From a physical point of view the boundary problem (6), (7) describes a longitudinal bar vibrations in non-linear elastic medium.

We introduce a small parameter in the following way

$$u_{tt} = u_{xx} - (\omega^2 - 1)u^{1+2\delta} \tag{8}$$

A solution to the boundary value problem (8), (7) has the form

$$u = u_0 + \delta u_1 + \delta^2 u + \dots \tag{9}$$

The time variable t is changed according to Poincaré-Lindstedt method

$$t = \tau / \omega, \tag{10}$$

$$\omega^2 = 1 + \alpha_1 \delta + \alpha_2 \delta^2 + \dots \tag{11}$$

After introducing of (9) - (11) to the boundary value problem (7), (8) and splitting in regard to δ the following recurrent system of equations is obtained

$$u_{0\tau\tau} = u_{0xx} - (\omega^2 - 1)u_0 , \tag{12}$$

$$u_{1\tau\tau} + \alpha_1 u_{0\tau\tau} = u_{1xx} - (\omega^2 - 1)u_1 - (\omega^2 - 1)u_0 \ln(u_0{}^2) , \tag{13}$$

$$\begin{aligned} u_{2\tau\tau} + \alpha_2 u_{0\tau\tau} + \alpha_1 u_{1\tau\tau} &= u_{2xx} - (\omega^2 - 1)u_2 - \\ &- (\omega^2 - 1)\{u_1 \ln(u_0{}^2) + 2u_1 + 0.5u_0[\ln(u_0{}^2)]^2\}, \end{aligned} \tag{14}$$

$$u_i(0,t) = u_i(\pi,t) = 0, \quad i = 0, 1, 2, \dots \tag{15}$$

We are going to find a periodic solution to the boundary value problem (7), (8) taking into account the following initial conditions

$$u(x,0) = \sin x; \tag{16}$$

$$u_t(x,0) = 0\ . \tag{17}$$

In order to satisfy the initial conditions (16), (17) two different approaches might be applied. One of them is related to a series development of initial conditions in regard to δ. Thus, we get in each of the approximation (excluding zero order solution) the homogeneous boundary conditions. In the second approach a zero order solution is obtained with an accuracy of an arbitrary constant, which is then defined by a final solution. This approach represents a kind of renormalization method and seems to be more suitable for our purpose.

Let us take a zero order solution of (12), (15)-(17) in the form

$$u_0 = A \sin x \cos(\omega\tau),$$

where the constant A will be further defined.

The first order solution has the form

$$\begin{aligned} u_{1\tau\tau} - u_{1xx} + (\omega^2 - 1)u_1 = L \equiv A \sin x \cos(\omega\tau)\{\alpha_1\omega^2 + \ln A - \\ -(\omega^2 - 1)[\ln(\sin^2 x) + \ln(\cos^2(\omega\tau))]\}. \end{aligned} \tag{18}$$

The constant α_1 is found using a condition of avoiding a secular term:

$$\int_0^{\pi/2} \int_0^{\pi/(2\omega)} L \sin x \cos(\omega\tau) dx d\tau = 0$$

which yields

$$\alpha_1 = -\frac{\ln A}{\omega^2} - \frac{\omega^2 - 1}{2\omega^2}(2\ln 2 - 1)(1 + \omega)\ .$$

Now the functions $\sin x \ln(\sin^2 x)$ and $\cos(\omega\tau)\ln(\cos^2 \omega\tau)$ are developed into the Fourier series, and L reads

$$\begin{aligned} L = -A(\omega^2 - 1)\Bigg[\sin x \sum_{j=2}^{\infty} T_j \cos(j\omega\tau) + \\ + \cos(\omega\tau)\sum_{k=2}^{\infty} X_k \sin kx\Bigg], \end{aligned} \tag{19}$$

where:

$$T_j = -\frac{4}{j^2 - 1}, \quad j = 3, 5, 7...$$
$$X_k = -\frac{4}{k^2 - 1}, \quad k = 3, 5, 7... \tag{20}$$

