Plane Motion of a Rigid Body Suspended on Nonlinear Spring-Damper

Roman Starosta, Grażyna Sypniewska-Kamińska and Jan Awrejcewicz

Abstract The paper deals with the analytical investigation of the behaviour of the harmonically excited physical pendulum suspended on the nonlinear spring. The asymptotic method of multiple scales (MS) has been used to derive approximate solutions in the analytical form. The applied approach allows one to perform a qualitative analysis of the behaviour of the system. MS method gives possibility, among others, to recognize resonance conditions which can appear in the system.

1 Introduction

Although pendulums are relatively simple systems, they can be used to simulate the dynamics of a variety of engineering devices and machine parts. The behaviour of pendulum-type mechanical systems with nonlinear and parametric interactions is complicated, and hence its understanding and prediction are important from a point of view of both the theory and application. The coupling of the equations of motion results in a possibility of autoparametric excitation and is connected to the energy exchange between vibration modes [6]. Various kinds of pendulums are widely discussed in numerous references and analytical investigations are recently of great interest of many researchers. Main and parametric resonances of the kinematically driven spring pendulum are studied in the paper [2]. The nonlinear response of a system including a double pendulum and having three degrees of freedom (DOFs) is analytically investigated in the paper [5]. Stationary and non-stationary resonant

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dynamics of the harmonically forced pendulum is the subject of the paper [4]. The physical pendulum suspended on the spring-damper which has linear features has been modelled and discussed in the article [1]. The present paper extends these investigations and presents further development of the model and results of asymptotic analysis.

2 Problem Formulation and Equations of Motion

Plane motion of a rigid body mounted on a spring-damper suspension is analyzed in the paper. The scheme of the system is presented in Fig. 1. The spring is assumed to be massless and nonlinear. The non-linearity is of the cubic type, and k_1 and k_2 are constant elastic coefficients. L_0 denotes the spring length in the non-stretched state. There is a purely viscous damper having a damping coefficient C_1, and the damper and the spring are arranged in parallel. The rigid body of mass m is connected to this system via a pin joint A. The distance between the point A and the mass centre C of the body is denoted by S and called further the eccentricity. The body moment of inertia relative to the axis passing through the centre of mass C and perpendicular to the plane of motion is equal to I_C. In the direction compatible with the main axis of the suspension system acts the known force **F** the magnitude of which changes harmonically $F(t) = F_0 \cos(\Omega_1 t)$. Besides, the system is loaded by two known harmonically changing torques $M_1(t) = M_{01} \cos(\Omega_2 t)$ and $M_2(t) = M_{02} \cos(\Omega_3 t)$ shown in Fig. 1. There are also assumed two torques of the viscous nature attenuate the swing vibration related to the angles Φ and Ψ, where C_2 and C_3 are their viscous coefficients. The body is free in its plane motion, so it has three degrees of freedom. The time functions $X(t)$, $\Phi(t)$ and $\Psi(t)$ are assumed as the generalized coordinates. The coordinate $X(t)$ is understood as the total elongation of the spring involving also the static elongation X_r that satisfies the equilibrium condition

$$k_2 X_r^3 + k_1 X_r = mg. \tag{1}$$

The kinetic and potential energy of the system are

$$\mathrm{T} = m\,S\,\dot{X}\,\dot{\Psi}\,\sin(\Phi - \Psi) + S\,m\,\dot{\Phi}\,\dot{\Psi}(L_0 + X)\cos(\Phi - \Psi) + \frac{1}{2}R_A^2 m\dot{\Psi}^2 + \frac{m}{2}(L_0 + X)^2\dot{\Phi}^2 + \frac{m}{2}\dot{X}^2, \tag{2}$$

$$\mathrm{V} = \frac{1}{2}k_1 X^2 + \frac{1}{4}k_2 X^4 - mg((L_0 + X)\cos(\Phi) + S\,\cos(\Psi)). \tag{3}$$

In Eq. (2) occurs the quantity denoted by R_A which is a radius of gyration of the body with respect to the axis passing through the joint A and perpendicular to the plane of motion. The radius R_A is related to the inertia moment I_C by commonly known parallel axis theorem

$$mR_A^2 = I_C + mS^2. \tag{4}$$

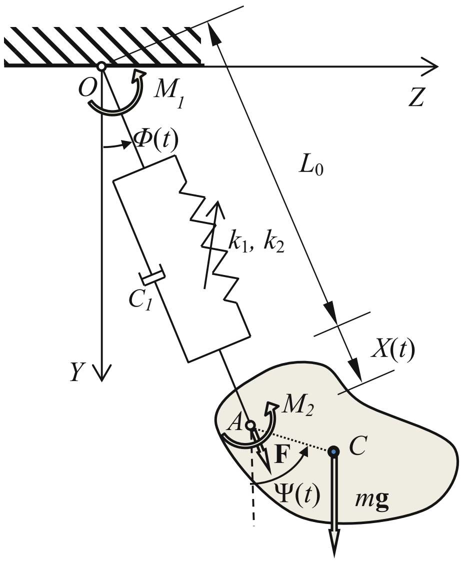

Fig. 1 Mass-spring-damper system

In order to obtain the equations of motion we write the equations Lagrange equations of the second kind

