Vibration of nonlinear lumped systems with serially connected elastic elements

Jan Awrejcewicz, Roman Starosta, Grażyna Sypniewska-Kamińska

Abstract: The mechanical system with the nonlinear springs connected in series is considered in the paper. The mathematical model of that kind of systems consists of the differential and algebraic equations (DAEs). Adequately modified multiple scales method (MSM) in time domain have been applied to solve effectively the problem of harmonically forced vibration governed by DAEs. The obtained approximate solution in the analytical form allows for qualitative study of the considered system, among others for identification of the resonance conditions. The case of the main resonance is analysed in details. The modulation equations of the amplitudes and phases which are the integral part of the MSM solution allow one to study both steady and unsteady resonant motion. The stability of the resonant curves concerning the steady states has been tested and verified by comparison with the numerically obtained solutions.

1. Introduction

The massless springs in various configurations serve as a widely used models of the elastic effects in many structures. They occur not only in pure mechanical systems but also in mechatronical devices and in micro-electro-mechanical systems as well. The springs arranged in various configurations can be a source of manifold and sometimes unexpected dynamical phenomena, especially near resonances.

Our research deals with the one dimensional lumped system containing two springs with nonlinear properties and connected in series. The system seems to be quite simple, however its governing equations contain both differential and algebraic equation, therefore the appropriate modification of the asymptotic approach is necessary. We are focused on the forced vibration both far from resonance as well as in the resonance conditions. The alike system but containing one nonlinear and one linear spring was analyzed by Telli and Kopmaz [1]. The one dimensional oscillator with two nonlinear springs connected in series was analyzed in the paper [2], where the solutions dealing with only the non-resonant case are analyzed.

2. Mechanical system and mathematical model

Let us consider a body of mass *m* attached to the immovable wall by two springs connected in series, which can move in the horizontal path. The physical model of the analyzed system is given in Fig.1.

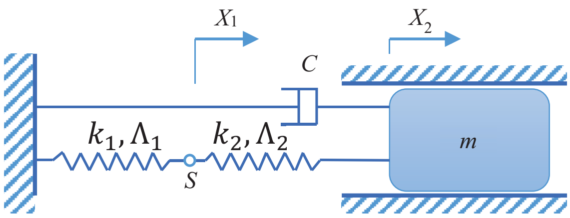

Figure 1. The analyzed mechanical system.

Let X_1 and X_2 are the elongation of the springs, whose nominal length are L_{01} and L_{02}, therefore the absolute displacement of the body equals $X_1 + X_2$. We assume the nonlinear character of the restoring forces in the springs in the following form

$$F_i = k_i(X_i + \Lambda_i X_i^3) \text{ for } i = 1, 2\,, \tag{1}$$

where k_i is the stiffness coefficient and Λ_i stands for the nonlinearity parameter for the *i*-th spring.

The kinetic energy of the system is

$$T = \frac{1}{2}m\left(\dot{X}_1 + \dot{X}_2\right)^2, \tag{2}$$

while the potential energy

$$V = k_1\left(\frac{1}{2}X_1^2 + \frac{1}{4}\Lambda_1 X_1^4\right) + k_2\left(\frac{1}{2}X_2^2 + \frac{1}{4}\Lambda_2 X_2^4\right). \tag{3}$$

The forces connected with the external excitation and the damping effects are introduced into model as generalized force

$$Q = F_0 \cos(\Omega\, t) + C(\dot{X}_1 + \dot{X}_2). \tag{4}$$

Since the springs are connected serially, the equilibrium equation for the weightless connecting point *S* is as follows

$$k_1 X_1(1 + \Lambda_1 X_1^2) = k_2 X_2(1 + \Lambda_2 X_2^2). \tag{5}$$

The equation of motion, derived using the Lagrang'e formalism, and the equation (5) are transferred to the convenient dimensionless form, so the governing equations supplemented with the initial conditions take the following form

$$\ddot{x}_1 + \ddot{x}_2 + c(\dot{x}_1 + \dot{x}_2) + (1 + \lambda)x_2(1 + \alpha_2 x_2^2) = f_0 \cos(p\,\tau), \tag{6}$$

$$x_1(1 + \alpha_1 x_1^2) = \lambda x_2(1 + \alpha_2 x_2^2), \tag{7}$$

$$x_1(0) + x_2(0) = x_0,\;\; \dot{x}_1(0) + \dot{x}_2(0) = v_0, \tag{8}$$

where

$$\lambda = \frac{k_2}{k_1},\ \alpha_1 = \Lambda_1 L^2,\ \alpha_2 = \Lambda_2 L^2,\ c = \frac{C}{m\,\omega},\ f_0 = \frac{F_0}{L\,m\,\omega^2},\ p = \frac{\Omega}{\omega},\ L = L_{01} + L_{02}.$$

The overdot denotes the differentiation with respect to the dimensionless time $\tau = t\,\omega$, where $\omega = \sqrt{k_e/m}$ and $k_e = \frac{k_1\,k_2}{k_1+k_2}$ is the effective stiffness of two linear springs connected in series which plays the role of a characteristic coefficient.

