Dimension reduction method in nonlinear equations of mathematical physics (MEMS/NEMS problems)

Vadim A. Krysko-jr, Jan Awrejcewicz, Maxim V. Zhigalov, Vadim. A. Krysko

Abstract To simulate the static and dynamic behavior of nanoplates, higher-order continuum theories have been developed: modified couple stress theory of elasticity, nonlocal theory of elasticity, gradient theory of elasticity, and surface elasticity theory. It should be noted that when using these theories, the equations describing the behavior of the plates have a high order, and the desired functions depend on two or more variables. In this regard, there is a need to create methods that can reduce the dimension of the desired functions, i.e. reduce the solution of the partial differential equation to the solution of an ordinary differential equation. The paper provides an overview of methods for reducing partial differential equations to ordinary differential equations based on the Kantorovich-Vlasov method. One such method is the variational iteration method. MVI was widely used by many researchers in solving problems of the theory of shells and plates. The authors of this work, since the 70s of the last century, has been used this method to solve geometrically, physically nonlinear and contact problems of the theory of plates and shells for full-size systems. In a number of their works, the authors provided a justification of this method for a class of equations described by positive definite operators. In the presented paper, MVI is used in plate nanomechanics problems and a proof of MVI convergence for the problems under consideration is given. A numerical example is also added.

1. Introduction

A number of computational methods aimed to solve the most diverse problems of mathematical physics and technology are based on the ideas of Russian scientists I.G. Bubnov and B.G. Galerkin. To date, the Bubnov-Galerkin methods have been applied in solving numerous problems of structural mechanics, structural dynamics, hydromechanics, hydromechanical stability theory, magneto hydrodynamics, heat and mass transfer theory, acoustics, microwave propagation theory, neuron transfer theory, etc. Using the Bubnov - Galerkin approach, ordinary differential equations, partial differential equations, and integral equations were studied. The application of the Bubnov-Galerkin method is connected, among other things, with the search for stationary points of some functional, which is a certain integral of the differential expressions generated by the original problem, which makes it possible to decrease the order of differential operators in the integrand.

The origin of the Bubnov-Galerkin method is usually associated with the name of the outstanding Russian scientist Ivan G. Bubnov (1872 - 1919). Together with A.N. Krylov, he was the creator of the Russian Navy. S.P. Timoshenko in a paper published in 1907 [1] using the example of a central compressed rod considered the stability problem based on minimizing the potential energy of the rod.

This work was sent for feedback to professors N.A. Belelyubsky, S.I. Beletsky, I.G. Bubnov, V.L. Kirpichev and G.V. Kolosov, which were published in 1913 in the "Collection of the Institute of Railway Engineers" [2]. This date is considered the date of the official birth of the Bubnov method, as a general method for solving differential equations. I.G. Bubnov gave two options for solving the problem of reducing partial differential equations (or their systems), i.e. either to algebraic equations (or their systems), or to an ordinary differential equation (or their systems).

In Western literature, this method is associated with the B.G. Galerkin's article [3] published in 1915. The article was devoted to the elastic equilibrium of rods and thin plates. But the above analysis of publications devoted to this method suggests that I.G. Bubnov as a true genius proposed an idea that occurred to him when he was working on a review of S.P. Timoshenko paper. In this review, he already established the identity between the energy method (called the Rayleigh – Ritz – Timoshenko method) and his approach (called the Bubnov – Galerkin method). I.G. Bubnov subsequently used this approach extremely little. This method is well known in the scientific literature due to the works of B.G. Galerkin and his colleagues.

Applying the Bubnov – Galerkin methodology in one of the variables, if the desired function depends on two variables, we arrive at the solution of the ordinary differential equation in the other variable, accordingly, to the Kantorovich – Vlasov method [4, 5]. Such a procedure linked these two distinguished names, and the method they developed became known as the Kantorovich-Vlasov method (MKV). This method, by its ideology, couples the Fourier method (MF) based on the separation of variables, and the Bubnov-Galerkin method (MBG), which gave impetus to a number of modifications (Fig. 1). Modifications are based on the Weindiner method (MV), the variational iteration method (MVI), the Agranovsky – Baglai – Smirnov method (MABS), and their

