HOPF BIFURCATION IN DUFFING'S OSCILLATOR

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ABSTRACT

The paper presents a simple analytical method of determining the new solutions occuring in the course of Hopf bifurcation in nonautonomous systems,on the example of Duffing's oscillator.The method is a combination of the classical method of harmonic balancing and the perturbation method.The following cases have been considered:non-resonance,main resonance and resonance of the n-th order.

INTRODUCTION

The harmonic balancing method as well as the perturbation method are popular methods used to solve the nonlinear vibrations problems.The first method, although often employed by many authors, requires an a priori information regarding the behaviour of the analysed system[1].The latter, starting from the fundamental work[2], belongs to more universal methods, and is widely described eg. in[3].Recently, in the field of nonlinear vibrations an increasing interest has been noted in the Hopf bifurcation.After Hopf's original work[4], many authors have investigate this problem.In particular, such works as[5-7] should be pointed out, containing rich bibliography on this subject.The problem of the resonance dynamic sensitivity of Hopf bifurcation in a forced nonlinear oscillator, with its short history is described in[8].

This paper presents the method of determining a new solution after Hopf bifurcations in nonlinear system with harmonic excitation.The following assumptions have been made: in the case of lack of external excitation, when the bifurcation parameter $\eta<\eta_c$(η_c- critical point) the system has a static equilibrium path.On the other hand in the critical point, after linearization of the nonlinear differential equations, in the neighbourhood of the equilibrium path, their characteristic equation has two complex conjugate eigenvalues which, with the increase of η , cross the imaginary axis with nonzero velocity, ie. fulfill the basic assumption of Hopf[5].

THE VIBRATING SYSTEM

Let us consider the vibrating mechanical system with one degree of freedom

$$m\frac{d^2x}{dt^2} - (c_1 - c_2x^2)\frac{dx}{dt} + kx + k_1x^3 = P_0\cos\omega_1 t, \tag{1}$$

where m is the mass of the vibrating body, c_1 and c_2 are damping factors, k and k_1 - rigidity factors, and P_0 and ω_1 are respectively, amplitude and external excitation frequency.

The eq. (1) assumes the dimensionless form

$$\frac{d^2y}{d\tau^2} - \eta\frac{dy}{d\tau} + \delta y^2\frac{dy}{d\tau} + \omega_c^2 y + \xi y^3 = \lambda\cos\tau, \tag{2}$$

where

$$\tau = \omega_1 t,\ x = \frac{P_{max}}{m\omega_1^2} y,\ \eta = \frac{c_1}{m\omega_1},\ \delta = \frac{c_2 P_{max}^2}{m^3\omega_1^5},\ \alpha_0^2 = \frac{k}{m},$$

$$\omega_c^2 = \frac{\alpha_0^2}{\omega_1^2},\ \xi = \frac{k_1 P_{max}^2}{m^3\omega_1^6},\ \lambda = \frac{P_0}{P_{max}}. \tag{3}$$

The eq. (2) is a particular type of the Van der Pol - Duffing eq., and for $\eta=0$ it circumscribes the vibrations of Duffing's oscillator.It can be presented of the form

$$\begin{aligned}\frac{dx_1}{d\tau} &= x_2 - \frac{\delta x_1^3}{3} + \eta x_1,\\ \frac{dx_2}{d\tau} &= -\omega_c^2 x_1 - \xi x_1^3 + \lambda\cos\tau,\end{aligned} \tag{4}$$

where $x_1 = y$.The characteristic eq. obtained from (4), in the case of lack of external excitation, has the roots $\delta_{1,2} = 1/2(\eta \pm \sqrt{\eta^2 - 4\omega_c^2})$.As in this case $\eta_c=0$ then in the critical point $\delta_{1,2} = \pm i\omega_c$ and $d\delta_{1,2}/d\eta|_c = 1/2$, so that condition of occurrence of the Hopf bifurcation are fulfilled.

NON-RESONANCE CASE ($\alpha_0/\omega_1 \neq k/l;\ k,l \in N$)

Let $\lambda = \varepsilon\lambda_1$; then the eq.(4) will have the form

$$\begin{aligned}\frac{dx_1^{(1)}}{d\tau} &= x_2^{(1)} - \frac{\delta(x_1^{(1)})^3}{3} + \eta x_1^{(1)},\\ \frac{dx_2^{(1)}}{d\tau} &= -\omega_c^2 x_1^{(1)} - \xi(x_1^{(1)})^3 + \varepsilon\lambda_1\cos\tau.\end{aligned} \tag{5}$$

The solutions of (5) are searched in the form of series

$$x_i^{(1)} = x_{ic}^{(1)} + \varepsilon x_i^{(1)'} + \frac{1}{2}\varepsilon^2 x_i^{(1)''} + \frac{1}{6}\varepsilon^3 x_i^{(1)'''} + \ldots,\ i=1,2 \tag{6}$$

$$\eta = \eta_c + \varepsilon\eta' + \frac{1}{2}\varepsilon^2\eta'' + \frac{1}{6}\varepsilon^3\eta''' + \ldots, \tag{7}$$

$$\omega = \omega_c + \varepsilon\omega' + \frac{1}{2}\varepsilon^2\omega'' + \frac{1}{6}\varepsilon^3\omega''' + \ldots \tag{8}$$