Differentiating twice the algebraic equation (7) the following relation between the second derivatives of the unknown functions is obtained

$$\ddot{x}_1(1 + 3\alpha_1 x_1^2) + 6\alpha_1 x_1 \dot{x}_1^2 = \ddot{x}_2 \lambda (1 + 3\alpha_2 x_2^2) + 6\alpha_2 \lambda x_2 \dot{x}_2^2, \tag{9}$$

which allows to eliminate, for example, the function $x_1(\tau)$ from the equation (6).

3. Analytical solution to the problem

The differential-algebraic problem (6) – (8) is solved in the asymptotic way using the Multiple Scale Method (MSM). Since we assume the smallness of some parameters, so we formally introduce the parameters with the tilde over the symbol:

$$\alpha_1 = \varepsilon\tilde{\alpha}_1,\ \alpha_2 = \varepsilon\tilde{\alpha}_2,\ c = \varepsilon\tilde{c},\ f_0 = \varepsilon\tilde{f}_0, \tag{10}$$

where $0 < \varepsilon \ll 1$ is a so called small parameter.

Each of the solutions are assumed in the form of the sum containing the new unknown functions dependent on two time scales, i.e. we have

$$x_1(\tau;\varepsilon) = \xi_{10}(\tau_0,\tau_1) + \varepsilon\,\xi_{11}(\tau_0,\tau_1), \tag{11}$$

$$x_2(\tau;\varepsilon) = \xi_{20}(\tau_0,\tau_1) + \varepsilon\,\xi_{21}(\tau_0,\tau_1), \tag{12}$$

where $\tau_0 = \tau$ is the fast time scale, and $\tau_1 = \varepsilon\tau$ is the slow time scale. The differential operators take the form

$$\frac{d}{d\tau} = \frac{\partial}{\partial\tau_0} + \varepsilon\frac{\partial}{\partial\tau_1},\quad \frac{d^2}{d\tau^2} = \frac{d}{d\tau}\left(\frac{d}{d\tau}\right) = \frac{\partial^2}{\partial\tau_0^2} + 2\varepsilon\frac{\partial^2}{\partial\tau_0\partial\tau_1} + o(\varepsilon^2). \tag{13}$$

Substituting expressions (10) – (13) into equations (6) – (7) yields the algebraic-differential system in which the small parameter ε appears in various powers. This leads to the first and the second order approximation equations:

- approximation of the order ε^0

$$(1+\lambda)\xi_{20} + \frac{\partial^2\xi_{10}}{\partial\tau_0^2} + \frac{\partial^2\xi_{20}}{\partial\tau_0^2} = 0, \tag{14}$$

$$\lambda\xi_{20} - \xi_{10} = 0, \tag{15}$$

- approximation of the order ε^1

$$\frac{\partial^2\xi_{11}}{\partial\tau_0^2} + \frac{\partial^2\xi_{21}}{\partial\tau_0^2} + (1+\lambda)\xi_{21} + (1+\lambda)\tilde{\alpha}\xi_{20}^3 + \tilde{c}\left(\frac{\partial\xi_{10}}{\partial\tau_0} + \frac{\partial\xi_{20}}{\partial\tau_0}\right) + 2\left(\frac{\partial^2\xi_{10}}{\partial\tau_0\partial\tau_1} + \frac{\partial^2\xi_{20}}{\partial\tau_0\partial\tau_1}\right) =$$
$$\tilde{f}_0\cos(p\,\tau_0), \tag{16}$$

$$\lambda\tilde{\alpha}_2\xi_{20}^3 + \lambda\xi_{21} - \tilde{\alpha}_2\xi_{10}^3 - \xi_{11} = 0. \tag{17}$$

The above set of the differential-algebraic equations is solved in the recursive way, i.e. the solutions to the lower order approximation equations are substituted into the higher order ones.