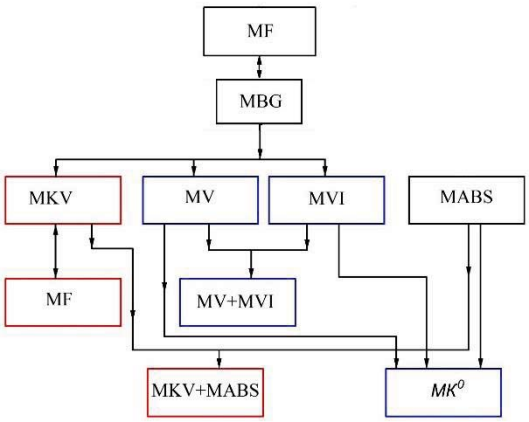

Fig. 1. Interrelation of Bubnov-Galerkin method, Fourier method, the method of Kantorovich-Vlasov and their modifications

combinations are described in a number of papers by the authors of this work [6–8]. These articles provide evidence of convergence and a comparative analysis of the results of these methods.

One of the methods included in the scheme is the variational iterations method (MVI), which saves the researcher from the need to build a system of approximating functions in the procedure while employing the Bubnov-Galerkin method. The functions initially specified in an arbitrary way (obviously satisfying certain well-known smoothness conditions) are refined in the process of calculations by MVI based on the solutions of the original system of differential equations.

This method was first proposed and applied in 1933 by T.E. Shunk [9] for calculating the bending of cylindrical panels. However, the work went unnoticed, and the method was rediscovered again in 1964 by E.E. Zhukov [10], who applied it in calculating thin rectangular plates. Later MVI was widely used by many researchers in solving problems of the theory of shells and plates (a bibliography on this subject is presented in [11]). The justification of this method for the class of equations described by positive definite operators is given in reference [12].

It should be noted the discrepancy in the names. In the Western scientific literature, the variational iterations method is called the extended Kantorovich method thanks to the work of A.D. Kerr [13-15] published 38 years after T.E. Shunk and 5 years after E.E. Zhukov. Thus, the method was reopened.

The variational iterations method (extended Kantorovich method) over the past half century has been used to solve problems of statics, stability, determination of natural frequencies and dynamics. A fairly complete review of Western publications in this area can be found in [16, 17]. In the USSR and Russia, this method was mainly used in the works of V.A.Krysko and his students. For the first time, this scientific group used the approach in 1968 to study the bending of flexible orthotropic plates [18], and it got its name from the variational iterations method in 1970 [19], devoted to the numerical study of flexible plates and comparison with experimental data. Later, scientists of this group used the variational iterations method to solve geometrically and physically nonlinear problems in the theory of shells and plates [20,21], in problems of designing optimal plates [22–24], and on other topics [25–27].

In this paper, the variational iterations method is extended to the study of the static bending of Kirchhoff-Love nano-shells taking into account Kármán geometric nonlinearity and based on a modified couple stress theory.

2. Mathematical background

Let us consider a shallow rectangular shell with dimensions a,b,h along axes x_1,x_2,x_3, respectively. For a spherical shell, the internal radius, expressed in the shell thickness, can be easily determined with a formula $f=k_1/8$, where k_1 stands for the shell curvature parameter [28].

The origin of the coordinate system is located in the upper left corner of the shell on its middle surface. The axes x_1, x_2 are parallel to the shell sides and the axis x_3 is directed towards the shell curvature (Fig. 2). In the given coordinate system, the shell is treated as a 3D region Ω defined by $\Omega = \{x_1, x_2, x_3 \ / \ (x_1, x_2, x_3) \in [0, a] \times [0, b] \times [-h/2, -h/2]\}$. The shell middle surface $x_3 = 0$ is defined as $\Gamma = \{x_1, x_2 / (x_1, x_2) \in [0, a] \times [0, b]\}$.

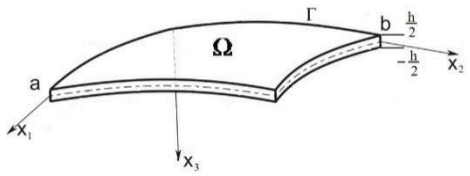

Fig.1. Scheme of the studied shell.