The particular solution of (18) can be presented in the form

$$u^{(1)} = u^{(11)} + u^{(12)}; \tag{21}$$

where:

$$u^{(11)} = A(\omega^2 - 1)\sin x \sum_{j=2}^{\infty} T_j^{(1)} \cos(j\omega\tau),$$
$$u^{(12)} = -A(\omega^2 - 1)\sin(\omega\tau) \sum_{j=2}^{\infty} X_k^{(1)} \sin kx, \tag{22}$$
$$T_j^{(1)} = \frac{T_j}{\omega^2(j^2 - 1)}; \quad X_k^{(1)} = \frac{1}{k^2 - 1}.$$

The function $u^{(1)}$ satisfies the initial condition (17). The function $u^{(11)}$ gives a condition to define the constant A in this approximation

$$u_0 + \delta u^{(11)} = 1$$

Therefore

$$1 = A + A\delta(\omega^2 - 1)\sum_{j=2}^{\infty} T_j^{(1)}. \tag{23}$$

In order to compensate residual function in the initial condition (16) introduced by the function $u^{(12)}$ it is necessary to modify a general solution of a homogeneous equation of the first order approximation

$$u_{\tau\tau}^{(2)} - u_{xx}^{(2)} + (\omega^2 - 1)u^{(2)} = 0 .$$

In result one obtains

$$u^{(2)} = A(\omega^2 - 1)\sum_{k=2}^{\infty} X_k^{(1)} \sin(kx)\cos(\omega_k^{(0)}\tau), \tag{24}$$

where: $\omega_k^{(0)} = \sqrt{k^2 + \omega^2 - 1}$.

Let us consider a second order approximation equation of the form

$$u_{2\tau\tau} - u_{2xx} + (\omega^2 - 1)u_2 =$$
$$= \alpha_2\omega^2 \sin x \cos(\omega\tau) + (\omega^2 - 1)(L_1 + L_2 + L_3), \tag{25}$$

where:

$$L_1 = (2 + \ln A)u_1^{(1)} - \alpha_1 u_{\tau\tau}^{(1)},$$

$$L_2 = 0.5u_0[\ln(u_0)^2]^2 + u^{(11)}\ln(\cos^2(\omega\tau)) + u^{(12)}\ln(\sin x),$$

$$L_3 = [2 + \ln(u_0^{\ 2})]u^{(2)} - \alpha_1 u_{\tau\tau}^{(2)} + u^{(11)}\ln(\sin^2 x) + u^{(12)}\ln(\cos^2(\omega\tau)).$$

The terms occurring in the right-hand side of the function L_1 do not include resonance harmonics. In contrary, in the function L_2 there occur, only resonance harmonic sin x cos($\omega\tau$). In the function L_3, inspite of the already mentioned harmonic, there appears also the resonance of the form sin(kx) cos ω_kx, k = 2, 3,... .

A proper choice of the constant α_2 allows for omitting sin x cos($\omega\tau$). In order to avoid problems concerning other resonance harmonics the following procedure can be applied.

The following change of variables is introduced

$$\omega_k = \omega_k^{(0)} + \beta_k^{(1)}\delta + \beta_k^{(2)}\delta^2 + \dots.$$

and thus

$$u^{(2)} = A(\omega^2 - 1)\sum_{k=2}^{\infty} X_k \sin(kx)\cos(\omega_k^{(0)}\tau) +$$
$$+ A(\omega^2 - 1)\delta\sum_{k=2}^{\infty} X_k \beta_k^{(1)} \sin(kx)\cos(\omega_k^{(0)}\tau).$$

Now, by a proper choice of the constant $\beta_k^{(1)}$ one can avoid all the resonance terms appearing in the right-hand side of equation (25).

4. Conclusions

As it has been shown, the small δ method can be applicable for construction of periodic solutions to nonlinear partial differential equations to the second approximation order.

Here only a construction of solution is presented. The problems of convergence, accuracy estimation, as well as construction of higher order approximation need further investigations.