The solution to the equations (14) – (15) are

$$\xi_{10} = \lambda B(\tau_1)e^{i\tau_0} + \lambda\bar{B}(\tau_1)e^{-i\tau_0}, \tag{18}$$

$$\xi_{20} = B(\tau_1)e^{i\tau_0} + \bar{B}(\tau_1)e^{-i\tau_0}, \tag{19}$$

where $B(\tau_1)$ and its complex conjugate $\bar{B}(\tau_1)$ are the unknown complex functions.

4. Vibration far from resonance

After substituting the solutions (18) – (19) into the equations (16) – (17), the secular terms should be eliminated, which leads to the following solvability conditions

$$2\,i\frac{\partial B}{\partial\tau_1} + i\,B\tilde{c} + \frac{3B^2\bar{B}(\lambda^3\tilde{\alpha}_1+\tilde{\alpha}_2)}{1+\lambda} = 0, \tag{20}$$

$$2\,i\frac{\partial\bar{B}}{\partial\tau_1} + i\,\bar{B}\tilde{c} - \frac{3\bar{B}^2B(\lambda^3\tilde{\alpha}_1+\tilde{\alpha}_2)}{1+\lambda} = 0. \tag{21}$$

Substituting solutions (18) and (19) into (16) – (17) and taking into consideration the conditions (20) – (21), the following solution to the second order approximation equations is found

$$\xi_{11} = -\frac{e^{i\,p\tau_0}\lambda\tilde{f}_0}{2(p^2-1)(1+\lambda)} + \frac{e^{3i\tau_0}\lambda B^3(\lambda^2(\lambda-8)\tilde{\alpha}_1+9\,\tilde{\alpha}_2)}{8(1+\lambda)} - 3e^{i\tau_0}\lambda B^2\bar{B}(\lambda^2\tilde{\alpha}_1 - \tilde{\alpha}_2) + CC \tag{22}$$

$$\xi_{21} = -\frac{e^{i\,p\tau_0}\tilde{f}_0}{2(p^2-1)(1+\lambda)} + \frac{e^{3i\tau_0}B^3(9\lambda^3\tilde{\alpha}_1-8\,\tilde{\alpha}_2)}{8(1+\lambda)} + CC \tag{23}$$

where *CC* stands for the complex conjugates.

There is convenient to express the complex functions $B(\tau_1)$ and $\bar{B}(\tau_1)$ in the exponential form

$$B(\tau_1) = \frac{1}{2}a(\tau_1)e^{i\psi(\tau_1)},\ \bar{B} = \frac{1}{2}a(\tau_1)e^{-i\psi(\tau_1)}, \tag{24}$$

where $a(\tau_1)$ and $\psi(\tau_1)$ are unknown real valued functions and stand for the amplitude and the phase of the oscillations, respectively.

Introducing relationships (24) into the solvability conditions (20) – (21), returning to the original notations according to (10) and using the definition of operator $(13)_1$ allow us to write the modulation equations in the following form

$$\frac{da(\tau)}{d\tau} = -\frac{1}{2} c\, a(\tau), \tag{25}$$

$$\frac{d\psi(\tau)}{d\tau} = \frac{3(\alpha_2+\lambda^3\alpha_1)a(\tau)^2}{8(1+\lambda)}. \tag{26}$$

Assuming the initial conditions in the form

$$a(0) = a_0, \quad \psi(0) = \psi_0, \tag{27}$$

we obtain the solution to the problem (25) – (27) as follows

$$a(\tau) = a_0 e^{-\frac{c\tau}{2}}, \; \psi(\tau) = \psi_0 + \frac{3a_0^2(1-e^{-c\tau})(\alpha_2+\alpha_1\lambda^3)}{8c(1+\lambda)}. \tag{28}$$

The amplitude and phase determined by (28) are then introduced into the solutions (18) – (19), (22) – (23) of the first and second order approximation. Then the relationships (24) are taken into account. Afterwards, using expressions (11) – (12) and returning to the original denotations according to (10), we obtain the approximate solution to the original problem (6) – (8). The absolute dimensionless displacement of the body obtained in this way follows

$$x(\tau) = x_1(\tau) + x_2(\tau) = -\frac{f_0 \cos(p\tau)}{(p^2-1)} + (1+\lambda)a(\tau)\cos\big(\tau+\psi(\tau)\big) +$$

$$\frac{1}{32} a(\tau)^3 \cos\big(\tau+\psi(\tau)\big)\Big(24\alpha_2\lambda - \alpha_2 - 25\alpha_1\lambda^3 + 2(\alpha_2+\alpha_1\lambda^3)\cos\Big(2\big(\tau+\psi(\tau)\big)\Big)\Big), \tag{29}$$

where $a(\tau)$ and $\psi(\tau)$ are the solutions (28) to the modulation problem (25) – (27).

