TENTH WORLD CONGRESS ON THE THEORY OF MACHINE AND MECHANISMS
Oulu, Finland, June 20-24, 1999

OPTIMIZATION OF PLATE AND SHELL SURFACES

Jan Awrejcewicz* and Vadim A. Krysko†
*Technical University of Łódź, Division of Control and Biomechanics (I-10)
1/15 Stefanowskiego St., 90-924 Łódź, POLAND.
†Saratov State University, Department of Mathematics, 96a, 410054 Saratov, RUSSIA

Keywords: Shells theory, vibroisolation, optimization

1 Introduction

In this paper we discuss a problem of optimal vibroisolation of a general shell with varying thickness and made from the orthotropic material. This problem is more complicated in comparison to the previous discussed cases. We shortly discuss a general approach using the non-linear model, and then we consider a linear case and apply successfully a superposition rule. Contrary to mostly used approaches we show that an optimal vibroisolation can be achieved by minimization external forces work instead of rather commonly used natural frequency optimizations.

2 Fundamental assumptions and relations

Let us assume that a conical shell covers the finite space Ω with the boundary S. The following equations govern behavior of a shell with a varying thickness and made from orthotropic material [Krysko 1976]:

$$k_x \frac{\partial^2 F}{\partial x^2} + k_y \frac{\partial^2 F}{\partial y^2} - L(w,F) + \frac{2}{3}\bar{\lambda}_1 \frac{\partial}{\partial x}\left[h\left(\gamma_x + \frac{\partial w}{\partial x}\right)\right] + \\ + \frac{2}{3}\bar{\lambda}_2 \frac{\partial}{\partial y}\left[h\left(\gamma_y + \frac{\partial w}{\partial y}\right)\right] + h\frac{\partial^2 w}{\partial t^2} = -q(x,y,t), \tag{1}$$

$$\frac{1}{12}\frac{\partial}{\partial x}\left[h^3\left(\lambda^{-2} A_{1111}\frac{\partial \gamma_x}{\partial x} + A_{1122}\frac{\partial \gamma_y}{\partial y}\right)\right] + \frac{1}{12}A_{1212}\frac{\partial}{\partial y}\left[h^3\left(\frac{\partial \gamma_x}{\partial y} + \right.\right. \\ \left.\left. + \frac{\partial \gamma_y}{\partial x}\right)\right] - \frac{2}{3}\bar{\lambda}_1^2 h\left(\gamma_x + \frac{\partial w}{\partial x}\right) - \frac{1}{12}\lambda_1^2 h^3 \frac{\partial^2 \gamma_x}{\partial t^2} = 0,$$

$$\frac{1}{12}\frac{\partial}{\partial y}\left[h^3\left(\lambda^2 A_{2222}\frac{\partial\gamma_y}{\partial y}+A_{1122}\frac{\partial\gamma_x}{\partial x}\right)\right]+\frac{1}{12}A_{1212}\frac{\partial}{\partial x}\left[h^3\left(\frac{\partial\gamma_x}{\partial y}+\right.\right.$$

$$\left.\left.+\frac{\partial\gamma_y}{\partial x}\right)\right]-\frac{2}{3}\overline{\lambda_2^2}h\left(\gamma_y+\frac{\partial w}{\partial y}\right)-\frac{1}{12}\lambda_2^2 h^3\frac{\partial^2\gamma_y}{\partial t^2}=0,$$

$$\frac{\partial^2}{\partial x^2}(k_y w)+\frac{\partial^2}{\partial y^2}(k_x w)+\lambda^{-4}a_{1111}\frac{\partial^2}{\partial x^2}\left(h^{-1}\frac{\partial^2 F}{\partial x^2}\right)+$$

$$+\lambda^4 a_{2222}\frac{\partial^2}{\partial y^2}\left(h^{-1}\frac{\partial^2 F}{\partial y^2}\right)+a_{1122}\left[\frac{\partial^2}{\partial x^2}\left(h^{-1}\frac{\partial^2 F}{\partial y^2}\right)\right]+$$

$$+\frac{\partial^2}{\partial y^2}\left(h^{-1}\frac{\partial^2 F}{\partial x^2}\right)-a_{1212}\left[\frac{\partial^2}{\partial x\partial y}\left(h^{-1}\frac{\partial^2 F}{\partial x\partial y}\right)\right]+\frac{1}{2}L(w,w)=0.$$

where:

$$L(w,w)=\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 F}{\partial x^2}-2\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2 F}{\partial x\partial y}+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 F}{\partial y^2},$$

$2h_0$ - thickness of a shell measured in its centre; $k_x = R_x^{-1}$, $k_y = R_y^{-1}$ - shell curvatures in x and y directions, correspondingly; F - stress function; w - normal displacement of shell's averaged surface in z direction; γ_x, γ_y - rotation angles of the, averaged surface in surfaces xz and yz, correspondingly; γ - weight by volume of the shell's material; g - gravity acceleration; t - time; x, y, z - Descartes co-ordinates.

