Complex vibrations of flexible beam NEMS elements, taking into account Casimir's forces under additive white noise

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Abstract: A mathematical model of the vibrations of the sensing element NEMS in the form of a flexible size-dependent rigidly clamped beam connected to the electrode at a distance h_0 is developed. A transverse uniformly distributed alternating load and additive white noise act on the beam. Geometric nonlinearity is taken into account according to the theory of Kármán. The equations of motion of an element of a mechanical system, as well as the corresponding boundary and initial conditions, are derived based on both the Hamilton principle a modified couple stress theory taking into account the Euler-Bernoulli hypothesis. It was revealed that the size-dependent parameter significantly affects the dynamics of the beam under the action of a transverse alternating load and additive white noise. The dynamic stability loss is investigated.

1. Formulation of the problem

In recent years, interest in physical phenomena, known under the general name "Casimir effect" has steadily increased. Quantum theory has shown that vacuum is an extremely dynamic, continuously changing substance, from virtually born and right there dying elementary particles [1]. The combination of these effects and the fact that a mechanical device often integrates directly with electronics provides both problems and opportunities for studying the dynamics of NEMS. We note a number of works in this direction.

The static and dynamic behavior of carbon nanotube-based switches using the van der Waals interaction is described in [2, 3]. The influence of the Van der Waals forces and the Casimir forces on the stability of electrostatic torsion of NEMS accelerometers was studied in [4].

A study of the influence of self-affine roughness in terms of the retraction parameters for NEMS switches taking into account the Casimir force was carried out in [5]. A theoretical analysis of the influence of the Casimir forces on the nonlinear behavior of nanoscale electrostatic accelerometers is given in [6]. The study of the forces of Casimir and van der Waltz in cantilever beams is the subject of studies of references [7-10]. An analysis of the influence of the Casimir force on the instability of retraction in micro-membranes was described in [11] and various forms of plates were studied in [12, 13]. The electrostatic instability of nanobeams with allowance for the forces of Casimir and Van der Wals was investigated in [14]. In paper [15], a numerical algorithm is proposed that can predict the static and dynamic behavior of circular NEMS devices under the influence of electrostatic and Casimir forces. Analytical modeling of the retracting instability of a CNT probe with van der Waals force was analyzed in [16], whereas Casimir effects are discussed in [17-19]. Zhang et al. [19], considered the theoretical details of Casimir effects, as well as experimental observations and applications were reported.

The study of the dynamic instability of a cantilever actuator made of a conductive cylindrical nanowire with a circular cross section, at the presence of Casimir power was carried out in [20]. The nano-beam is modeled on the basis of the nonlocal gradient theory of deformation and the Euler – Bernoulli hypothesis taking into account the Casimir forces in [21]. Jia and Yang [22] investigated the retracting instability of microswitches under combined electrostatic and intermolecular forces and axial residual stress, taking into account the force nonlinearity and geometric nonlinearity that arises from the extension of the middle plane. Theoretical formulations are based on the theory of the Bernoulli-Euler beam and geometric non-linearity of the Theodore von Kármán type. These solutions were confirmed by direct comparisons with experimental and other existing results. A parametric study was carried out taking into account the combined effects of geometric non-linearity, the ratio of the gap to the thickness of the structure, the Casimir force, the axial residual stress and the composition of the material with retracting instability.

Nayfeh [23] presented a nonlinear model of electric drive microbeams taking into account the electrostatic effect of the air gap condenser, the restoring force of the microbeam, and the axial load applied to the microbeam.

The boundary-value problem that describes the static deflection of a micro-object under the influence of electrostatic force due to constant polarizing voltage was solved. The eigenvalue problem, which described the vibration of a microsphere around its statically deflected position, was solved numerically for eigenfrequencies and modes. A comparison of the results obtained by this model with the experimental results showed excellent agreement, thus checking the model. The results indicated that the inability to take into account the extension of the midplane in the recovery effort of the micropulses leads to an underestimation of the stability limits. It was also demonstrated that the ratio of the width of the air gap to the thickness of the beam can be configured to expand the region of the linear relationship between the polarization voltage of the direct current and the fundamental natural frequency. This fact and the ability of the nonlinear model to accurately predict the natural frequencies for any constant polarization voltage allow developers to use a wider range of polarized DC voltages in the resonators. A review of the literature showed that the issue of the dynamics of beams under Casimir's action, vibration load and additive white noise was not considered.