The comparison of the time course of the body displacement determined by the solution (29) with the analogic one obtained numerically is presented in Fig.2 and Fig.3 for the transient and the steady state vibration, respectively.

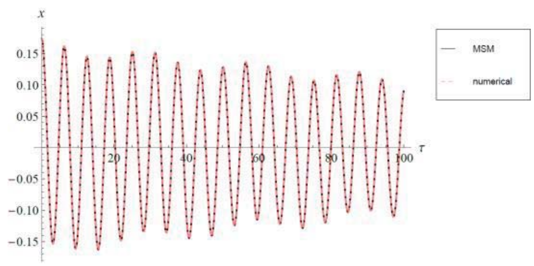

Figure 2. Body displacement in time for the transient non-resonant vibration.

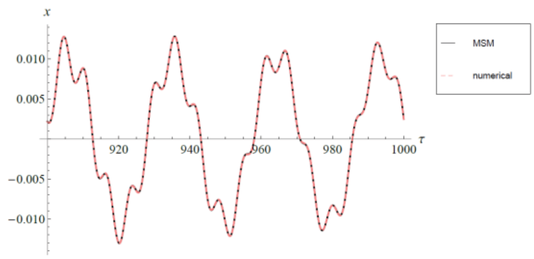

Figure 3. Body displacement in time for the steady-state non-resonant vibration.

The results presented are obtained for the following data: $\alpha_1 = 0.87, \alpha_2 = 1.21, \lambda = 0.63, f_0 = 0.01, p = 0.215, c = 0.009, a_0 = 0.1, \psi_0 = 0$. The compatibility of the two approaches, i.e. MSM and numerical solution, is very high which confirms the correctness of the derived analytical solutions. The relationship between initial conditions (8) and (27) has been determined using the analytical form of the solution (29).

5. Vibration at resonance

Let us analyze the case of the main resonance when $p \approx 1$. In order to investigate the behaviour of the system near resonance, the detuning parameter σ is introduced as follows

$$p = 1 + \sigma. \tag{30}$$

The assumption (30) is inserted into equation (6) and then the procedure similar to that of the previous section is carried out. In result, the modulation equations are obtained of the following form

$$\frac{da(\tau)}{d\tau} = -\frac{1}{2} c\, a(\tau) + \frac{f_0 \sin(\sigma\tau - \psi)}{2(1+\lambda)}, \tag{31}$$

$$\frac{d\psi(\tau)}{d\tau} = \frac{3(\alpha_2 + \lambda^3 \alpha_1) a(\tau)^2}{8(1+\lambda)} - \frac{f_0 \cos(\sigma\tau - \psi)}{2(1+\lambda) a(\tau)}. \tag{32}$$

Observe that equations (31) – (32), supplemented by initial conditions (27), cannot be solved analytically. The numerical treatment is required in this case. The absolute dimensionless body displacement obtained in the way similar to the one described in the previous section is as follows

$$x(\tau) = x_1(\tau) + x_2(\tau) = (1+\lambda) a(\tau) \cos\big(\tau + \psi(\tau)\big) +$$

$$\frac{1}{32} a(\tau)^3 \cos\big(\tau + \psi(\tau)\big)\Big(24\alpha_2\lambda - \alpha_2 - 25\alpha_1\lambda^3 + 2(\alpha_2 + \alpha_1\lambda^3)\cos\Big(2\big(\tau + \psi(\tau)\big)\Big)\Big), \tag{33}$$

where $a(\tau)$ and $\psi(\tau)$ denote the solutions to the modulation equations (31) – (32).

Time history of the body displacement in the case of the main resonance is presented in Figs. 4, 5 for the transient and the steady state. The data assumed for the calculations are as follows: $\alpha_1 = 0.87, \alpha_2 = 1.21,\ \lambda = 0.63, f_0 = 0.009, \sigma = 0.008, c = 0.008, \mathrm{a0} = 0.2, \psi 0 = 0$.