We denote shell displacements along the axes x_1, x_2, x_3 by u_1, u_2, u_3, respectively, where $u_3 = u_3(x_1, x_{2)})$. All components of the displacement are assumed to be essentially smaller than the characteristic shell dimension; deformations in the shell middle surface $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}$ are assumed to be negligible with respect to a unit (however, it does not mean that the relationship between displacements and deformations must be linear). Owing to the Kirchhoff-Love hypothesis, the following relations between the deformations of the middle surface ε_{ij} and an arbitrary surface e_{ij} are valid [29]

$$e_{ii} = \varepsilon_{ii} + x_3 \xi_{ii}, \ i = 1,2, \quad e_{12} = \varepsilon_{12} + x_3 \xi_{12}, \tag{1}$$

where

$$\varepsilon_{ij} = \frac{\partial u_i}{\partial x_i} + \frac{1}{2}\left(\frac{\partial u_3}{\partial x_i}\right)^2 - k_i u_3, i = 1,2, \quad \varepsilon_{12} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_1} + \frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_2},$$
$$\xi_{ij} = -\frac{\partial^2 u_3}{\partial x_i^2}, i = 1,2, \quad \xi_{12} = -2\frac{\partial^2 u_3}{\partial x_1 \partial x_2}, \tag{2}$$

and k_1, k_2 denote the shell curvatures.

In the modified coupled stress theory [30], the deformation energy U_1 of an elastic body occupying the space Ω, taking into account small deformation, reads

$$U_1 = \frac{1}{2}\int_{\Omega}\left(\sigma_{ij}\varepsilon_{ij} + m_{ij}\chi_{ij}\right)dv, \tag{3}$$

where: ε_{ij} – deformation tensor components, χ_{ij} – components of an asymmetric tensor of the curvature gradient. The components are defined as follows

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{m,i}u_{m,j}\right), \quad \chi_{ij} = \frac{1}{2}\left(\theta_{i,j} + \theta_{j,i}\right), \theta_i = \frac{1}{2}\left(rot\left(u\right)\right)_i . \tag{4}$$

Here u_i stands for the components of the displacement vectors **u**, **θ** stands for an infinitely small rotation vector with elements θ_i and δ_{ij} is the Kronecker symbol. In the case of an isotropic elastic material, stresses generated by kinematic parameters occurred in (4) are yielded by the following state equations [30]:

$$\sigma_{ij} = \lambda\varepsilon_{mm}\delta_{ij} + 2\mu\varepsilon_{ij}, \; m_{ij} = 2\mu l^2\chi_{ij} , \tag{5}$$

where $\sigma_{ij}, \varepsilon_{ij}, m_{ij}$ и χ_{ij} denote components of the classical tensor of stresses **σ**, deformation tensor **ε**, deviator part of the symmetric tensor of the higher order **m** and a symmetric part of the curvature tensor **χ**, respectively; $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \mu = \frac{E}{2(1+\nu)}$ are the Lamè parameters; $E\left(x,y,z,e_i\right), \nu\left(x,y,z,e_i\right)$ is Young's modulus and Poisson's coefficient, respectively, $\rho\left(x,y,z,e_i\right)$ - density of the beam material; e_i - the strength of the deformation.

In this model, in addition to the classical Lamè parameters, the additional scale parameter of the length *l* is employed [30]. This is a simple consequence of the fact that in the couple stress theory, the density of the deformation energy depends only on the deformation tensor and the symmetric curvature tensor. The latter does not explicitly depend on the rotation (nonsymmetric part of the deformation gradient) and the non-symmetric part of the curvature tensor [30].

To obtain the initial differential equations in mixed form we introduce the force function *F*:

$$T_{ij} = -\frac{\partial^2 F}{\partial x_i \partial x_j} , \tag{6}$$

and then the equations in mixed form will be written with respect to u_3 and the force function *F*.