Equations (1) have the non-dimensional form:

$$\bar{x}=\frac{x}{a},\quad \bar{y}=\frac{y}{a},\quad \bar{w}=\frac{w}{2h_0},\quad (2\bar{h})=\frac{2h}{2h_0},\quad \bar{F}=\frac{F}{A_{1111}(2h_0)^3},$$

$$\lambda_1=\frac{a}{2h_0},\quad \lambda_2=\frac{b}{2h_0},\quad \lambda=\frac{a}{b},\quad \bar{k}_x=\frac{k_x a^2}{2h_0},\quad \bar{k}_y=\frac{k_y b^2}{2h_0},$$

$$\bar{\gamma}_x=\lambda_1\gamma_x,\quad \bar{\gamma}_y=\lambda_2\gamma_y,\quad \bar{\gamma}_1=\lambda_2^2\bar{A}_{1313},\quad \bar{\gamma}_2=\lambda_1^2\bar{A}_{2323},$$

$$t=\frac{2h_0}{ab}\left(\frac{A_{1111}g}{\gamma}\right)^{\frac{1}{2}}t,\quad \bar{A}_{ijmk}=A_{ijmk}A_{1111}^{-1},\quad \bar{a}_{ijmk}=a_{ijmk}A_{1111}.$$

and the bars above the non-dimensional quantities are omitted.

The boundary conditions on the edge S are considered in a general form. Let $S=S_1+S_2+S_3$ and S_1 corresponds to the free part, S_2 corresponds to the free support and S_3 to the sliding support. Therefore we have

$$Q_n = M_n = M_\tau = 0 \qquad \text{on } S_1, \tag{2}$$

$$w = M_n = \gamma_\tau = 0 \qquad \text{on } S_2, \tag{3}$$

$$w = \gamma_n = \gamma_\tau = 0 \qquad \text{on } S_3, \tag{4}$$

$$F=\frac{\partial^2 F}{\partial n^2}=0 \qquad \text{on } S. \tag{5}$$

The following initial conditions are attached

$$\bar{u}(x,y,0)=\bar{u}^0(x,y), \qquad \frac{\partial\bar{u}(x,y,0)}{\partial t}=\bar{u}^1(x,y), \tag{6}$$

where: $\bar{u}(x,y,t)=(w,\gamma_x,\gamma_y)$ - three components of vector of displacements.

The following assumptions are applied to the coefficients A_{ijmk}, a_{ijmk} and the shell's thickness h(x,y):

$$\text{a) } 0<A_n \le A_{ijmk} \le A_b; \qquad 0<a_n \le a_{ijmk} \le a_b; \tag{7}$$

A_{ijmk}, a_{ijmk} $(i,j,m,k=1,2,3)$ - bounded functions on Ω;

b) for $\forall(x,y)\in\Omega$ and therefore $\forall\xi,\eta\in R^1$, $\exists C_0>0$,

$$A_{1111}\xi^2+2A_{1111}\xi\eta+A_{2222}\eta^2 \ge C_0(\xi^2+\eta^2) \tag{8}$$

as well as $\exists C_1>0$, which means that

$$a_{1111}\xi^2+(2a_{1122}-a_{1212})\xi\eta+a_{2222}\eta^2 \ge C_1(\xi^2+\eta^2) \tag{9}$$

c) h(x, y) - bounded function on Ω for $\forall(x,y)\in\Omega$:

$$0\le h_H \le h(x,y) \le h_b. \tag{10}$$

3 Non-linear vibration of shells governed by Timoshenko type model

In this item two theorems will be formulated. The first one is related to the important property of the operator L(w, F).

THEOREM 1. If $w\in H_0^1(\Omega)$, then $\forall F\in H_0^2(\Omega)$ and if $L(w,F)\in H^{-1}(\Omega)$ then the following relation is satisfied

$$\int_\Omega L(w,F)w\,d\Omega=\int_\Omega L(w,w)F\,d\Omega. \tag{11}$$

and consequently $L(w,w)\cdot F\in L^1(\Omega)$.

The functions space is denoted by $H_0^2(\Omega)$ and $H_0^2(\Omega)$ is a closure in $H^2(\Omega)$ of the functions manifold

$$V=\{F\in C^\infty(\Omega)\mid F=\frac{\partial^2F}{\partial n^2}=0 \text{ on } S\}.$$

The theorem 2 shows in which sense and in which spaces of functions the problems of (1) and (6) can be solved.