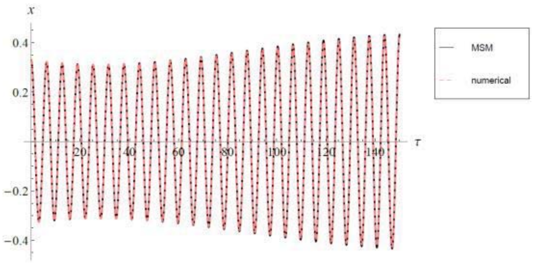

Figure 4. Time courses of the body (transient vibration) obtained analytically and numerically (resonance case).

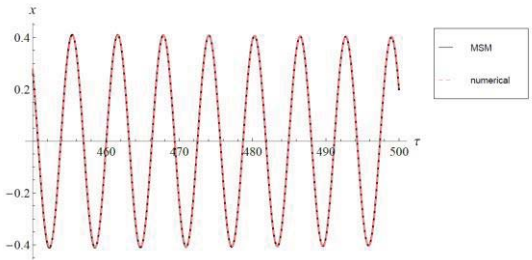

Figure 5. Time course of the body (steady state vibration) obtained analytically and numerically (resonance case).

The obtained results clearly exhibit powerful of the employed approximate analytical method.

5.1. Steady-state resonant responses

When the transient processes disappear, the forced system can reach the steady state oscillations. In order to study this case it is convenient to introduce the modified phase $\theta = \sigma\tau - \psi(\tau)$ into equations (31) – (32) which allows to transform them into the following counterpart autonomous form

$$\frac{da(\tau)}{d\tau} = -\frac{1}{2} c\, a(\tau) + \frac{f_0 \sin(\theta)}{2(1+\lambda)}, \tag{34}$$

$$\frac{d\theta(\tau)}{d\tau} = \sigma - \frac{3(\alpha_2 + \lambda^3 \alpha_1) a(\tau)^2}{8(1+\lambda)} + \frac{f_0 \cos(\theta)}{2(1+\lambda) a(\tau)}. \tag{35}$$

Fixation of the values of the amplitude and the modified phase is characteristic for the steady state solutions. Consequently, zeroing the derivatives of both the amplitude and the modified phase in modulation equations (34) - (35) yields the conditions of the steady state in the form of the set of two following equations

$$-\frac{1}{2}c\,a(\tau) + \frac{f_0 \sin(\theta)}{2(1+\lambda)} = 0, \tag{36}$$

$$\sigma - \frac{3(\alpha_2+\lambda^3\alpha_1)a(\tau)^2}{8(1+\lambda)} + \frac{f_0 \cos(\theta)}{2(1+\lambda)a(\tau)} = 0. \tag{37}$$

The resonance curves with regard to the amplitude and the modified phase, obtained through equations (36) – (37) are presented in Figs. 6, 7 for the following fixed parameters: $\alpha_1 = 0.87, \alpha_2 = 1.21, \lambda = 0.63, f_0 = 0.009, c = 0.008$.

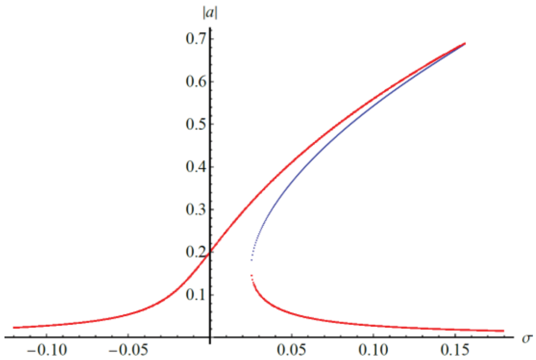

Figure 6. Resonance curve for the amplitude of $x(\tau)$.

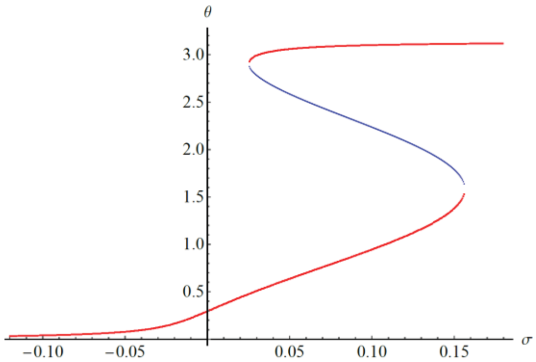

Figure 7. Resonance curve for the modified phase of $x(\tau)$.

In Figures 6 - 7 the stable branches are depicted in red color, whereas unstable ones in blue color.