$$\frac{d}{dt}\left(\frac{\partial \mathrm{L}}{\partial \dot{X}}\right)-\left(\frac{\partial \mathrm{L}}{\partial X}\right)=Q_X,\ \frac{d}{dt}\left(\frac{\partial \mathrm{L}}{\partial \dot{\Phi}}\right)-\left(\frac{\partial \mathrm{L}}{\partial \Phi}\right)=Q_\Phi,\ \frac{d}{dt}\left(\frac{\partial \mathrm{L}}{\partial \dot{\Psi}}\right)-\left(\frac{\partial \mathrm{L}}{\partial \Psi}\right)=Q_\Psi, \tag{5}$$

where $\mathrm{L} = T - V$ is the Lagrange function, and the general forces are given by

$$\begin{aligned} Q_X &= F_0\cos(\Omega_1 t) - C_1\dot{X},\\ Q_\Phi &= M_{02}\cos(\Omega_2 t) - C_2\dot{\Phi},\\ Q_\Psi &= M_{03}\cos(\Omega_3 t) - C_3\dot{\Psi}. \end{aligned} \tag{6}$$

The dimensionless form of the equations of motion derived from (5) is as follow

$$\ddot{\xi} + c_1\dot{\xi} + \xi + \alpha\,\xi^3 + 3\xi_r\alpha\,\xi^2 + 3\xi_r^2\alpha\,\xi - w_2^2(\cos\varphi - 1) - (1+\xi)\dot{\varphi}^2 \\ - s\,\cos(\varphi-\gamma)\dot{\gamma}^2 + s\,\sin(\varphi-\gamma)\ddot{\gamma} = f_1\cos(p_1\tau), \tag{7}$$

$$\ddot{\varphi}\left(1 + 2\xi + \xi^2\right) + w_2^2\sin\varphi(1+\xi) + c_2\dot{\varphi} + 2\dot{\xi}\dot{\varphi} + 2\xi\,\dot{\xi}\dot{\varphi} \\ + s\,\sin(\varphi-\gamma)(1+\xi)\dot{\gamma}^2 + s\,\cos(\varphi-\gamma)(1+\xi)\ddot{\gamma} = f_2\cos(p_2\tau), \tag{8}$$

$$\ddot{\gamma} + w_3^2\sin\gamma + c_3\dot{\gamma} + 2\frac{w_3^2}{w_2^2}\cos(\varphi-\gamma)\dot{x}\dot{\varphi} - \frac{w_3^2}{w_2^2}(1+\xi)\sin(\varphi-\gamma)\dot{\varphi}^2 \\ + \frac{w_3^2}{w_2^2}\sin(\varphi-\gamma)\ddot{\xi} + \frac{w_3^2}{w_2^2}(1+\xi)\cos(\varphi-\gamma)\ddot{\varphi} = f_3\cos(p_3\tau). \tag{9}$$

The frequency $\omega_1 = \sqrt{\frac{k_1}{m}}$ and the spring length $L = L_0 + X_r$ in the static equilibrium position are assumed as the reference quantities. The functions $\varphi(\tau)$ and $\gamma(\tau)$ of the dimensionless time $\tau = \omega_1 t$ are generalized coordinates related to $\Phi(t)$ and $\Psi(t)$, respectively, whereas $\xi(t)$ is associated with $X(t)$ by the relation $\xi(t) = X(t)/L$. The others dimensionless quantities are defined as follows:

$$s = S/L, \xi = X/L, \xi_r = X_r/L,$$

$c_1 = \frac{C_1}{m\omega_1}$, $c_2 = \frac{C_2}{mL^2\omega_1}$, $c_3 = \frac{C_3}{\omega_1 m r_A^2 L^2}$, $f_1 = \frac{F_0}{mL\omega_1^2}$, $f_2 = \frac{M_{01}}{mL^2\omega_1^2}$, $f_3 = \frac{M_{02}}{\omega_1^2 m R_A^2 L^2}$, $\alpha = \frac{k_2 L^2}{\omega_1^2 m}$, $w_2 = \frac{\omega_2}{\omega_1}$, $w_3 = \frac{\omega_3}{\omega_1}$, $p_1 = \frac{\Omega_1}{\omega_1}$, $p_2 = \frac{\Omega_2}{\omega_1}$, $p_3 = \frac{\Omega_3}{\omega_1}$, where $\omega_2^2 = \frac{g}{L}$ and $\omega_3^2 = \frac{S g}{R_A}$. Dimensionless counterpart of condition (1) is

$$\alpha \xi_r^3 + \xi_r = w_2^2. \tag{10}$$

Equations (7)–(9) are supplemented by the initial conditions related the generalized coordinates and their first derivatives

$$\xi(0) = u_{01}, \dot{\xi}(0) = u_{02}, \varphi(0) = u_{03}, \dot{\varphi}(0) = u_{04}, \gamma(0) = u_{05}, \dot{\gamma}(0) = u_{06}, \tag{11}$$

where dimensionless quantities $u_{01}, \ldots, u_{06}$ are known.