In the critical point $\eta_c = x_{ic}^1 = 0$.The bifurcation solutions $x_i^{(1)}(\tau,\varepsilon)$ are represented by the Fourier series

$$\begin{aligned}x_i^{(1)} = &\sum_{k=0}^{K}(p_{ik0}^s(\varepsilon)\sin k\tau_1 + p_{ik0}^c\cos k\tau_1) + \sum_{l=0}^{L}(p_{io1}^c(\varepsilon)\cos l\tau +\\ &p_{io1}^s(\varepsilon)\sin l\tau) + \frac{1}{2}\sum_{l=1}^{L}\sum_{k=1}^{K}[p_{ikl}^{cc}(\varepsilon)[\cos(k\tau_1 + l\tau) + \cos(k\tau_1 - l\tau)] +\\ &p_{ikl}^{cs}(\varepsilon)[\sin(k\tau_1 + l\tau) + \sin(l\tau - k\tau_1)] + p_{ikl}^{sc}(\varepsilon)[\sin(k\tau_1 + l\tau) +\end{aligned} \tag{9}$$

$$\sin(k\tau_1 - l\tau)\Big] + p^{ss}_{ikl}(\varepsilon)\Big[\cos(k\tau_1 - l\tau) - \cos(k\tau_1 + l\tau)\Big]\Big\} ,$$

where: $\tau_1 = \omega\tau$,

$$p^{(*)}_{i(\cdot)} = p^{(*)}_{ic(\cdot)} + \varepsilon p^{(*)}_{i(\cdot)} + \frac{1}{2}\varepsilon^2 p^{(*)}_{i(\cdot)} + \frac{1}{6}\varepsilon^3 p^{(*)}_{i(\cdot)} + \ldots \tag{10}$$

The following parameter - frequency relations and the dependence of bifurcation parameter on amplitude are obtained

$$\eta = \frac{\delta}{2}\left(\frac{1}{2}A^2 + \frac{\lambda^2}{(\omega_c^2 - 1)^2}\right), \tag{11}$$

$$\omega = \omega_c + \frac{3}{4}\frac{\xi}{\omega_c}\left(\frac{1}{2}A^2 + \frac{\lambda^2}{(\omega_c^2 - 1)^2}\right) \tag{12}$$

where: $A^2 = (\varepsilon p^{c'}_{110})^2 + (\varepsilon p^{s'}_{110})^2$.

MAIN RESONANCE

Let $\lambda = \varepsilon^3\lambda_2$; then eq. (4) gives

$$\begin{aligned} \frac{dx_1^{(2)}}{d\tau} &= x_2^{(2)} - \frac{\delta(x_1^{(2)})^3}{3} + \eta x_1^{(2)}, \\ \frac{dx_2^{(2)}}{d\tau} &= -\omega_c^2 x_1^{(2)} - \xi(x_1^{(2)})^3 + \varepsilon^3\lambda_2\cos\tau. \end{aligned} \tag{13}$$

Let us assume that frequency ω_c and external excitation frequency differ from each other only slightly, ie. let $\omega_c \cong 1 - 1/2a'$, where $a' = \varepsilon^2 a$.Now we act analogously to the non - resonance case and we obtain

$$\eta = \frac{\delta}{4}A^2 - \frac{\lambda}{A}\sin\varphi, \tag{14}$$

$$\omega = \omega_c - \frac{1}{2}\frac{\lambda\cos\varphi}{A} + \frac{3}{8}\xi A^2, \tag{15}$$

where: $p^{s'}_{110}\varepsilon = A\sin\varphi$; $p^{c'}_{110}\varepsilon = A\cos\varphi$.

RESONANCE OF THE N-TH ORDER

In this case it is assumed that the following relation exists between the frequency and external excitation frequency: $\omega_c \cong 1/n(1 - 1/2a')$.The eq. (4), after taking $\lambda = \varepsilon\lambda_1$, $\tau = n\tau_2$, gives

$$\begin{aligned} \frac{dx_1^{(3)}}{d\tau} &= x_2^{(3)} - n\frac{\delta}{3}(x_1^{(3)})^3 + n\eta x_1^{(3)}, \\ \frac{dx_2^{(3)}}{d\tau} &= -x_1^{(3)} + a'x_1^{(3)} - n^2\xi(x_1^{(3)})^3 + \varepsilon n^2\lambda_1\cos n\tau_2. \end{aligned} \tag{16}$$

In this case we obtain

$$\eta = \frac{1}{2}\delta\left(\frac{1}{2}A^2 + \frac{\lambda^2 n^4}{(1-n^2)^2}\right), \tag{17}$$

$$\omega = n\omega_c + \frac{1}{2}n^2\xi\left(\frac{3}{4}A^2 + \frac{3}{2}\frac{\lambda^2 n^4}{(1-n^2)^2}\right). \tag{18}$$

CONCLUDING REMARKS

On the example of forced Van der Pol-Duffing's oscillator, the paper presents an analytical method of determining the post-critical family of solutions after Hopf bifurcation in nonlinear nonautonomous oscillators with one bifurcation parameter. It is assumed that Hopf's conditions are fulfilled for a certain value of the bifurcation parameter when no forcing exists in the system. The bifurcated solutions are sought in the form of a particular Fourier series. The practical application of the series consists in solving perturbation eqs of the first, second and k-th order by means of the harmonic balancing method. The exemplary calculations have been reduced to k=3, as an approximate bifurcation solution is determined in principle near the critical point, while the nearer it we are the most exact is the solution. In the investigated case the amplitude is connected with the small perturbation parameter . The exciting force amplitudes are of the order of the amplitude A, or are smaller.

In the case of no resonance as well as in that of the resonance of n-th order, the parameter ξ has no influence on the dependence of bifurcation parameter-amplitude-amplitude of the exciting force, whereas the parameter δ has no influence on the dependence of amplitude-frequency-amplitude of the exciting force.

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