In the classical theory of elasticity, the work of deformation and the strain energy depend on the stress tensor and do not depend on the rotation vector due to material independence. However, the rotation vector gradient can be a significant factor in the equations of state. Based on the modified couple stress theory presented by Yang et al. [24], the strain energy density is a function of both the stress tensor (conjugate to the strain tensor) and the curvature tensor (conjugate to the moment stress tensor). In one or another deformed isotropic linear elastic material located in the region Ω, the strain energy Π is expressed by the formula:

$$\Pi = \int_{\Omega} (\sigma_{ij}\,\varepsilon_{ij} + m_{ij}\eta_{ij})d\Omega \quad (i,j = 1,2,3) \tag{1}$$

where: σ_{ij} is a Cauchy stress tensor, ε_{ij} is a stress tensor, m_{ij} represents the deviator component of the moment stress tensor, and η_{ij} symmetric curvature tensor. These tensors are determined by the formulas:

$$\sigma_{ij} = \lambda tr(\varepsilon_{ij})I + 2\mu\varepsilon_{ij}, \tag{2}$$

$$\varepsilon_{ij} = \frac{1}{2}[\nabla u + (\nabla u)^T], \tag{3}$$

$$m_{ij} = 2l^2\mu\chi_{ij}, \tag{4}$$

$$\eta_{ij} = \frac{1}{2}\left[\nabla\varphi + \left(\nabla_{\varphi}\right)^T\right], \tag{5}$$

where: u — vector moving, λ = Eν/(1+ν) (1-2ν) и μ = E/2 (1+ν) — Lamé constants, E, ν are respectively Young's modulus and Poisson's ratio for the beam material, φ — this is a rotation vector, presented as $\varphi_{i=}\frac{1}{2}rot(u_i)$. l — this is a parameter of the length scale of the material, understood as a property of the material, characterizing the effect of moment stress [24]. The material length scale parameter related to the microstructures of the material is designed to interpret the size effect in a non-classical model of Bernoulli-Euler beams.

From the analysis of equations (3) and (5) it follows that the stress tensor ε_{ij} and symmetric curvature tensor η_{ij} are symmetric, and therefore it follows from equations (2) and (4) that the stress tensor σ_{ij} and the deviator component of the moment stress tensor m_{ij} also symmetrical. Considered structure represents a beam located at a distance of h_0, a two-dimensional region of space R^2 with a Cartesian coordinate system, introduced as follows: in the body of the nanobalk, a cast line, called the midline, is fixed: $z = 0$, axis OX is directed from left to right along the midline, axis OZ – down, perpendicular to OX. In the indicated coordinate system, a structure of two beams, as a two-dimensional region Ω determined by in the following way:

$$\Omega = \left\{(x,z)|\ x \in [0,a], -\frac{h}{2} \le z \le \frac{h}{2}\right\} \text{ (Fig. 1)};\ 0 \le t \le \infty .$$

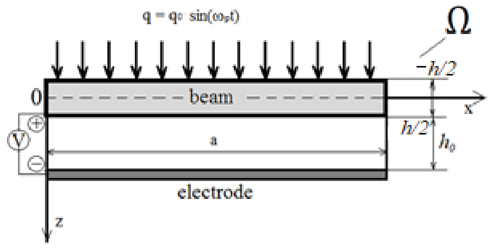


Fig. 1 Computational scheme.

At getting equations size-dependent beams connected to the electrode at a distance h_0, the following hypotheses are used:

- single-layer beam, isotropic, Hooke's law holds;
- the longitudinal size of the beams significantly exceeds their transverse dimensions;
- the beam axis is a straight line, the Euler-Bernoulli kinematic model is used, the normal stresses at sites parallel to the axis are negligible;
- the load acts in the direction of the OZ axis and external forces do not change their direction during beam deformation;
- geometric nonlinearity is taken into account in the form of von Kármán.

According to the Hamilton principle, we have

$$\int_{t_0}^{t_1}(\delta K - \delta \Pi + \delta' W)dt = 0. \tag{7}$$

Here K, Π are the kinetic and potential energy, respectively, $\delta' W$ is the work of external forces. Using the methods of calculus of variations, a system of differential equations of the theory of flexible beams is obtained taking into account the modified couple stress theory of elasticity [24]:

$$\begin{gathered} -\left(\frac{1}{12}+\bar{D}_1\right)\frac{\partial^4 \bar{w}}{\partial \bar{x}^4} + L_1(\bar{u},\bar{w}) + \left(\frac{h_0}{h}\right)^2 L_2(\bar{w},\bar{w}) - \frac{\partial^2 \bar{w}}{\partial \bar{t}^2} + \bar{q}(t) + \frac{\bar{q}_k}{(1-\bar{w})^4} = 0, \\ \left(\frac{h}{a}\right)^2 \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \left(\frac{h_0}{h}\right)^2 \left(\frac{a}{h}\right)^2 L_3(\bar{w},\bar{w}) - \frac{\partial^2 \bar{u}}{\partial \bar{t}^2} = 0, \end{gathered} \tag{8}$$