We introduce the well-known notation for differential operators

$$\nabla_k^{\,2}(.) = k_2\frac{\partial^2(.)}{\partial x_1^2} + k_1\frac{\partial^2(.)}{\partial x_2^2}; \; L((.),(.)) = \frac{\partial^2(.)}{\partial x_1^2}\frac{\partial^2(.)}{\partial x_2^2} + \frac{\partial^2(.)}{\partial x_2^2}\frac{\partial^2(.)}{\partial x_1^2} - 2\frac{\partial^2(.)}{\partial x_1 dx_2}\frac{\partial^2(.)}{\partial x_1 dx_2} . \tag{7}$$

Using the Hamilton principle, we come to a system of differential equations composed of the equations of motion of the nano-shell (8) and the equations of compatibility of deformations (9):

$$\tilde{D}\nabla^4 u_3 - L(u_3, F) - \nabla_k^2 F - q = 0 , \tag{8}$$

$$\frac{1}{Eh}\nabla^4 F = -\nabla^2{}_k u_3 - \frac{1}{2}L(u_3, u_3) , \tag{9}$$

where $\tilde{D} = D + \frac{El^2}{4(1+\nu)}$ (D – cylindrical stiffness, l - size-dependent length parameter of the material).

Scheme of MVI can be formally described in the following way. We are aimed at finding a solution to equation $Aw(x,y) = q(x,y);\quad x,y \in \Omega(x,y)$, where A stands for a certain operator defined on the manifold $D(A)$ of the Hilbert space $L_2(\Omega)$; $q(x, y)$ stands for a given function of two variables x, y, and $w(x, y)$ is a searched function; $\Omega(x, y)$ is a space associated with variations of x and y.

If $\Omega(x, y) = X \times Y$ (X – a certain bounded set of variables x; Y - a bounded set of y), then a solution to equation has the following form $w_N(x,y) = \sum_{i=1}^{N} u_i(x)v_i(y)$, where the functions $u_i(x)$ and $v_i(y)$ are defined by the following system of equations

$$\int_X (Aw_N - q)u_1(x)dx = 0, \qquad \int_Y (Aw_N - q)v_1(y)dy = 0,$$

.....

$$\int_X (Aw_N - q)u_N(x)dx = 0, \qquad \int_Y (Aw_N - q)v_N(y)dy = 0,$$

in the following way. A certain system composed of N functions with respect to one of the variables, for instance, $u_1^0(x), u_2^0(x),..., u_N^0(x)$ is given. Then, the first N equations of the system yield N functions $v_1^1(x), v_2^1(x),..., v_N^1(x)$. Next, the obtained functions are employed to create a new set of functions $x - u_1^2(x), u_2^2(x),..., u_N^2(x)$, which is further used to construct a set of new functions with respect to the variable y, i.e. $v_1^3(x), v_2^3(x),..., v_N^3(x)$, and so on. In the case of the iterational procedure MVI [12], proves of the theorems constituting the theoretical background of the MVI convergence were given for the problems of the theory of plates.

Theorem 1. If A is a positively defined operator with its action space $D(A) \subset H_A$, then the sequence of elements $\alpha_k = \left\| w_1^k(x,y) - w_0 \right\|_{H_T}$ is monotonously decreasing, i.e. for arbitrary i and j if only $i \geq j$, then $\left\| w_1^j - w_0 \right\|_{H_T} \leq \left\| w_1^j - w_0 \right\|_{H_T}$.

Theorem 2. Let each element of the basis system of the space $\overset{0}{W}{}_2^m(X \times Y)$ has the form $\theta_i(x,y) = \varphi_i(x)\psi_i(y)$, where $\{\phi_i(x)\}$ is a basis system in the space $\overset{0}{W}{}_2^m(X)$, and respectively $\overset{0}{W}{}_2^m(X)$ in the space $\overset{0}{W}{}_2^m(Y)$, and in order to get an arbitrary N-th approximation regarding MVI, the

components of the elements of the basis system $\{\theta_i(x,y)\}$ are taken as initial conditions. Then, for sufficiently large *N*, the MVI gives a unique approximate solution w_N, and the sequence $\{w_N\}$ is convergent with regard to the norm of the space $\overset{0}{W}{}_2^m(X \times Y)$ and tends to the exact solution w_0 independently of a number of steps *k*, which can be defined for each of the N-th approximation, i.e. $\left\| w_N^k - w_0 \right\|_{\overset{0}{W}{}_2^m} \to 0, \quad N \to \infty$.

The resulting system of nonlinear partial differential equations can be solved by one of the methods shown in Fig. 1. Boundary and initial conditions are given in [31].