3 Multiple Scales Method

The oscillations of the system are investigated in the neighborhood of the equilibrium position, hence the trigonometric functions of the generalized coordinates can be substituted by their power series approximations

$$\sin\varphi \approx \varphi - \varphi^3/6, \cos\varphi \approx 1 - \phi^2/2, \sin\gamma \approx \gamma - \gamma^3/6, \cos\gamma \approx 1 - \gamma^2/2. \tag{12}$$

The method of multiple scales (MSM) is used to solve the initial value problem described by (7)–(9) and (11). According to this method, we introduce two scales related to the dimensionless time as follows: the fast scale $\tau_0 = \tau$ and the slow scale $\tau_1 = \varepsilon\tau$, where ε is the small parameter. Then, taking into account the existence of three scales we assume the following expansion of the functions ξ, φ, and γ in the power series of the small parameter

$$\xi = \sum_{k=1}^{2} \varepsilon^k x_k(\tau_0, \tau_1) + O(\varepsilon^3), \ \varphi = \sum_{k=1}^{2} \varepsilon^k \phi_k(\tau_0, \tau_1) + O(\varepsilon^3), \ \gamma = \sum_{k=1}^{2} \varepsilon^k \chi_k(\tau_0, \tau_1) + O(\varepsilon^3). \tag{13}$$

The ordinary derivatives with respect to time τ are equivalent to the following differential operators for the two introduced time scales

$$\begin{aligned} \frac{d}{d\tau} &= \frac{\partial}{\partial\tau_0} + \varepsilon\frac{\partial}{\partial\tau_1}, \\ \frac{d^2}{d\tau^2} &= \frac{\partial^2}{\partial\tau_0^2} + 2\varepsilon\frac{\partial^2}{\partial\tau_0\partial\tau_1} + \varepsilon^2\frac{\partial^2}{\partial\tau_1^2} + O(\varepsilon^3). \end{aligned} \tag{14}$$

Moreover, the amplitudes of generalized forces, all damping coefficients, and the eccentricity are assumed to be small, therefore they are expressed in the form [3]

$$c_i = \varepsilon\,\tilde{c}_i,\ f_i = \varepsilon^2\tilde{f}_i,\ i = 1, 2, 3,\ s = \varepsilon\,\tilde{s}, \alpha = \varepsilon\,\tilde{\alpha}, \tag{15}$$

where each of the quantities $\tilde{c}_i$, $\tilde{f}_i$, $\tilde{s}$, $\tilde{\alpha}$ can be understood as $O(1)$ as $\varepsilon \to 0$.

Substituting, in sequence (12), (13) and (15) into the original Eqs. (7)–(9) and replacing the ordinary derivatives by the differential operators (14) we obtain the equations in which the small parameter ε appears in the first, second, and higher powers. These equations should be satisfied for any value of the small parameter. So, after ordering each of these equations according to the powers of small parameter we require that the coefficients of each order of ε equal to zero. Omitting the coefficients of order higher than two, we obtain a sequence of six equations that should be satisfied instead of the original equations. We can organize them into two groups:

- equations of the first order approximation

$$\frac{\partial^2 x_1}{\partial\,\tau_0^2} + x_1 = 0, \tag{16}$$

$$\frac{\partial^2\phi_1}{\partial\tau_0^2} + w_2^2\phi_1 = 0, \tag{17}$$

$$\frac{\partial^2\chi_1}{\partial\tau_0^2} + w_3^2\chi_1 + \frac{w_3^2}{w_2^2}\frac{\partial^2\phi_1}{\partial\tau_0^2} = 0, \tag{18}$$

- equations of the second order approximation

$$\frac{\partial^2 x_2}{\partial\tau_0^2} + x_2 = \tilde{f}_1\cos(\tau_0 p_1) - 3\xi_r^2\tilde{\alpha}x_1 - \frac{1}{2}w_2^2\phi_1^2 - \tilde{c}_1\frac{\partial x_1}{\partial\tau_0} - 2\frac{\partial^2 x_1}{\partial\tau_0\partial\tau_1} + \left(\frac{\partial\phi_1}{\partial\tau_0}\right)^2, \tag{19}$$

$$\frac{\partial^2\phi_2}{\partial\tau_0^2} + w_2^2\phi_2 = \tilde{f}_2\cos(\tau_0 p_2) - \tilde{s}\frac{\partial^2\chi_1}{\partial\tau_0^2} - w_2^2 x_1\phi_1 - \tilde{c}_2\frac{\partial\phi_1}{\partial\tau_0} - 2\,x_1\frac{\partial^2\phi_1}{\partial\tau_0^2} - 2\frac{\partial x_1}{\partial\tau_0}\frac{\partial\phi_1}{\partial\tau_0} - 2\frac{\partial^2\phi_1}{\partial\tau_0\partial\tau_1}, \tag{20}$$

$$\begin{aligned} &\frac{\partial^2\chi_2}{\partial\tau_0^2} + w_3^2\chi_2 + \frac{w_3^2}{w_2^2}\frac{\partial^2\phi_2}{\partial\tau_0^2} = \tilde{f}_3\cos(\tau_0 p_3) - \tilde{c}_3\frac{\partial\chi_1}{\partial\tau_0} - \frac{w_3^2}{w_2^2}\left((\phi_1 - \chi_1)\frac{\partial^2 x_1}{\partial\tau_0^2} + x_1\frac{\partial^2\phi_1}{\partial\tau_0^2}\right) - \\ &2\frac{\partial^2\chi_1}{\partial\tau_0\partial\tau_1} - 2\frac{w_3^2}{w_2^2}\left(\frac{\partial^2\phi_1}{\partial\tau_0\partial\tau_1} + \frac{\partial x_1}{\partial\tau_0}\frac{\partial\phi_1}{\partial\tau_0}\right), \end{aligned} \tag{21}$$

The applied procedure replace, in the approximate meaning, the original equations of motion (7)–(9) with the system of six partial differential Eqs. (16)–(21). This system is solved recursively i.e. solutions of the equations of the lower order are introduced into the equations of higher order approximation. It is worth to notice that differential operators are the same for each step of approximation. The operators of two first equations in every group are mutually independent what significantly simplifies the solving. In every group, the third equation is coupled with the second one. This dependence demands solving at first the first two equations at every step of approximation procedure. Next these solutions are introduced into Eqs. (18) and (21). The general solution of Eqs. (16)–(17) can be presented in the form

$$x_1 = B_1 e^{\mathrm{i}\,\tau_0} + \bar{B}_1 e^{-\mathrm{i}\,\tau_0}, \tag{22}$$

$$\phi_1 = B_2 e^{\mathrm{i}\,w_2\tau_0} + \bar{B}_2 e^{-\mathrm{i}\,w_2\tau_0}, \tag{23}$$

where i denotes the imaginary unit.