where: $L_1(\bar{u},\bar{w}) = \frac{\partial^2 \bar{u}}{\partial \bar{x}^2}\frac{\partial \bar{w}}{\partial \bar{x}} + \frac{\partial^2 \bar{w}}{\partial \bar{x}^2}\frac{\partial \bar{u}}{\partial \bar{x}}$; $L_2(\bar{w},\bar{w}) = \frac{3}{2}\frac{\partial^2 \bar{w}}{\partial \bar{x}^2}\left(\frac{\partial \bar{w}}{\partial \bar{x}}\right)^2$; $L_3(\bar{w},\bar{w}) = \frac{\partial^2 \bar{w}}{\partial \bar{x}^2}\frac{\partial \bar{w}}{\partial \bar{x}}$. The boundary and initial conditions should be added to the system of nonlinear partial differential equations (8). The dimensionless quantities are introduced as follows:

$$\bar{x} = \frac{x}{a}; \quad \bar{w} = \frac{w}{h_0}; \quad \bar{u} = \frac{ua}{h_0^2}; \quad \bar{t} = t\sqrt{\frac{D_0}{\rho h a^4}}; \quad \bar{q} = \frac{a^4}{E h_0 h^3} q; \quad D_0 = \frac{E \nu h^3}{(1+\nu)(1-2\nu)}; \quad \bar{q}_k = \frac{\hbar c \pi^2 a^4}{240 D_0 h_0^5};$$

$$\bar{D}_1 = \gamma \frac{(1-2\nu)}{2\nu}; \quad \gamma = \frac{l^2}{h^2}.$$

We have used the following notation: t - time; w – deflection, u – function axis movements x; h – beam thickness; h_0 – the distance between the electrode and nanobalk; q_0 – amplitude of load, ν – Poisson's ratio, E – elastic modulus, l – size dependent parameter, ρ – density plate material, a – radius, $\hbar$ - Planck constant.

The system of nonlinear partial differential equations reduces to the Cauchy problem by the finite difference method with approximation of the second order of accuracy. The Cauchy problem is solved by methods of the Runge-Kutta type (4th, 6th, 8th order of accuracy) and the Newmark method [25], [26].

2. Numerical results

Consider the vibrations of a rigidly clamped at both ends of the nanobeam under the action of the Casimir force, an alternating load $q = q_0 \sin(\omega_p t)$ and white additive noise [27], [28]:

$$\bar{w}(0,\bar{t}) = \bar{w}(1,\bar{t}) = u(0,\bar{t}) = u(1,\bar{t}) = \frac{\partial w(0,\bar{t})}{\partial \bar{x}} = \frac{\partial \bar{w}(1,\bar{t})}{\partial \bar{x}} = 0\ , \tag{9}$$

with zero initial conditions: $$\bar{w}(\bar{x})_{|t=0} = 0,\ \bar{u}(\bar{x})_{|t=0} = 0,\ \frac{\partial \bar{w}(\bar{x})}{\partial \bar{t}}_{|t=0} = 0,\ \frac{\partial \bar{u}(\bar{x})}{\partial \bar{t}}_{|t=0} = 0\ , \tag{10}$$

Geometric and physical parameters of nanobeams: length $a = 4\cdot10^{-7}$m, thickness $h = 4\cdot10^{-9}$m, density $\rho = 19320$ kg/m^3 and Young's modulus $E = 1{,}224\cdot10^{7}$ kgF/m^2 (gold), Poisson's modulus $\nu = 0.44$, size-dependent parameter $l = 0.5$, distance between electrode and beam $h_0 = 6\cdot10^{-9}$ m.

Microbeam geometric parameters: length $a = 4\cdot10^{-4}$ m, thickness $h = 4\cdot10^{-6}$ m, size-dependent parameter $l = 0$, distance between electrode and beam $h_0 = 6\cdot10^{-6}$ m.