THEOREM 2. Let the shell's curvatures k_x, k_y are the bounded functions on Ω together with their second derivatives and suppose that conditions (7) - (10) are satisfied. Then for $\forall q(x,y,t)\in L^2(0,T,L^2(\Omega))$, $\bar{u}^0(x,y)=V_0$, $\bar{u}^1(x,y)\in(L^2(\Omega))^3$ a weak solution to the problem defined by (1) and (6) exists and

$$\vec{u}(x,y,t) \in L^{\infty}(0,T;V_0),$$
$$F(x,y,t) \in L^{\infty}(0,T;H_0^2(\Omega)).$$

Above $L_2(\Omega)$ denotes the Hilbert space.

4 Optimal vibroisolation against the harmonic load

The governing equations (1) have the following operator form

$$A[h]\vec{u} + B[h]\vec{u}'' = q(x,y,t), \tag{12}$$

where: A[h] - non-linear differential operator related to the shell's deformation energy, B[h] - operator related to the shell's mass distribution (it includes the inertial effects), q(x,y,t) - external excitation, $\vec{u} = (w, \gamma_x, \gamma_y, F)$.

We assume that the shell's thickness h(x, y) and its plane Ω are not fixed. They are chosen on the given manifolds $U_{\partial 1}$ and $U_{\partial 2}$ in order to realize minimum dynamical effects of the load $\vec{q}(x,y,t)$ on the shell as well as the required processes from an economical point of view.

Suppose that we are going to find $h^*(x,y) \in U_{\partial 1}$ and $\Omega^* \in U_{\partial 2}$. Both manifolds $U_{\partial 1}$ and $U_{\partial 2}$ are defined by the technological requirements and additional factors, but they also should satisfy the "mathematical constraints", which guarantee a solvability of the problem.

We are going to minimize the shell's weight and we require that a work of the elastic forces in the time period [0, T] overlaps with that applied to a shell with a priori given configuration Ω_0 and with the thickness h (x, y) = 1. In other words, we are going to find $h^*(x,y) \in U_{\partial 1}$, $\Omega^* \in U_{\partial 2}$ which realize min $I_1(h, \Omega)$, where

$$I_1(h,\Omega) = \int_{\Omega} h d\Omega + \frac{1}{\varepsilon}\left|\int_0^T \int_{\Omega} \vec{q}\vec{u} d\Omega dt - P_0\right|^2, \tag{13}$$

and $\vec{u}$ - solution to (12) for a shell with thickness h and plane Ω, $P_0 = \int_0^T \int_{\Omega} \vec{q}\vec{u}_0 d\Omega dt$ - work of external forces on displacement $\vec{u}_0$ of the initial shell. In our considerations the constraint $P=P_0$ is achieved using a penalty multiplier ε^{-1}.

For linear cases many conclusions about strength of constructions can be obtained analytically using properties of solution superposition. Developing $\vec{q}(x,y,t)$ in the Fourier series in relation to time t we get

$$\vec{q}(x,y,t) = \sum_{n=1}^{\infty} \vec{q}_n(x,y) \sin \omega_n t$$

for the periodic excitation.

Furthermore, we are going to analyze the shell's behavior subjected to only one of the harmonic component $\vec{q}_n \sin \omega_n t$ in the stable regime. It is evident, that also $\vec{u}_n$ can be presented in a similar way $\vec{u}_n = \vec{u}_n(x,y) \sin \omega_n t$, where $\vec{u}_n(x,y)$ fulfils the equation

$$A[h]\vec{u}_n - \omega_n^2 B[h]\vec{u}_n = \vec{q}_n(x,y). \tag{14}$$

Consider a shell with an arbitrary plane Ω and the thickness h. If the eigenvalue problem corresponding to free shell's vibration of the form

$$A[h]\vec{\psi}_i - \lambda_i^2 B[h]\vec{\psi}_i = 0$$

is solved, then the solution $\vec{u}_n$ and the load $\vec{q}_n$ can be sought in the form of a series related to the eigenmodes

$$\begin{aligned}\vec{u}_n(x,y) &= \sum_{i=1}^{\infty} a_{in}\vec{\psi}_i(x,y),\\ \vec{q}_n(x,y) &= \sum_{i=1}^{\infty} f_{in}\vec{\psi}_i(x,y),\end{aligned} \tag{15}$$

where f_{in} are defined by the following equations

$$\sum f_{in}(\vec{\psi}_i, \vec{\psi}_j) = (\vec{q}_n, \vec{\psi}_j), \qquad j = 1,2,\dots .$$