The solution (23) is then introduced into Eq. (18) what allows to obtain its solution

$$\chi_1 = B_3 e^{\mathrm{i}\tau_0 w_3} + \bar{B}_3 e^{-\mathrm{i}\tau_0 w_3} + \frac{w_3^2}{w_2^2 - w_3^2}\left(B_2 e^{\mathrm{i}\,w_2\tau_0} + \bar{B}_2 e^{-\mathrm{i}\,w_2\tau_0}\right). \tag{24}$$

The symbol B_i for $i = 1, 2, 3$ in (22)–(24) denotes the unknown complex-valued functions of time scale τ_1, whereas the bar over the symbol denotes its complex conjugate function.

After introducing solutions (22)–(23) into equations of the second order (19)–(20), the secular terms appear. Elimination of them leads to the solvability conditions

$$2\mathrm{i}\frac{dB_1}{d\tau_1} + 3B_1\tilde{\alpha}\,z_r^2 + \mathrm{i}\tilde{c}_1 B_1 = 0, \tag{25}$$

$$2\mathrm{i}w_2\frac{dB_2}{d\tau_1} + \mathrm{i}\tilde{c}_2 w_2 B_2 + \frac{w_2^2 w_3^2}{w_2^2 - w_3^2}\tilde{s}B_2 = 0. \tag{26}$$

There exist also two equations that are conjugate to Eqs. (25)–(26).

Taking advantage of solutions (22)–(24) and conditions (25)–(26), the solutions to Eqs. (19)–(20) are as follows

$$x_2 = \frac{3e^{2\mathrm{i}\,w_2\,\tau_0} w_2^2 B_2^2}{2\left(-1 + 4w_2^2\right)} + w_2^2 B_2\bar{B}_2 + \frac{e^{\mathrm{i}\,p_1\tau_0}\,\tilde{f}_1}{2\left(1 - p_1^2\right)} + CC, \tag{27}$$

$$\phi_2 = -\frac{e^{\mathrm{i}(1+w_2)\tau_0} w_2(2 + w_2)}{1 + 2w_2} B_1 B_2 + \frac{e^{\mathrm{i}(-1+w_2)\tau_0} w_2(-2 + w_2)}{-1 + 2w_2}\bar{B}_1 B_2 + \frac{e^{\mathrm{i}\,w_3\tau_0} w_3^2\tilde{s}}{w_2^2 - w_3^2} B_3 + \frac{e^{\mathrm{i}\,p_2\tau_0}\,\tilde{f}_2}{2(w_2^2 - p_2^2)} + CC. \tag{28}$$

Substituting all the previously obtained solutions i.e. (22)–(24), (27)–(28) and conditions (25)–(26) into Eq. (21), and then demanding of elimination of secular terms lead to the following solvability condition

$$-\mathrm{i} w_3 B_3 \tilde{c}_3 + \frac{w_3^6 \tilde{s}}{w_2^2\left(w_2^2 - w_3^2\right)} - 2\mathrm{i} w_3 \frac{\partial B_3}{\partial \tau_1} = 0. \tag{29}$$

Beside, we obtain also the condition which is conjugate to (29).

The solution to the Eq. (21) in the following general form

$$\begin{aligned}
\chi_2 = & \frac{e^{\mathrm{i}(1+w_2)\tau_0} w_3^2\left(-1+2w_2^3+w_2^4-w_2^2w_3^2-2w_2\left(1+w_3^2\right)\right)B_1B_2}{(1+2w_2)(w_2-w_3)(1+w_2-w_3)(w_2+w_3)(1+w_2+w_3)} + \frac{e^{\mathrm{i}(1+w_3)\tau_0} w_3^2 B_1 B_3}{w_2^2(1+2w_3)} \\
& + \frac{e^{\mathrm{i}(-1+w_2)\tau_0} w_3^2\left(1+2w_2^3-w_2^4+w_2^2w_3^2-2w_2\left(1+w_3^2\right)\right)\bar{B}_1B_2}{(-1+2w_2)(w_2-w_3)(-1+w_2-w_3)(w_2+w_3)(-1+w_2+w_3)} + \frac{e^{\mathrm{i}(-1+w_3)\tau_0} w_3^2 \bar{B}_1 B_3}{w_2^2(-1+2w_3)} \\
& - \frac{\mathrm{i}\, e^{i\, w_2\tau_0} w_3^2\left(\left(w_3^4-w_2^2w_3^2\right)\tilde{c}_2 + w_2\left(w_2^3-w_2w_3^2\right)\tilde{c}_3 + \mathrm{i}\, w_2 w_3^4 \tilde{s}\right)B_2}{(-1+2w_2)\left(w_2^2-w_3^2\right)(-1+w_2-w_3)(-1+w_2+w_3)} + \frac{e^{\mathrm{i}\, p_2\tau_0} p_2^2 w_3^2 \tilde{f}_2}{2w_2^2\left(p_2^2-w_2^2\right)\left(p_2^2-w_3^2\right)} \\
& - \frac{e^{\mathrm{i}\, p_3\tau_0} \tilde{f}_3}{2\left(p_3^2-w_3^2\right)}
\end{aligned} \tag{30}$$

has been obtained analytically.