For a numerical example, the variational iteration method and the combination of the variational iteration method and the Agranovsky – Baglay – Smirnov method [MABS] were used. The Agranovsky-Baglay-Smirnov method is proposed and substantiated in the works of Agranovsky et al. [32]. Let's consider the scheme of application of the Agranovsky-Baglai-Smirnov method on the example of the operator equation:

$$A\left[w(x_1,x_2) \right] = q(x_1,x_2). \tag{10}$$

A solution to equation (10) in the first approximation ($w_1 = \varphi_1^{(k-1)}(x)\ \psi_1^{(k)}(y)$) is searched in a way similar to the MVI.

The new equation is defined as follows

$$Aw_2(x,y) = q(x,y) - Aw_1(x,y), \tag{11}$$

i.e. we have changed the right hand side of equation (10). Equation (11) is solved again with the help of MVI, and its first approximation yields

$$w_2(x,y) = \phi_2^{(k-1)}(x)\psi_2^{(k)}(y). \tag{12}$$

The next new equation follows

$$Aw_3(x,y) = q(x,y) - Aw_1(x,y) - Aw_2(x,y). \tag{13}$$

and then one employs the MVI again in the first approximation, and so on.

Finally, the following series is used as the input solution:

$$w(x,y) = \sum_{n=1}^{N} w_n(x,y). \tag{14}$$

3. Results and discussions.

As a numerical example, we consider the application of the described approaches to solve the problem of bending the nanoplate ($k_1 = k_2 = 0$) and without taking into account the geometric nonlinearity, then equations (8, 9) in dimensionless form will be written as

$$\tilde{D}\Delta\Delta w(x_1, x_2) = q(x_1, x_2), \tag{15}$$

where $\tilde{D} = \frac{1}{12}\frac{(1-\nu)}{(1+\nu)(1-2\nu)} + \frac{1}{4(1+\nu)}\gamma^2$, $\gamma = \frac{l}{h}$ - the dimensionless form of the dimension-dependent coefficient , ν is the Poisson's ratio. We consider the boundary conditions of two kinds:

$$w(x_1, x_2)\big|_{\Gamma} = 0,\ \Delta w(x_1, x_2)\big|_{\Gamma} = 0, \tag{16}$$

$$w(x_1, x_2)\big|_{\Gamma} = 0,\ \partial w(x_1, x_2)/\partial n\big|_{\Gamma} = 0. \tag{17}$$

The load is constant and distributed over the entire surface of the plate and is equal to 50. Ordinary differential equations obtained after the use of MSI and MSI + ABS are reduced by the finite difference method of 2 order of accuracy to a system of algebraic equations which is solved by the Gauss method.

Numerical results for the variational iteration (MVI) and combination (MVI+ABS) method are given in Table 1. The exact solution obtained in double trigonometric series is also reported in [8]:

Table 1.

Boundary condition	Exact solution	γ	MVI			MVI+MABS		
			N=4	N=10	N=20	N=4	N=10	N=20
(16)	0.2028	γ=0	*0.2031*	*0.2031*	*0.2030*	*0.2029*	*0.2029*	*0.2028*
	Relates error		-//-	-//-	*0.1%*	-//-	-//-	*0%*
(17)	*0.0661*		*0.0650*	*0.0651*	*0.0653*	*0.0660*	*0.0660*	*0.0661*
	Relates error		*1.66%*	*1.51%*	*1.21%*	-//-	-//-	*0%*
(16)		γ=0.5	0.1074	0.1074	0.1073	0.1066	0.1066	0.1065
(17)			*0.0374*	*0.0375*	*0.0376*	*0.0382*	*0.0382*	*0.0383*

Here N is the number of partitions of the plate NxN. As can be seen from the table, the use of MABS significantly increases the accuracy of the solution and for the value γ=0, the numerical solution is equal to the exact one. Taking into account the size-dependent behavior leads to the fact that the deflection decreases almost twice, i.e. the plate becomes more rigid.