Using a superposition rule only one component $\vec{q}_{nk} = f_{nk}\vec{\psi}_k(x,y)$ can be applied. We are going to find a solution to the equation (14) for this case. Substituting (15) to (14) we get

$$\sum_{i=1}^{\infty} a_{ink}(A[h]\vec{\psi}_i(x,y) - \omega_n^2 B[h]\vec{\psi}_i(x,y)) = f_{nk}\vec{\psi}_k(x,y). \tag{16}$$

Multiplying (16) by $\vec{\psi}_j(x,y)$ and integrating on Ω one obtains

$$\sum_{i=1}^{\infty} a_{ink}(\lambda_i^2 - \omega_n^2)(B[h]\vec{\psi}_i, \vec{\psi}_j) = f_{nk}(\vec{\psi}_k, \vec{\psi}_j).$$

Introducing the norm for the functions $\vec{\psi}_i$ in the form $(B[h]\vec{\psi}_i, \vec{\psi}_j) = \delta_{ij}$, we get

$$a_{ink} = \frac{f_{nk}(\vec{\psi}_k, \vec{\psi}_i)}{(\lambda_i^2 - \omega_n^2)},$$

or

$$\vec{u}_{nk}(x,y) = \sum_{i=1}^{\infty} \frac{f_{nk}(\vec{\psi}_k, \vec{\psi}_i)}{(\lambda_i^2 - \omega_n^2)} \vec{\psi}_i(x,y).$$

The work of external forces has the form

$$P_{nk} = \frac{\pi f_{nk}^2}{\omega_n} \sum_{i=1}^{\infty} \frac{(\vec{\psi}_k, \vec{\psi}_i)^2}{(\lambda_i^2 - \omega_n^2)}.$$

In the above formulae $\vec{\psi}_i$, $\vec{\psi}_k$, f_{nk}, λ_i depend on both h and Ω. In this case we can assume that f_{nk} and $\vec{\psi}_i$ do not depend on h, and the problem of minimization of the external forces is easily solved. The aim of approach is to minimize the expression $(\vec{\psi}_k, \vec{\psi}_i)^2/(\lambda_i^2 - \omega_n^2)$. It means that λ_k^2 should be maximally shifted away from the excitation frequencies ω_n^2. In this simple formulation a problem of vibroisolation is reduced to that of spectrum optimization. In a general case, even for a harmonic excitation with the frequency ω_n^2 the full work has the form

$$P_n = \sum_{k=1}^{\infty} P_{nk} = \frac{\pi}{\omega_n} \sum_{k=1}^{\infty} \frac{(\vec{\psi}_k, \vec{\psi}_i)^2}{(\lambda_i^2 - \omega_n^2)}. \tag{17}$$

It is clear, according to (17), that in practice it is impossible to give conditions on the shell's spectrum λ_i, because they strongly depend on f_{nk}^2. In addition, with a plane change also $\vec{\psi}_i$ are strongly changed and that the sum of the all harmonics work $P_n = \sum_{n=1}^{\infty} P_{nk}$. The above remarks lead to conclusions that a practical realization of a construction with a maximal difference between the excitation and resonance frequencies is rather difficult or even impossible. The construction can work on one of the resonance frequencies only if the corresponding component f_{nk} does not appear in a load. The problem is more complicated in the non-linear case. In spite of the mentioned remarks the frequency of vibration ω_n depends on the amplitudes, i.e. on the thickness distribution as well as on the plane shape of a shell.

The above considerations lead to the following conclusion. A problem of the optimal vibroisolation should be considered not as that of constraints formulations applied to the natural vibration spectrum, but in a more broadband manner defined as the external forces work minimization.

However, due to new formulation of the problem many difficulties occur relating either to calculation abilities or to a search problem formulation. During a seek process the shell's natural frequencies may lie in a neighborhood of one of the excitation frequencies. In this case a solution to the problem (12) simply does not exist. As a result all of the algorithms stop to work. The constraints on $U_{\partial 1}$ and $U_{\partial 2}$ excluding those effects are very complex in general.

Therefore we consider only a problem of optimal vibroisolation in the linear case with a harmonic excitation and with a given frequency. The problem is formulated via (14), where the linear differential operators L[h] and M[h] should substitute the operators A[h] and B[h], correspondingly.

The results given above allow to apply the algorithms used for the spectrum optimization problems. The only difference is that on each calculation step (instead of spectrum) a work of external forces should be calculated.

In a case of shells optimization, in regard to a set of parameters h(x, y) and Ω, the most effective is the finite element method.

References

Krysko, V.A., Nonlinear Static and Dynamics of Non-Homogeneous Plates. Saratov State University, Saratov, 1976 (in Russian).