The solution of the considered problem, given by (22)–(24), (27)–(28) and (30), is valid when the oscillations take place away from any resonance. However, the analytical form of the approximate solution of the problem allows to recognize the parameters of the system for which the resonances occur. The resonance case appears when any of the polynomials that stand in the denominators of the solutions (27)–(28) and (30) tends to zero. The resonances detected in this way can be selected as:

(i) primary external resonance, when $p_1 = 1$, $p_2 = w_2$, $p_3 = w_3$;
(ii) internal resonance, when $w_2 = 1/2$, $w_2 = w_3$, $w_3 = 1/2$, $p_2 = w_3$, $w_2 + w_3 = 1$, $w_2 – w_3 = 1$.

Satisfying of one or more of the conditions listed above, implies the need to modify the method of solution, what is described in Sect. 5.

4 Non-resonant Vibration

The solvability conditions (25), (26) and (29) (together with their complex conjugated forms) constitute a set of constraints with respect to unknown functions $B_1(\tau_1)$, $\bar{B}_1(\tau_1)$, $B_2(\tau_1)$, $\bar{B}_2(\tau_1)$, $B_3(\tau_1)$, $\bar{B}_3(\tau_1)$. They have form of the ordinary differential equations with respect to these functions. Let us postulate that the unknown complex-valued functions $B_i(\tau_1)$ are of the following exponential form

$$B_i = \frac{\tilde{a}_i(\tau_1)}{2} e^{\mathrm{i}\,\psi_i(\tau_1)}, \quad \text{and } a_i = \varepsilon \tilde{a}_i,\ i = 1, 2, 3, \tag{31}$$

where $a_i(\tau_1)$, $\psi_i(\tau_1)$ are real-valued functions and have the meaning of the vibration amplitudes and the phases, respectively.

Substituting relationships (31) into solvability conditions (25), (26) and (29), and then separating real and imaginary parts leads to the modulation equations of amplitudes and phases

$$\frac{da_1}{d\tau} = -\frac{1}{2}c_1 a_1, \quad \frac{d\psi_1}{d\tau} = \frac{3}{2}\xi_r^2 \alpha, \tag{32}$$

$$\frac{da_2}{d\tau} = -\frac{1}{2}c_2 a_2, \quad \frac{d\psi_2}{d\tau} = \frac{s\, w_2 w_3^2}{2\left(w_2^2 - w_3^2\right)}, \tag{33}$$

$$\frac{da_3}{d\tau} = -\frac{1}{2}c_3 a_3, \quad \frac{d\psi_3}{d\tau} = \frac{s w_3^5}{2w_2^2\left(w_2^2 - w_3^2\right)}. \tag{34}$$

Equations (32)–(34) are written after returning to the original denotations according to (15) and (31). The initial conditions supplementing the set (32)–(34) are

$$a_1(0) = a_{10},\ \psi_1(0) = \psi_{10},\ a_2(0) = a_{20},\ \psi_2(0) = \psi_{20},\ a_3(0) = a_{30},\ \psi_3(0) = \psi_{30}. \tag{35}$$

The sets of initial conditions (11) and (35) must be agreed one to another using the final analytical form of the solution.

Solution to the modulation Eqs. (32)–(34) follows

$$a_1 = a_{10}e^{-c_1\tau/2}, \quad \psi_1 = \frac{3}{2}\xi_r^2 \alpha\tau + \psi_{10} \tag{36}$$

$$a_2 = a_{20}e^{-c_2\tau/2}, \quad \psi_2 = \frac{s\, w_2 w_3^2 \tau}{2\left(w_2^2 - w_3^2\right)} + \psi_{20}, \tag{37}$$

$$a_3 = a_{30}e^{-c_3\tau/2}, \quad \psi_3 = -\frac{s\, w_3^5 \tau}{2w_2^2\left(w_2^2 - w_3^2\right)} + \psi_{30}. \tag{38}$$

Finally, expressing the complex-valued functions $B_i(\tau_1)$ by the real-valued functions $a_i(\tau_1)$, $\psi_i(\tau_1)$ according to (31) and then substituting (36)–(38) into solutions (22)–(24), (27), (28) and (30), one can obtain the approximate solution to the original problem (7)–(9) with (11). The solution has the following analytical form

$$\xi = a_1\cos(\tau + \psi_1) - \frac{f_1\cos(p_1\tau)}{p_1^2 - 1} + \frac{1}{4}w_2^2 a_2^2 + \frac{3w_2^2 a_2^2\cos(2w_2\tau + 2\psi_2)}{4(2w_2 - 1)(2w_2 + 1)} \tag{39}$$

$$\begin{aligned}\varphi = {} & a_2\cos(w_2\tau + \psi_2) - \frac{f_2\cos(p_2\tau)}{p_2^2 - w_2^2} + \frac{s w_3^2 a_3\cos(w_3\tau + \psi_3)}{w_2^2 - w_3^2} \\ & - \frac{w_2 a_1 a_2\left((3w_2 + 2 - 2w_2^2)\cos(\tau - w_2\tau + \psi_1 - \psi_2) + (3w_2 - 2 + 2w_2^2)\cos(\tau + w_2\tau + \psi_1 + \psi_2)\right)}{2(4w_2^2 - 1)}\end{aligned} \tag{40}$$