5.2. Stability of the resonance curves

In order to examine the stability of the steady-state solution in the sense of Lyapunov, we analyze the non-stationary solutions of equations (34) – (35) that are close to the steady state solutions (a_s, θ_s).

Introducing the functions $\tilde{a}(\tau), \tilde{\theta}(\tau)$ that can be treated as small perturbations, one can assume the following non-stationary solution

$$a(\tau) = a_s + \tilde{a}(\tau), \quad \theta(\tau) = \theta_s + \tilde{\theta}(\tau). \tag{38}$$

Next, substituting expressions (38) into equations (34) – (35), linearizing the obtained equations and noting that (a_s, θ_s) are the steady-state solutions, we get

$$\frac{d\tilde{a}(\tau)}{d\tau} = -\frac{1}{2} c\, \tilde{a}(\tau) + \frac{f_0 \cos(\theta_s)}{2(1+\lambda)} \tilde{\theta}(\tau), \tag{39}$$

$$\frac{d\tilde{\theta}(\tau)}{d\tau} = -\frac{3a_s(\alpha_2+\lambda^3\alpha_1)\tilde{a}(\tau)}{4(1+\lambda)} - \frac{f_0 \cos(\theta_s)}{2(1+\lambda)a_s^2} \tilde{a}(\tau) - \frac{f_0 \sin(\theta_s)}{2(1+\lambda)a_s} \tilde{\theta}(\tau). \tag{40}$$

The characteristic matrix of the homogeneous differential equations (39) – (40) has the form

$$\mathbf{A} = \begin{bmatrix} -\frac{c}{2} & \frac{f_0 \cos(\theta_s)}{2(1+\lambda)} \\ -\frac{3a_s(\alpha_2+\lambda^3\alpha_1)}{4(1+\lambda)} - \frac{f_0 \cos(\theta_s)}{2(1+\lambda)a_s^2} & -\frac{f_0 \sin(\theta_s)}{2(1+\lambda)a_s} \end{bmatrix}. \tag{41}$$

If the real parts of all eigenvalues of the matrix **A** are negative, then the fixed point (a_s, θ_s) relating to the steady state solution is asymptotically stable in the sense of Lyapunov.

The analytical form of equations (36) – (37) which determine the resonance response functions allows for predict behavior of the system in various conditions. In Figs. 8, 9 there is presented the influence of the external excitation amplitude on the shape of the response curves ($\alpha_1 = 0.87, \alpha_2 = 1.21, \lambda = 0.63, c = 0.008$).

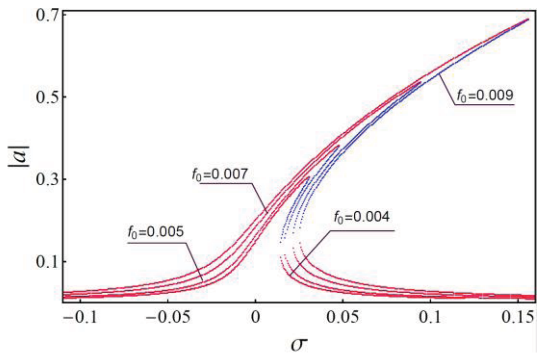

Figure 8. Influence of the external force amplitude on the system response amplitude.

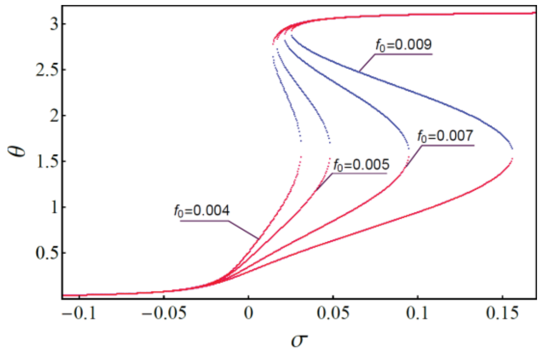

Figure 9. Influence of the external force amplitude on the modified system phase.

6. Conclusions

The dynamics of the lumped system containing two serially connected nonlinear springs has been investigated. The mathematical model consists of the differential and algebraic equations, which requires the appropriate modification of the asymptotic approach in order to deal with the considered mechanical system. The forced vibration in two cases have been analyzed: far from resonance and in the resonance conditions. The approximate analytical solution to the governing equations has been achieved. Its analytical form allows for quantitative and qualitative analysis of the behavior of the system for wide range of the characteristic parameters. The correctness of the results has been confirmed by the numerical calculations.

Acknowledgments

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