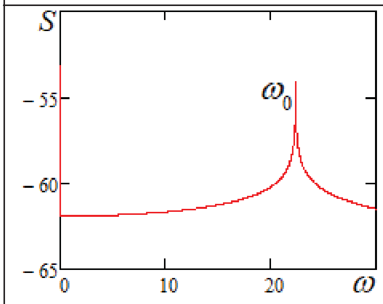
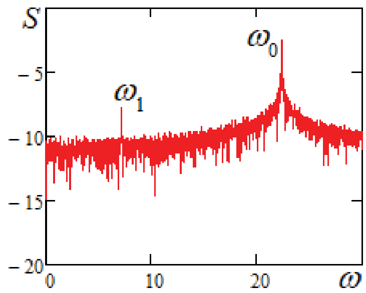
Beam is in a vacuum ($\varepsilon = 0$).

Case study 1.

Vibrations of a nanobeam under impact with an account of only the forces of Casimir. In this case, the periodic vibrations exhibit natural frequency ω_0. Table 1 shows the Fourier power spectra for the size-dependent parameter $\gamma = 0$ without white noise ($w_n = 0$) and taking into account white noise ($w_n = 5$). The presence of white noise leads the system to chaotic vibrations at the natural frequency ω_0 and independent frequency $\omega_1 = 7.1942$.

Consider the vibrations of microbeams ($\gamma = 0$) under the action of the Casimir force and transverse uniformly distributed load $q = q_0 \sin(\omega_p t)$ without white noise ($w_n = 0$). Table 2 presents the Fourier power spectra. Under action of the Casimir force and lateral load, the microbeam vibrates at a frequency ω_p and independent frequency ω_1 and their linear combinations $\omega_2 = \omega_p - \omega_1$. Taking into account the size-dependent parameter leads to a purification of the power spectrum in the region of the frequency of natural vibration. When the additive white noise load is taken into account in the power spectrum of the microbeam, the noise features are observed at low frequencies; for a nano-beam with the same loading parameters, an increase of the noise component in the power spectrum is observed.

Table 1. Fourier Power Spectra

$w_n = 0, \gamma = 0$	$w_n = 5, \gamma = 0$
	

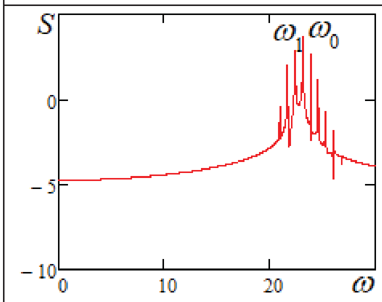
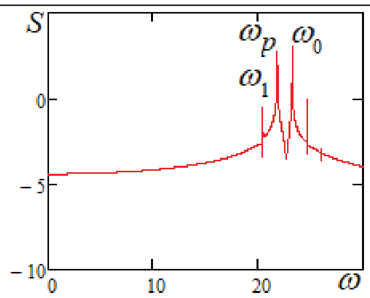
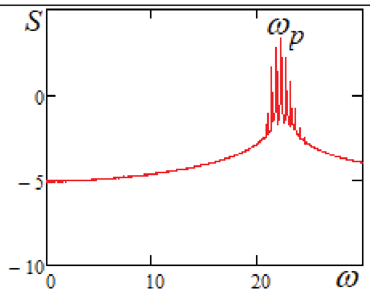
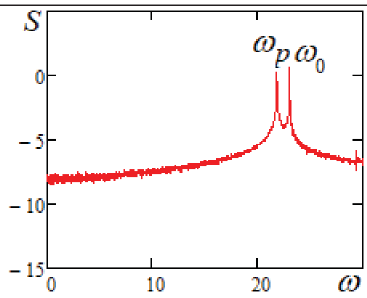
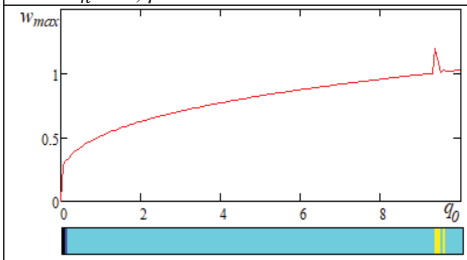
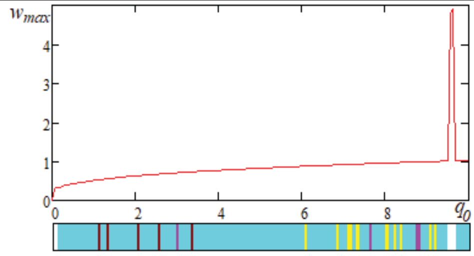
$w_n = 0, \gamma = 0, q_0 = 5, \omega_p = \omega_0 = 22.366$	$w_n = 0, \gamma = 0.5, q_0 = 5, \omega_p = \omega_0 = 22.366$
	
$w_n = 5, \gamma = 0, q_0 = 5, \omega_p = \omega_0 = 22.366$	$w_n = 0.5, \gamma = 0.5, q_0 = 5, \omega_p = \omega_0 = 22.366$
	