$$\gamma = a_3 \cos(w_2\tau + \psi_2) - \frac{f_3 \cos(p_3\tau)}{p_3^2 - w_3^2} + \frac{p_2^2 w_3^2 f_2 \cos(p_2\tau)}{w_2^2\left(p_2^2 - w_2^2\right)\left(p_2^2 - w_3^2\right)} + \frac{s\, w_3^6 a_2 \cos(w_2\tau + \psi_2)}{(w_2 - w_3)^3 (w_2 + w_3)^3}$$
$$+ \frac{w_3^2\left(-1 + 2w_2^3 + w_2^4 - w_2^2 w_3^2 - 2w_2\left(1 + w_3^2\right)\right) a_1 a_2 \cos(\tau + w_2\tau + \psi_1 + \psi_2)}{2(1 + 2w_2)(w_2 - w_3)(1 + w_2 - w_3)(w_2 + w_3)(1 + w_2 + w_3)}$$
$$- \frac{w_3^2\left(-1 + 2w_2 - 2w_2^3 + w_2^4 - w_2^2 w_3^2 + 2w_2 w_3^2\right) a_1 a_2 \cos(\tau - w_2\tau + \psi_1 - \psi_2)}{2(1 + 2w_2)(w_2 - w_3)(1 + w_2 - w_3)(w_2 + w_3)(1 + w_2 + w_3)}$$
$$- \frac{w_3^2 a_1 a_3 \cos(\tau - w_3\tau + \psi_1 - \psi_3)}{2w_2^2(1 - 2w_3)} + \frac{w_3^2 a_1 a_3 \cos(\tau + w_3\tau + \psi_1 + \psi_3)}{2w_2^2(1 + 2w_3)} - \frac{w_3^2 a_2 \cos(w_2\tau + \psi_2)}{(w_2 - w_3)(w_2 + w_3)} \quad (41)$$

As is earlier mentioned, the solutions (39)–(41) are valid only for non-resonant vibration. If the system is close to any of resonance cases then singularities appear in the analytical solution since some of the denominators in (39)–(41) tend to zero.

The correctness of the solution (39)–(41) is confirmed by numerical solution of the original problem (7)–(9) and (11) obtained using the functions of Mathematica software. The example of time histories of the system oscillations involving the strong influence of the initial conditions are presented in Fig. 2. Parameters fixed in calculations are as follows:

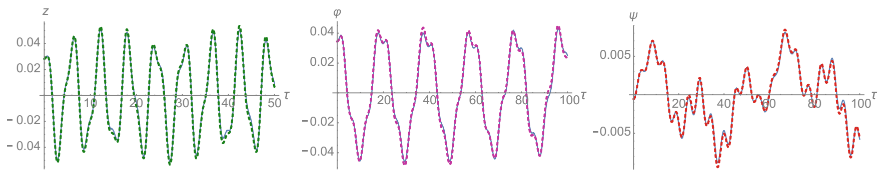

Fig. 2 Time history of vibration; solid curve—analytical solution, dashed curve—numerical solution

$\alpha = 2.25$, $f_2 = 0.01$, $f_3 = 0.002$, $f_1 = 0.05$, $c_1 = 0.001$, $c_2 = 0.001$, $c_3 = 0.001$, $w_2 = 0.32$, $w_3 = 0.09$, $p_1 = 2.3$, $p_2 = 1.28$, $p_3 = 1.18$, $e = 0.3$, $a_{10} = 0.04$, $a_{20} = 0.04$, $a_{30} = 0.004$, $\psi_{10} = 0.0$, $\psi_{20} = 0.0$, $\psi_{30} = 0.0$.

Figure 2 consists of three parts. Each of them present two solutions related to general coordinate $z(\tau)$, $\varphi(\tau)$ and $\gamma(\tau)$, (from left to right, respectively). These solutions are obtained in two ways as analytical solutions (39)–(41) using MSM and by numerical integration of the original equations.

5 Resonant Vibration

Let us consider the case of the three primary main resonances, induced by the triple external loading, occurring simultaneously i.e. $1 \approx p_1$, $w_2 \approx p_2$, and $w_3 \approx p_3$. The resonance effects are reflected in the secular generating terms. In order to deal with the resonance, the detuning parameters, as a measure of the distance of the system vibration from the strict resonance conditions, are introduced in the following way

$$p_1 = 1 + \sigma_1,\ p_2 = w_2 + \sigma_2, p_3 = w_3 + \sigma_3. \tag{42}$$

We assume the detuning parameters are of the order of small parameter, i.e. we take

$$\sigma_i = \varepsilon\tilde{\sigma}_i \quad i = 1, 2, 3. \tag{43}$$

The conditions (42)–(43) are introduced into Eqs. (7)–(9). Further procedure is analogous to this one described in the two previous sections. Therefore, we focus mainly on the secular terms generated by the resonance conditions (42). As a result of elimination of these secular terms we get the solvability conditions of the problem. They may be written as follows

$$2\mathrm{i}\frac{dB_1}{d\tau_1} + 3B_1\tilde{\alpha}\,\xi_r^2 + \mathrm{i}\tilde{c}_1 B_1 - \frac{1}{2}e^{\mathrm{i}\,\tau_1\sigma_1}\tilde{f}_1 = 0, \tag{44}$$