4. Concluding remarks

The paper deals with the application of the variational iteration method (extended Kantorovich method) to the solution of the problems of the Kirchhoff – Love nano-shells bending on the basis of the modified couple stress theory. Modifications of known methods and their relationship are given. The paper presents a scheme for proving the convergence of the variational iteration method. The numerical implementation of the variational iteration method and the Agranovsky-Baglay-Smirnov method is shown by the example of solving the Sophie-Germain-Lagrange equation. Numerical results prove fast convergence of the methods even with a small number of grid partitions.

Acknowledgements

This work has been supported by the Polish National Science Centre under the Grant PRELUDIUM 16 No. 2018/31/N/ST8/00707.

References

[1] Timoshenko S.P., On the longitudinal bending of rods in an elastic medium. News of St. Petersburg Polytechnic Institute. Department of Engineering, Natural Sciences and Mathematics 1907, vol. 7, no. 1, pp. 145 - 157.

[2] Reviews of professors Kirpichev, Beltsenko, Bubnov and Kolosov on the works of Professor Timoshenko, awarded the D.I. Zhuravsky. Collection of St. Petersburg Institute of Railway Engineers of Emperor Alexander I. Edition of the Institute of Railway Engineers. Printing house Yu.N. Erlich, 1913, vol. 81, pp. 1 - 40.

[3] Galerkin B.G., Rods and plates. Rows in some questions of elastic equilibrium of rods and plates. Vestnik Inzhenerov I Technikov. 915. T.I. № 19. pp. 897-908.

[4] Kantorovich L.V. Krylov V.I., Approximate methods of higher analysis. M.: Fizmatgiz, 1962.

[5] Vlasov V.V., General theory of shells. M: Gostekhizdat, 1949.

[6] Krysko A.V., Awrejcewicz J., Pavlov S.P., Zhigalov M.V., Krysko V.A., On the iterative methods of linearization, decrease of order and dimension of the Karman-type PDEs. The Scientific World Journal Volume 2014 (2014), Article ID 792829, pp. 15.

[7] Krysko A.V., Awrejcewicz J., Zhigalov M.V., Krysko V.A., On the contact interaction between two rectangular plates. Nonlinear Dynamics DOI 10.1007/s11071-016-2858-2, Vol 84, №4 June 2016, pp. 2729-2748.

[8] Awrejcewicz J., Krysko V.A., Zhigalov M.V., Krysko A.V., Contact interaction of two rectangular plates made from different materials with an account of physical non-linearity. Nonlinear Dynamics: An International Journal of Nonlinear Dynamics and Chaos in Engineering Systems. Volume 85, Number 4. 2016. pp. 2729-2748

[9] Schunk T.E. Zur Knienfestigkeit schwach gekrummter zylindrischer Schalen/ - Ing/ Arch., 1933, IV, pp. 394 – 414.

[10] Zhukov E.E., The variational method of successive approximations as applied to the calculation of thin rectangular plates. In the book: Calculation of thin-walled spatial structures / Ed. A.R. Rzhanitsina. - M.: Stroyizdat, 1964, pp. 27-35.

[11] Krysko V.A., Nonlinear statics and dynamics of inhomogeneous shells. - Saratov: Ed. Saratov. Univ., 1976. pp. 214.

[12] Kirichenko V. F., Krysko V. A., Substantiation of the variational iteration method in the theory of plates, Soviet Applied Mechanics, April 1981, Volume 17, Issue 4, pp. 366-37

[13] Kerr A.D., An extension of the Kantorovich method. Q. Appl. Math., vol. 26, 1968, pp. 219–229.

[14] Kerr A.D., Alexander H., An application of the extended Kantorovich method to the stress analysis of a clamped rectangular plate. Acta Mech., vol. 6, 1968, pp. 180–196.

[15] Kerr A.D., An extended Kantorovich method for the solution of eigenvalue problems, Int. J. Solids Struct. vol. 5, 1969, pp. 559–572.

[16] Singhatanadgid P., Singhanart T., The Kantorovich method applied to bending, buckling, vibration, and 3D stress analyses of plates: A literature review. Mechanics of Advanced Materials and Structures. 26(2), 2019, pp. 170-188.

[17] Banerjee M.M., Mazumdar, J., A Review of Methods for Linear and Nonlinear Vibration Analysis of Plates and Shells. 2016. Procedia Engineering. 144, pp. 493-503.