5. References

1. Struwe M. *Semilinear wave equation*. Bull. Amer. Math. Soc., vol. 26, N 1, p. 53-85.
2. Mac-Camy R. C., Mizel V. I. *Existence and nonexistence in the large of solutions of quasilinear wave equation.* Arch. Ration. Mechs and Anal., 1967, vol. 25, N 4, p. 299-320.
3. Kevorkian J., Cole J. D. *Multiple Scale and Singular Perturbation Methods.* Appl. Math. Sciences, 1996, vol. 114, Springer-Verlag, N. Y.
4. Chikwendu S. C. *Non-linear wave propagation solutions by Fourier transform perturbation.* Int. J. Non-Linear. Mechs. 1981, vol. 16, N 1, p. 117-128.
5. Witham G. B. *Linear and Non-linear Waves.* Wiley Intersc., N. Y., 1974.
6. Witham G. B. *A general approach to linear and non-linear dispersive waves.* J. Fluid Mechs., 1965, vol. 22, p. 273-283.
7. Luke J. C. *A perturbation method for non-linear dispersive wave propagation.* Proc. Royal Soc. London A, 1966, vol. 292, p. 403-412.
8. Eckhaus W. *New approach to the asymptotic theory of non-linear oscillations and wave propagation.* J. Math. Anal. Appl., 1975, vol. 49, p. 575-611.
9. Keller J. B., Kogelman S. *Asymptotic solutions of initial value problems for non-linear partial differential equations.* SIAM J. Appl. Maths., 1970, vol. 18, p. 748-758.
10. Lau S. L., Cheung Y. K., Chen Chuhui. *An alternative perturbation procedure of multiple scales for nonlinear dynamics systems.* Transaction of the ASME, J. of Applied Mechanics, 1989, vol. 56, N 3, p. 587-605.
11. Mitropolsky Yu., KhomA G., Gromak M. *Asymptotic Methods for Investigating Equations of Hyperbolic Type.* Kluwer Academic Publishers, N.Y., 1997.
12. Lidskij B. V., Schulman E. J. *Periodic solutions of the equation* $\ddot{u} = u_{xx} + u^3 = 0$. Functional Analysis and Applications, 1988, vol. 22, p. 332-333.
13. Rabinowitch P. *Free vibrations for a semilinear wave equation.* Communications of Pure and Applied Mathematics, 1977, vol. 30, p. 31-68.
14. Brezis H. *Periodic solutions of nonlinear vibrating strings and duality principles.* Bulletin of American Mathematical Society (N. S), 1983, vol. 8, p. 409-426.
15. Wayne C. E. *Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory.* Communications of Mathematical Physics, 1990, vol. 127, p. 479-528.
16. Craig W., Wayne C. E. *Newton's method and periodic solutions of nonlinear wave equations.* Communications of Pure and Applied Mathematics, 1993, vol. 46, p. 1409-1498.
17. Wayne C. E. *Periodic solutions of nonlinear partial differential equations.* Notices of the American Mathematical Society, 1997, N 7, p. 895-902.
18. Bourgain J. *Construction of periodic solutions of nonlinear wave equations in higher dimension.* Geometry and Functional Analysis, 1995, vol. 5, p. 105-140.
19. Andrianov I. V., Danishevsky V. V. *Asymptotic investigation of the nonlinear dynamic boundary value problem for rods.* Technische Mechanik, 1995, vol. 15, No 1, p. 53-55.
20. Awrejcewicz J., Andrianov I. V., Manevitch L. I. *Asymptotic Approaches in Nonlinear Mechanics: New Trends and Applications*, Springer Verlag, Heidelberg, 1998.
21. Zhuravlev V. F., Klimov D. M. *Applied Methods in the Theory of Oscillations.* Moscow, Nauka, 1988 (in Russian).
22. Bender C. M., Milton K. A., Pinsky S. S., Simmons L. M.Jr., *Nonperturbative approach to nonlinear problems,* Journal of Mathematical Physics, 1989, vol. 30, No 7, p.1447-1445.
23. Bender C. M., Boettcher S., Milton K. A. *A new perturbative approach to nonlinear partial differential equations.* Journal of Mathematical Physics, 1991, vol. 32, No 11, p. 3031-3038.

Professor I. V. Andrianov
Prydneprovye State Academy of Civil Engineering and Architecture
24a Chernyshevskogo St. Dnepropetrovsk 320005, Ukraine
Professor J. Awrejcewicz
Division of Automatics and Biomechanics, Technical University of Lodz,
1/15 Stefanowskiego St., 90-924 Lodz, Poland
e-mail: awrejcew@ck-sg.p.lodz.pl.