$$2\mathrm{i}w_2\frac{dB_2}{d\tau_1} + \mathrm{i}\tilde{c}_2 w_2 B_2 + \frac{w_2^2 w_3^2}{w_2^2 - w_3^2}\tilde{s}B_2 - \frac{1}{2}e^{\mathrm{i}\,\tau_1\sigma_2}\tilde{f}_2 = 0, \tag{45}$$

$$-\mathrm{i}w_3 B_3\tilde{c}_3 + \frac{w_3^6\tilde{s}B_3}{w_2^2\left(w_2^2 - w_3^2\right)} - 2\mathrm{i}w_3\frac{dB_3}{d\tau_1} + \frac{1}{2}e^{\mathrm{i}\,\tau_1\sigma_3}\tilde{f}_3 = 0. \tag{46}$$

5.1 Modulation Problem Near Resonances

The solvability conditions (44)–(46) create a system of the ordinary differential equations with respect to unknown functions $B_1(\tau_1)$, $\bar{B}_1(\tau_1)$, $B_2(\tau_1)$, $\bar{B}_2(\tau_1)$, $B_3(\tau_1)$, $\bar{B}_3(\tau_1)$. After introducing the exponential form (31) for the complex-valued functions $B_i(\tau_1)$, it is convenient to define the modified phases in the following way

$$\begin{aligned}\theta_1(\tau_1) &= \tau_1\tilde{\sigma}_1 - \psi_1(\tau_1),\\ \theta_2(\tau_1) &= \tau_1\tilde{\sigma}_2 - \psi_2(\tau_1),\\ \theta_3(\tau_1) &= \tau_1\tilde{\sigma}_3 - \psi_3(\tau_1).\end{aligned} \tag{47}$$

After substitution the modified phases (47) into solvability conditions (44)–(46) and having returned to the original denotations according to (14)–(15), (31) and (43), the obtained modulation equations become autonomous of the following form

$$\frac{da_1}{d\tau} = -\frac{1}{2}a_1c_1 + \frac{f_1}{2}\sin(\theta_1), \tag{48}$$

$$\frac{d\theta_1}{d\tau}a_1 = a_1\sigma_1 - \frac{3}{2}\xi_r^2\alpha a_1 + \frac{f_1}{2}\cos(\theta_1), \tag{49}$$

$$\frac{da_2}{d\tau} = -\frac{1}{2}c_2a_2 + \frac{f_2}{2\mathrm{w}_2}\sin(\theta_2), \tag{50}$$

$$\frac{d\theta_2}{d\tau}a_2 = \sigma_2a_2 - \frac{s\,w_2w_3^2a_2}{2\left(w_2^2 - w_3^2\right)} + \frac{f_2}{w_2}\cos(\theta_2), \tag{51}$$

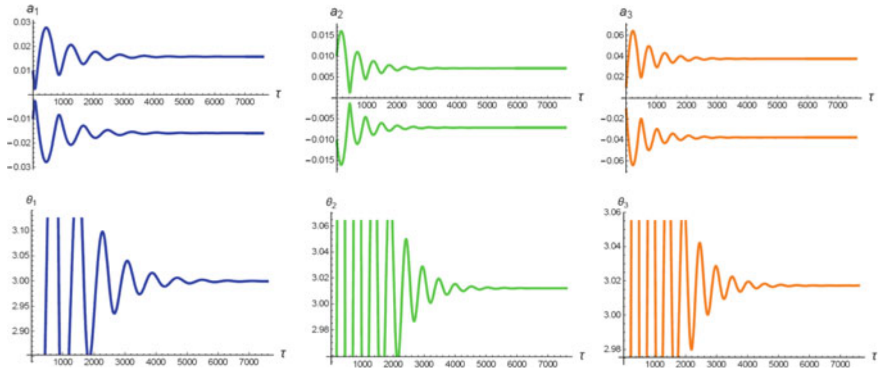

Fig. 3 Temporal behaviour of amplitudes and modified phases which tend to steady point

$$\frac{da_3}{d\tau} = -\frac{1}{2}c_3 a_3 + \frac{f_3}{2\mathrm{w}_3}\sin(\theta_3), \tag{52}$$

$$\frac{d\theta_3}{d\tau}a_3 = \sigma_3 a_3 + \frac{s\, w_3^5 a_3}{2w_2^2\left(w_2^2 - w_3^2\right)} + \frac{f_3}{w_3}\cos(\theta_3). \tag{53}$$

In contrary to the previously discussed case of the non-resonant vibration, the modulation Eqs. (48)–(53) cannot be solved in the analytical manner.

The initial conditions supplementing the set (48)–(53) are as follows

$$a_1(0) = a_{10},\ \psi_1(0) = \psi_{10},\ a_2(0) = a_{20},\ \psi_2(0) = \psi_{20},\ a_3(0) = a_{30},\ \psi_3(0) = \psi_{30}, \tag{54}$$

and must be compatible with the initial conditions (11).

The modulation curves describe the slow time changes in the motion. For some conditions vibration tends to the steady values of the amplitudes and phases. This case is presented in Fig. 3. The assumed parameters are:

$\sigma_1 = 0.01$, $\sigma_2 = 0.01$, $\sigma_3 = 0.01$, $w_2 = 0.293$, $w_3 = 0.055$, $s = 0.02$, $f_1 = 0.00025$, $f_2 = 0.00005$, $f_3 = 0.00005$, $c_1 = 0.00223$, $c_2 = 0.0031$, $c_3 = 0.003$, $\alpha = 0.2$, $a_{10} = 0.01$, $a_{20} = 0.01$, $a_{30} = 0.01$, $\psi_{10} = 0$, $\psi_{20} = 0$, $\psi_{30} = 0$.