[18] Krysko V.A., Amelchenko V.V., To the calculation of flexible orthotropic plates by the modified Vlasov-Kantorovich method using an electronic computer. For Technical Progress (Baku).-1968. No. 7. pp. 13-15.

[19] Krysko V.A., Amelchenko V.V., Calculation of flexible plates by the variational method of iterations and comparison with experimental data. Three young scientists: Materials of the 3rd inter-university. conf. Saratov, 1970. pp. 62-66.

[20] Krysko V.A., Fedorova A.G., Dynamics problems for elastically plastic flexible shallow shells. Applied Mechanics. USSR Academy of Sciences. 1979. T. XV, Iss. 2, pp. 71-76.

[21] Krysko V.A., Fedorova A.G., Stability of shallow shells under the influence of local loads, taking into account geometric and physical nonlinearities. Vestnik Universities. Construction and Architecture, 1978, No. 5, pp. 24-28.

[22] Krysko V.A., Bochkarev V.V., Designing plates and shells that are close to optimal in weight, taking into account the temperature effect. Applied Mechanics. Academy of Sciences of the Ukrainian SSR, 1981, T, XVII, No 11. pp. 54-59.

[23] Krysko V.A., Bochkarev V.V., Optimal design of plates and shells, taking into account physical non-linearity. Applied Mechanics. Academy of Sciences of the Ukrainian-SSR. 1982. T.XVIII, No 7. pp. 52-57.

[24] Krysko V.A., Pavlov S.P., The problem of optimal control of the natural frequency of inhomogeneous shells. Applied Mechanics. USSR Academy of Sciences. 1982. T.XVIII, No. 4. pp. 41-47.

[25] Krysko V.A., Bochkarev V.V., The stress-strain state of a plate of variable thickness with a flat front surface. Vestnik Vuzov Engineering. 1984. No. 12. pp. 231-235.

[26] Krysko V.A., Kirichenko V.F., Surova N.S., On projection methods for solving the problems of the theory of flexible gently sloping anisotropic multilayer shells. Materials II All-Union. conf. on scientific and technological products from composite materials. Yerevan, 1984. Part II. pp. 99-107.

[27] Krysko V.A., Zhigalov M.V. The method of linearization and lowering the order of systems of differential equations in the nonlinear mechanics of heterogeneous structures. Proc. doc. IV International conf. on the mechanics of heterogeneous structures. Ternopol, 1995. pp. 356.

[28] Vlasov V.Z., General Series of Shells and its Applications in Technique. Gostexizdat, Moscow (1949) (in Russian)

[29] Novozhilov V. V., Thin Shell Theory, Springer, New York, USA, 2nd edition, 1971.

[30] Yang F., Chong A. C. M., Lam D. C. C., and Tong P., Couple stress based strain gradient theory for elasticity, Int. J. Solids Struct. 39 (2002), pp. 2731–2743.

[31] Krysko-jr, V.A., Awrejcewicz, J., Zhigalov, M.V., Kirichenko, V.F., Krysko, A.V., Mathematical Models of Higher Orders. Shells in Temperature Fields. 1st ed. 2019, XII, p. 470.

[32] Baglay R. D., Smirnov K. K., On the processing of two-dimensional computer signals. Journal of Computational Mathematics and Mathematical Physics. 15:1. 1975. pp. 241–248.

Jan Awrejcewicz: Department of Automation, Biomechanics and Mechatronics, Lodz University of Technology, 1/15 Stefanowski St., 90-924 Lodz and Department of Vehicles, Warsaw University of Technology, 84 Narbutta Str., 02-524 Warsaw, POLAND; (*awrejcew@p.lodz.pl*). The author gave a presentation of this paper during one of the conference sessions

Vadim A. Krysko-jr: Department of Automation, Biomechanics and Mechatronics, Lodz University of Technology, 1/15 Stefanowski St., 90-924 Lodz *(vadimakrysko@gmail.com)*.

Maxim V. Zhigalov: Department of Mathematics and Modeling, Saratov State Technical University, Politehniekhskaya 77, 410054, Saratov, Russian Federation, (*max.zhigalov@gmail.com*).

Vadim A. Krysko: Department of Mathematics and Modeling, Saratov State Technical University, Politehniekhskaya 77, 410054, Saratov, Russian Federation, *(tak@san.ru)*