Table 2. Maximum deflection dependencies of the amplitude versus the driving load $w_{max}(q_0)$ and scales of vibrations

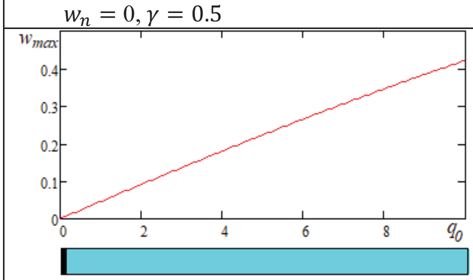
$w_n = 0, \gamma = 0$



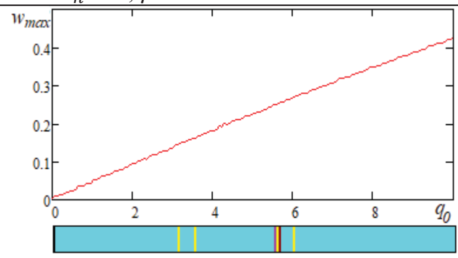
$w_n = 5, \gamma = 0$



$w_n = 0, \gamma = 0.5$



$w_n = 5, \gamma = 0.5$



	periodic vibrations with ω_p		subharmonic vibrations $\frac{\omega_p}{2n}$, where $n \in N$
	subharmonic vibrations with $\frac{\omega_p}{2}$		chaotic vibrations
	Quasi-periodic vibrations with 2 frequencies		subharmonic vibrations $\frac{\omega_p}{3}$

A general analysis of the study of vibrations of micro- and nano-beams taking into account the additive white noise of a transversely alternating load yields the following results: Table 3 presents the dependences of the maximum deflection in the center of the beam on the amplitude of the driving load $w_{max}(q_0)$ and character scales vibrations. In the range $q_0 \in (0; 9.4)$ beam exhibits vibrations at two independent frequencies, i.e. forcing load frequency ω_p and frequency ω_1. The first Lyapunov exponent is zero, and the rest are negative. With increasing load $q_0 \in [9.4; 9.6)$ there is a dynamic loss of stability, in which the signal exhibits subharmonic $\omega_p/23$ with a sharp increase in deflection, and a sharp change character of vibration (all Lyapunov exponents are close to zero (Le1 = – $58\cdot10^{-5}$, Le2 = $-61\cdot10^{-4}$, Le3 = - $57\cdot10^{-4}$, Le4 = - $11\cdot10^{-3}$). Spectrum of the Lyapunov exponents was calculated Jacobian method [29]. When taking into account the noise load, the nature of vibrations under changes: using the range of chaotic vibrations $q_0 \in (0; 0.1)$ - hyperchaos (Le1 = $18\cdot10^{-4}$, Le2 = $1.7494\cdot10^{-5}$, Le3 = $-39\cdot10^{-2}$, Le4 = $-46\cdot10^{-2}$). At the load $q_0 \in [0.1; 9.7)$, the microbeam vibrates at two independent frequencies. At frequencies $k\omega_p/2^n$ the periodicity windows appear, where $k, n \in$

N. At load $q_0 \in [9.7; 9.8)$ the dynamic loss of stability occurs, sharp increase deflection and transition from quasiperiodicity to hyperchaos takes place (Le1 = $48\cdot10^{-4}$, Le2 = $11\cdot10^{-3}$, Le3 = $-31\cdot10^{-2}$, Le4 = $-78\cdot10^{-2}$). Increase of the size dependent parameter implies changes within the whole interval $q_0 \in (0; 9.4)$.

3. Conclusions

A mathematical model of the nonlinear dynamics of the MEMS / NEMS beam element under the action of the Casimir force, under uniformly distributed alternating load and additive white noise is developed. The governing PDEs are yielded by the Hamilton principle for the Euler-Bernoulli kinematic model and the modified couple stress theory. For the first time, the phenomenon of stability loss with a transverse alternating load MEMS / NEMS is detected. As a dynamic criterion for the loss of stability, the Lyapunov approach is used by analyzing the spectrum of Lyapunov exponents. For microbeams, it was found that with a loss of stability, hyperchaotic oscillations are observed (the two highest Lyapunov exponents are positive). When the size-dependent parameter is taken into account in the equations, the amplitude of the oscillations and the nature of the vibrations at high loads decrease. It was revealed that the additive noise field inversely depends on the ratio of the amplitude of the driving load to the intensity of the noise field q_0/w_n.

Acknowledgments

The work was supported by the RSF № 19-19-00215.

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