Equations (48)–(53) describe effects related to the slow time scale. They allow to observe and follow non-steady oscillations, and to recognize and follow qualitative transitions in the character of motion. A good way of illustration of the dynamical behaviour of the system are trajectories depicted in a space the points of which are amplitudes and modified phases, and so the functions connected to the modulation equations. The projections of the trajectories onto the chosen planes of this space are shown in Fig. 4. The simulations are carried out for the same data as previously.

After the transient state, all trajectories achieve the stable steady state, although the duration of the transient vibration is various for the particular general coordinates. The steady state conditions correspond to the demand of vanish of time derivatives

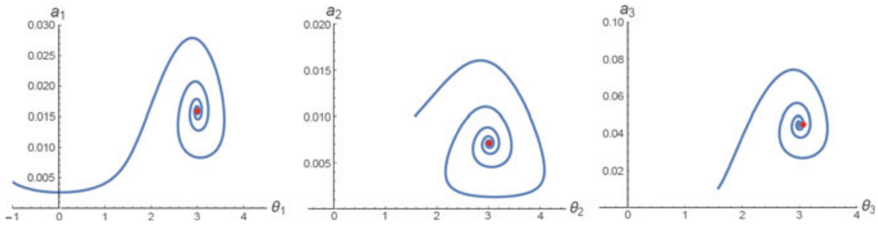

Fig. 4 Trajectories of motion in the amplitude-modified plane; red points indicate stable state

of amplitudes and modified phases in modulation Eqs. (48)–(53). They are governed by the following equations

$$-\frac{1}{2}a_1c_1 + \frac{f_1}{2}\sin(\theta_1) = 0, \tag{55}$$

$$a_1\sigma_1 - \frac{3}{2}\xi_r^2\alpha a_1 + \frac{f_1}{2}\cos(\theta_1) = 0, \tag{56}$$

$$-\frac{1}{2}c_2a_2 + \frac{f_2}{2\mathrm{w}_2}\sin(\theta_2) = 0, \tag{57}$$

$$\sigma_2a_2 - \frac{s\, w_2w_3^2a_2}{2\left(w_2^2 - w_3^2\right)} + \frac{f_2}{w_2}\cos(\theta_2) = 0, \tag{58}$$

$$-\frac{1}{2}c_3a_3 + \frac{f_3}{2\mathrm{w}_3}\sin(\theta_3) = 0, \tag{59}$$

$$\sigma_3a_3 + \frac{s\, w_3^5a_3}{2w_2^2\left(w_2^2 - w_3^2\right)} + \frac{f_3}{w_3}\cos(\theta_3) = 0. \tag{60}$$

Equations (55)–(60) stand for algebraic system with unknown values of amplitudes and modified phases a_1, a_2, a_3, θ_1, θ_2 and θ_3 in steady-state motion.

The fully explicit form of the approximate solution of the original problem in case of the resonance is usually impossible to achieve. The modulation equations due to their complexity are solved in numerical manner. Having however the solutions of the governing equations in the analytic form of functions of amplitudes and phases (or modified phases), we can substitute the numerical solutions into this analytical form. Time histories obtained in this way with comparison to the numerically obtained solutions are presented in Fig. 5. The results presented in Fig. 5 are obtained for the same values of system parameters as the ones listed above and demonstrated in Figs. 3 and 4.

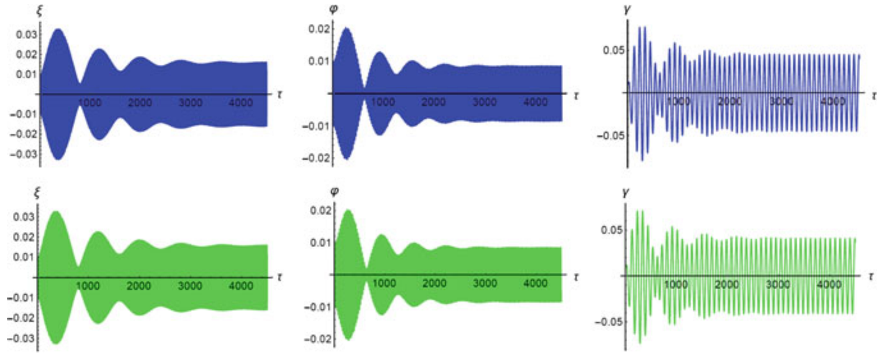

Fig. 5 Time histories; upper graphs are obtained analytically while the lower ones numerically

6 Conclusions

The approximate solution to the governing equations has been obtained using the multi scales method with two time scales. The analytical form of this solution is the main advantage of the applied approach, giving the possibility to the qualitative and quantitative study of the system dynamics in a wide range of the frequency spectrum. The approximate solution for non-resonant vibration has been obtained in fully analytical form because the modulation equations governing the evolution of amplitudes and phases in the slow time scale were solved analytically. Admittedly an approximate but however analytical form of this solution create, among others, the possibility to determine the conditions at which the resonances occur. The adequate conditions for possible resonances have been detected. The case of three primary main resonances occurring simultaneously has been considered.

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